CS473 - Algorithms I

Lecture 5
Quicksort

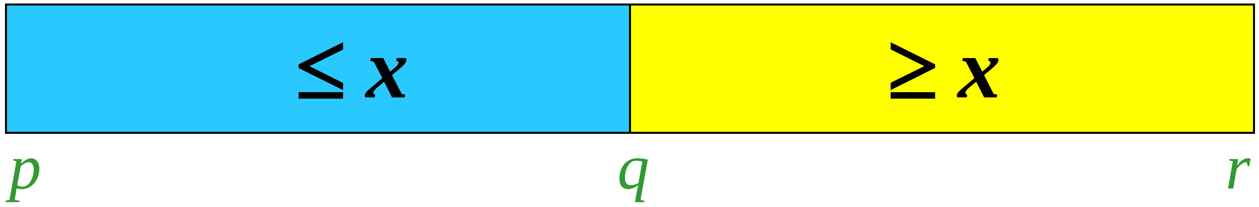
View in slide-show mode

Quicksort

- One of the most-used algorithms in practice
- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm
- In-place algorithm
  - The additional space needed is O(1)
  - The sorted array is returned in the input array
  - *Reminder: Insertion-sort is also an in-place algorithm, but Merge-Sort is not in-place.*
- Very practical

Quicksort

1. Divide: Partition the array into 2 subarrays such that elements in the lower part ≤ elements in the higher part



$\leq \boldsymbol{x}$ $\geq \boldsymbol{x}$

p q r

2. Conquer: Recursively sort 2 subarrays
3. Combine: Trivial (because in-place)

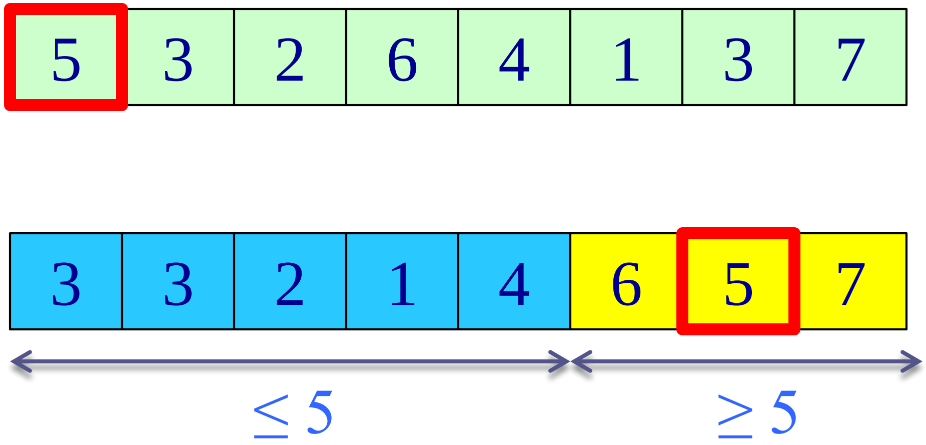
- Key: Linear-time ($\boldsymbol{\Theta}(n)$) partitioning algorithm

Divide: Partition the array around a pivot element

1. Choose a **pivot** element x
2. Rearrange the array such that:

   Left subarray: All elements ≤ x

   Right subarray: All elements ≥ x



Input:

5	3	2	6	4	1	3	7

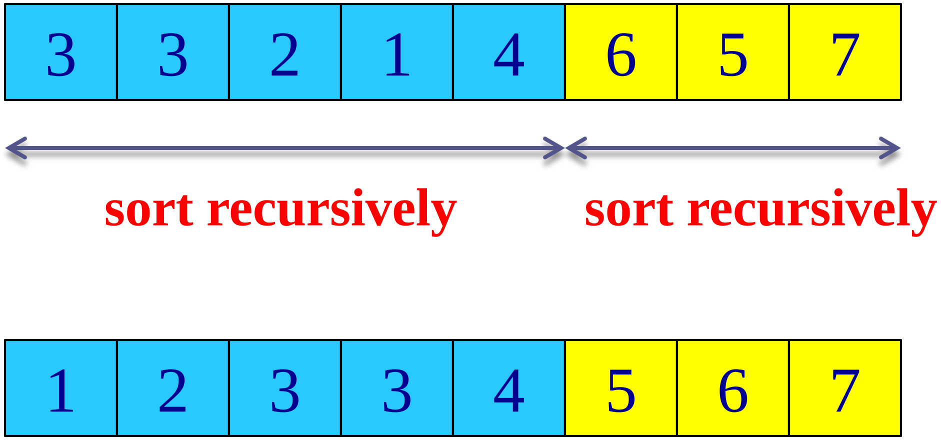
e.g. x = 5

After partitioning:

3	3	2	1	4	6	5	7

Conquer: Recursively Sort the Subarrays

Note: Everything in the left subarray ≤ everything in the right subarray



3	3	2	1	4	6	5	7

sort recursively sort recursively

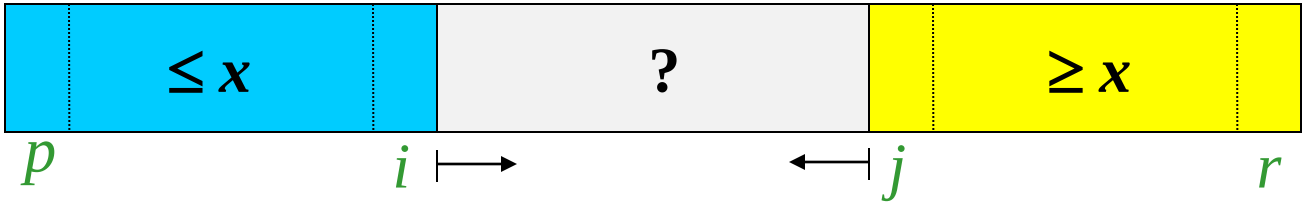
After conquer:

1	2	3	3	4	5	6	7

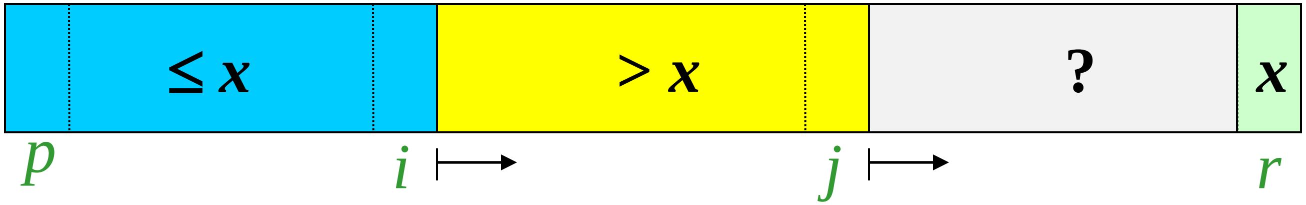
Note: Combine is trivial after conquer. Array already sorted.

Two partitioning algorithms

1. Hoare's algorithm: Partitions around the first element of subarray (*pivot* = *x* = A[*p*])



2. Lomuto's algorithm: Partitions around the last element of subarray (*pivot* = *x* = A[*r*])



Hoare's Partitioning Algorithm

1. Choose a pivot element: pivot = x = A[p]
2. Grow two regions:

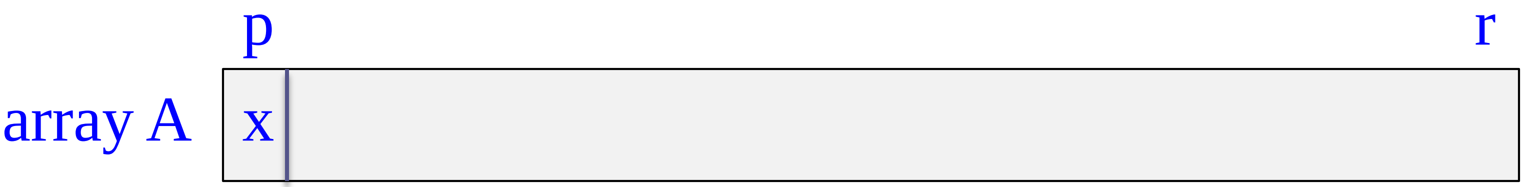
   from left to right: A[p..i]

   from right to left: A[j..r]

   such that:

   every element in A[p…i] ≤ pivot

   every element in A[j…r] ≥ pivot



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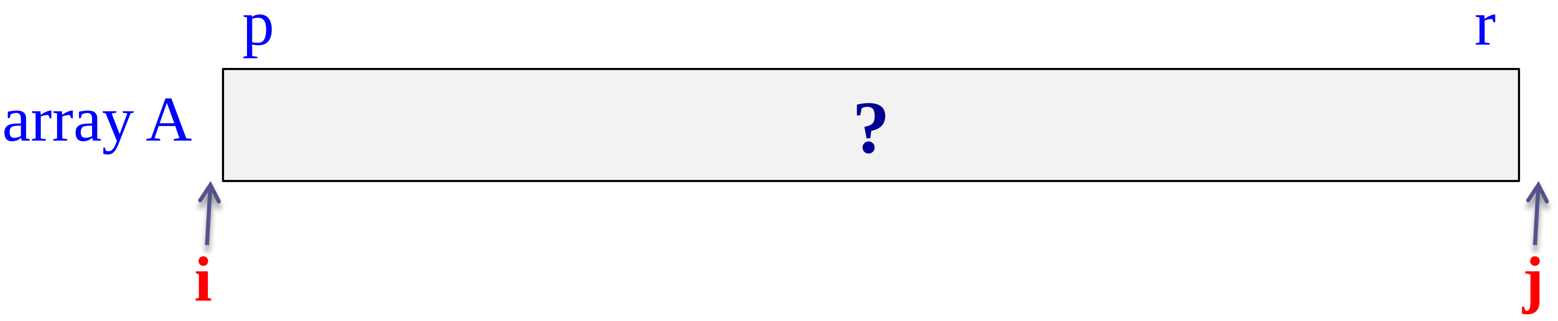

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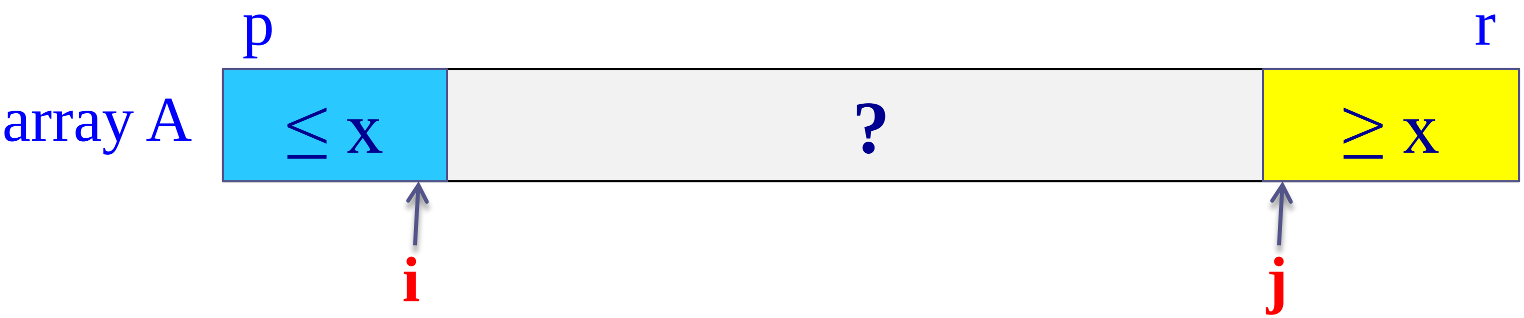

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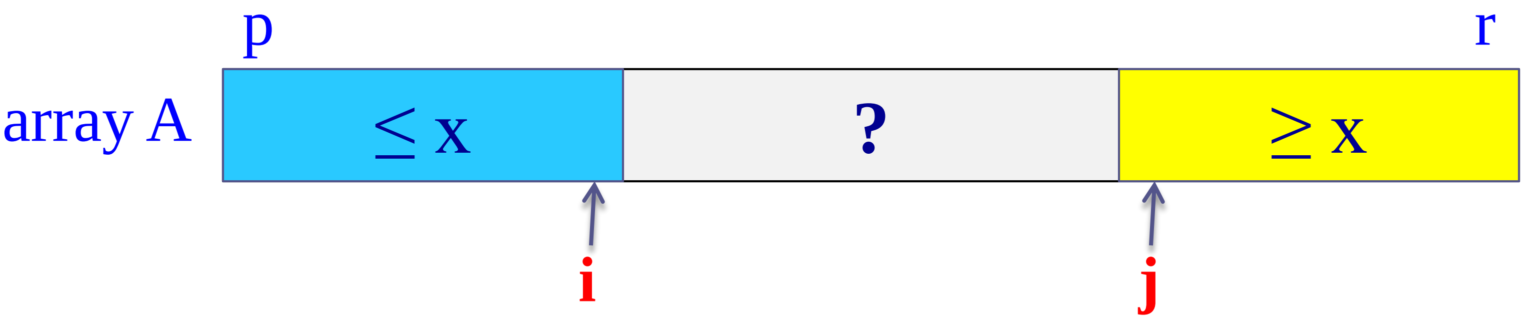

Hoare's Partitioning Algorithm

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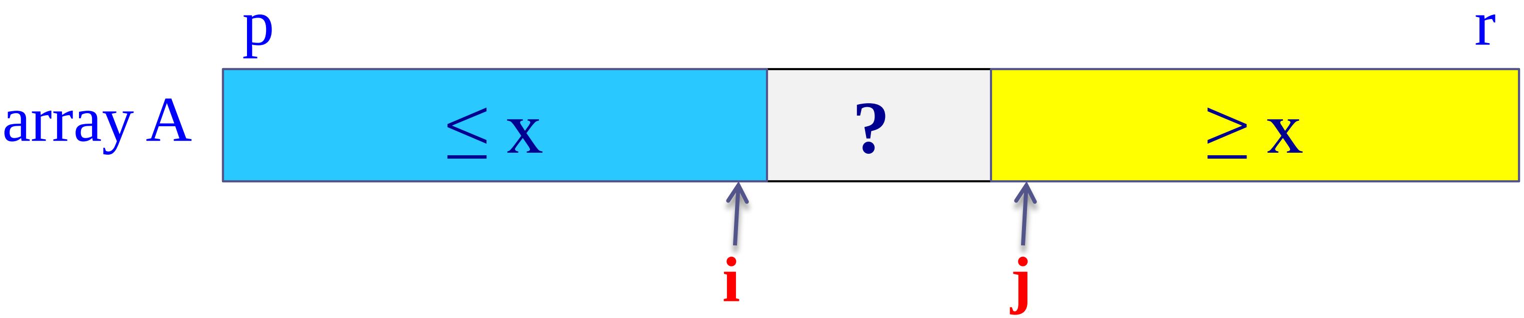

Hoare's Partitioning Algorithm

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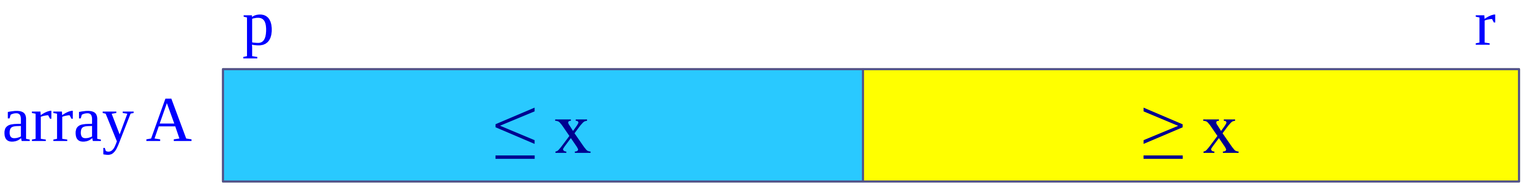

Hoare's Partitioning Algorithm

```
H-PARTITION (A, $p$, $r$)
    $pivot \leftarrow A[p]$
    $i \leftarrow p - 1$
    $j \leftarrow r + 1$
    while true do
        repeat $j \leftarrow j - 1$ until $A[j] \leq pivot$
        repeat $i \leftarrow i + 1$ until $A[i] \geq pivot$
        if $i < j$ then exchange $A[i] \leftrightarrow A[j]$
        else return $j$
```

	p							r
array A	5	3	2	6	4	1	3	7

pivot = 5

Hoare’s Partitioning Algorithm

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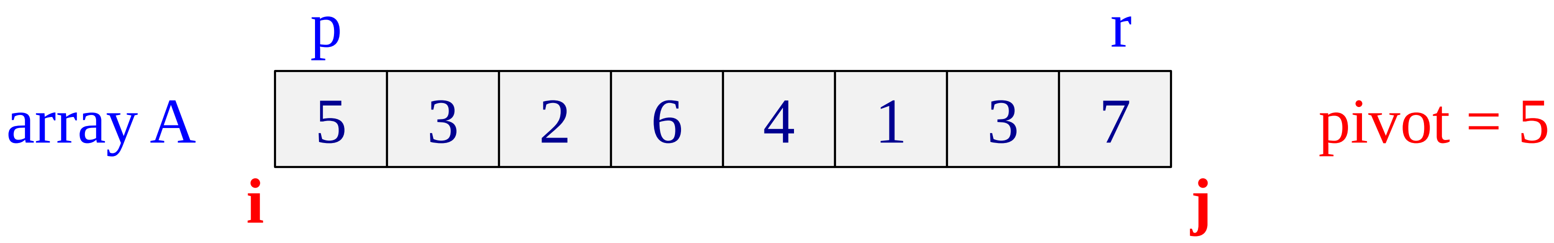

array A

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i j

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Hoare's Partitioning Algorithm

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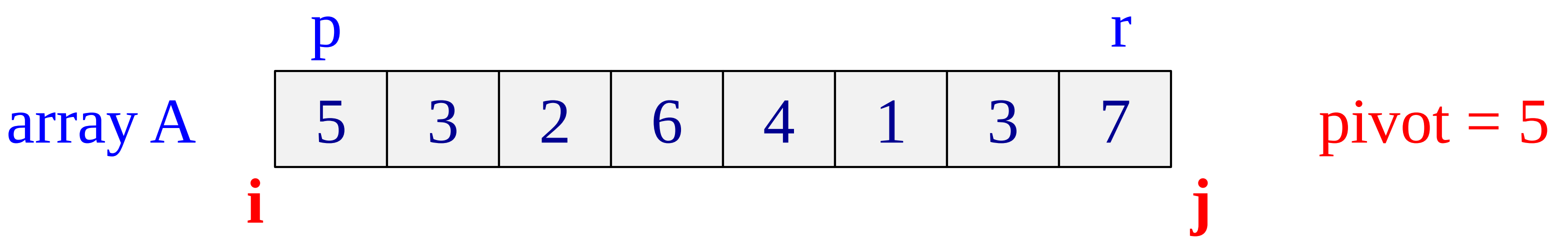

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	i							j	

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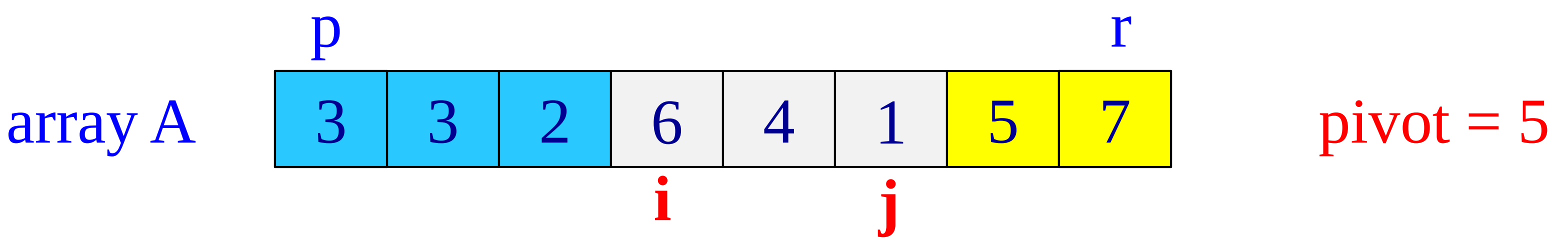

	p							r	
array A	3	3	2	6	4	1	5	7	pivot = 5
				i		j			

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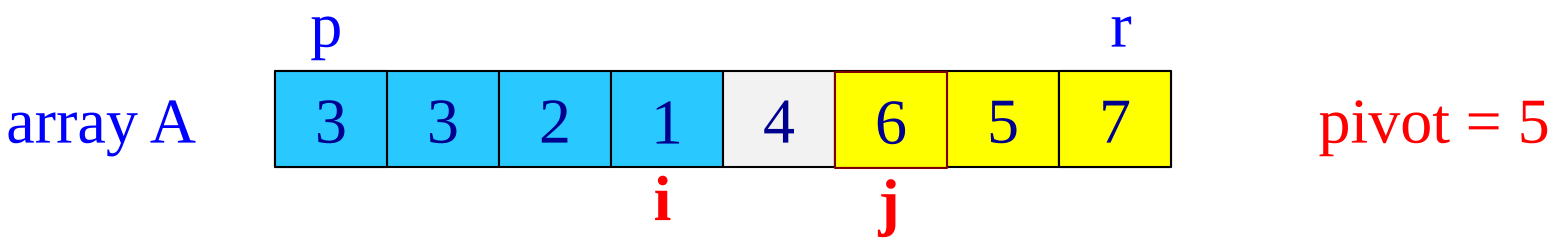

p r

array A

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i j

pivot = 5

Hoare’s Partitioning Algorithm

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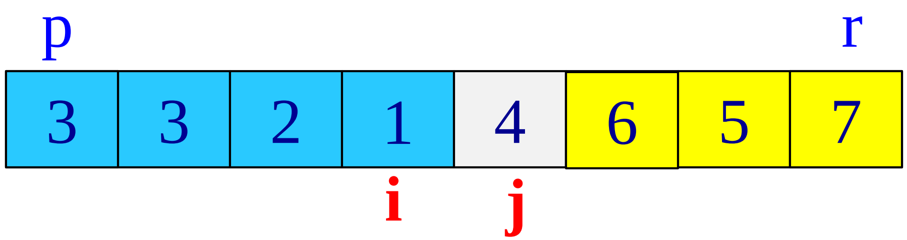
array A



p r

3 3 2 1 4 6 5 7

i j

pivot = 5

Hoare's Partitioning Algorithm

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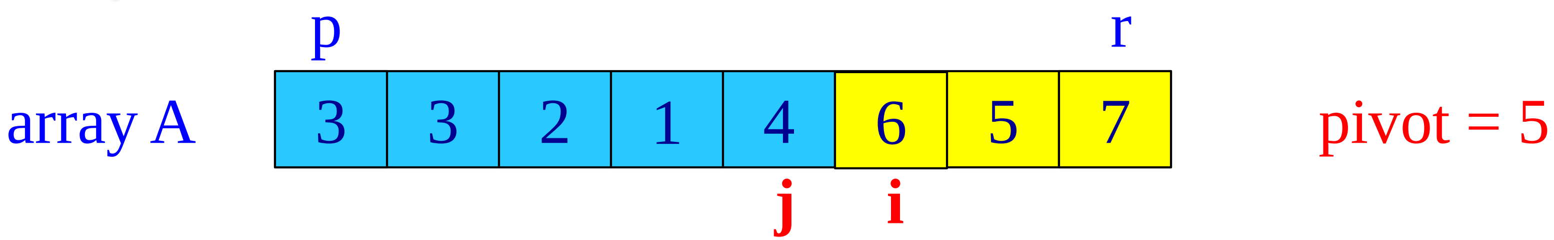

Hoare's Partitioning Algorithm - Notes

```
H-PARTITION (A, p, r)
    pivot ← A[p]
    i ← p – 1
    j ← r + 1
    while true do
        repeat j ← j – 1 until A[j] ≤ pivot
        repeat i ← i + 1 until A[i] ≥ pivot
        if i < j then exchange A[i] ↔ A[j]
        else return j
```

Elements are exchanged when

- A[i] is **too large** to belong to the left region
- A[j] is **too small** to belong to the right region

assuming that the inequality is strict

The two regions A[p..i] and A[j..r] grow until

$$A[i] \geq \text{pivot} \geq A[j]$$

Hoare's Partitioning Algorithm

```
H-PARTITION (A, p, r)
    $pivot \leftarrow A[p]$
    $i \leftarrow p - 1$
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    while true do
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        if $i < j$ then exchange $A[i] \leftrightarrow A[j]$
        else return $j$
```
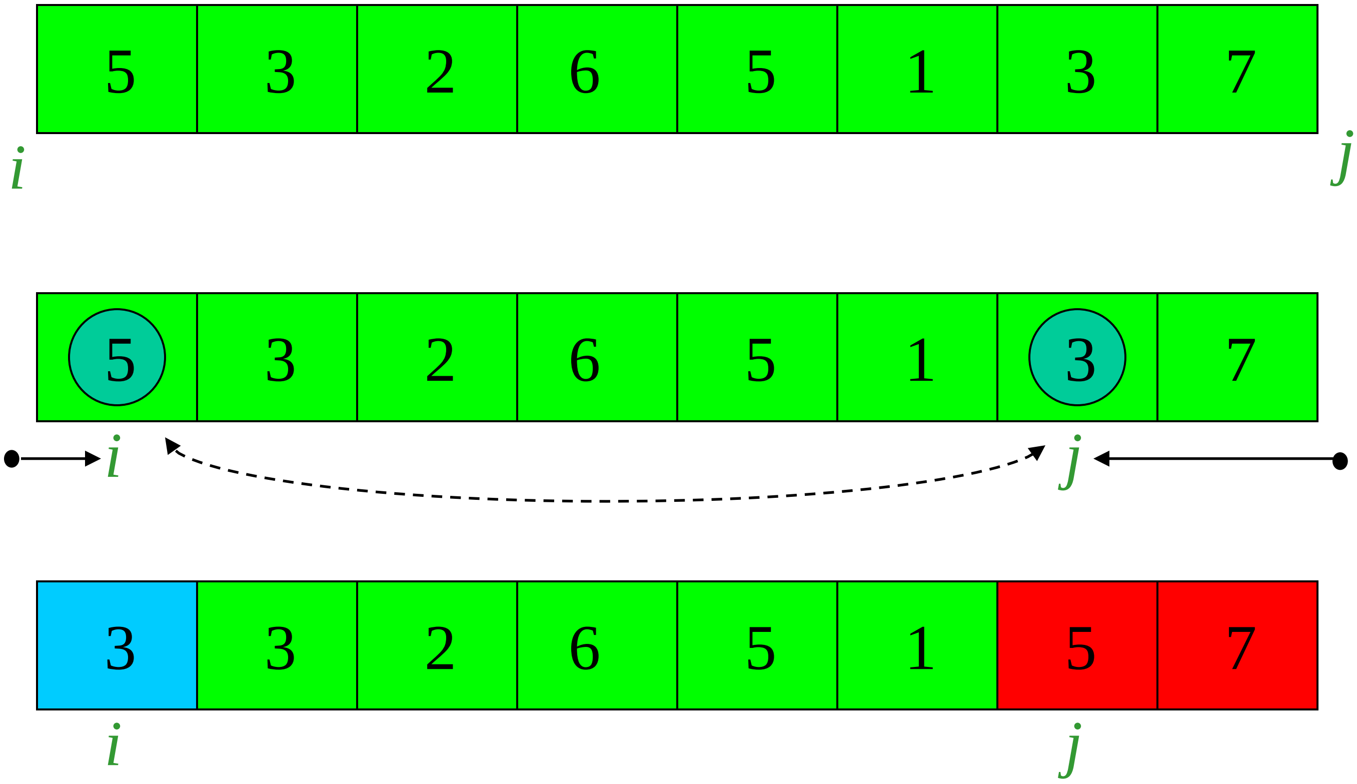

What is the asymptotic runtime of Hoare's partitioning algorithm?

$\Theta(n)$

Hoare’s Algorithm: Example 2 (pivot = 5)

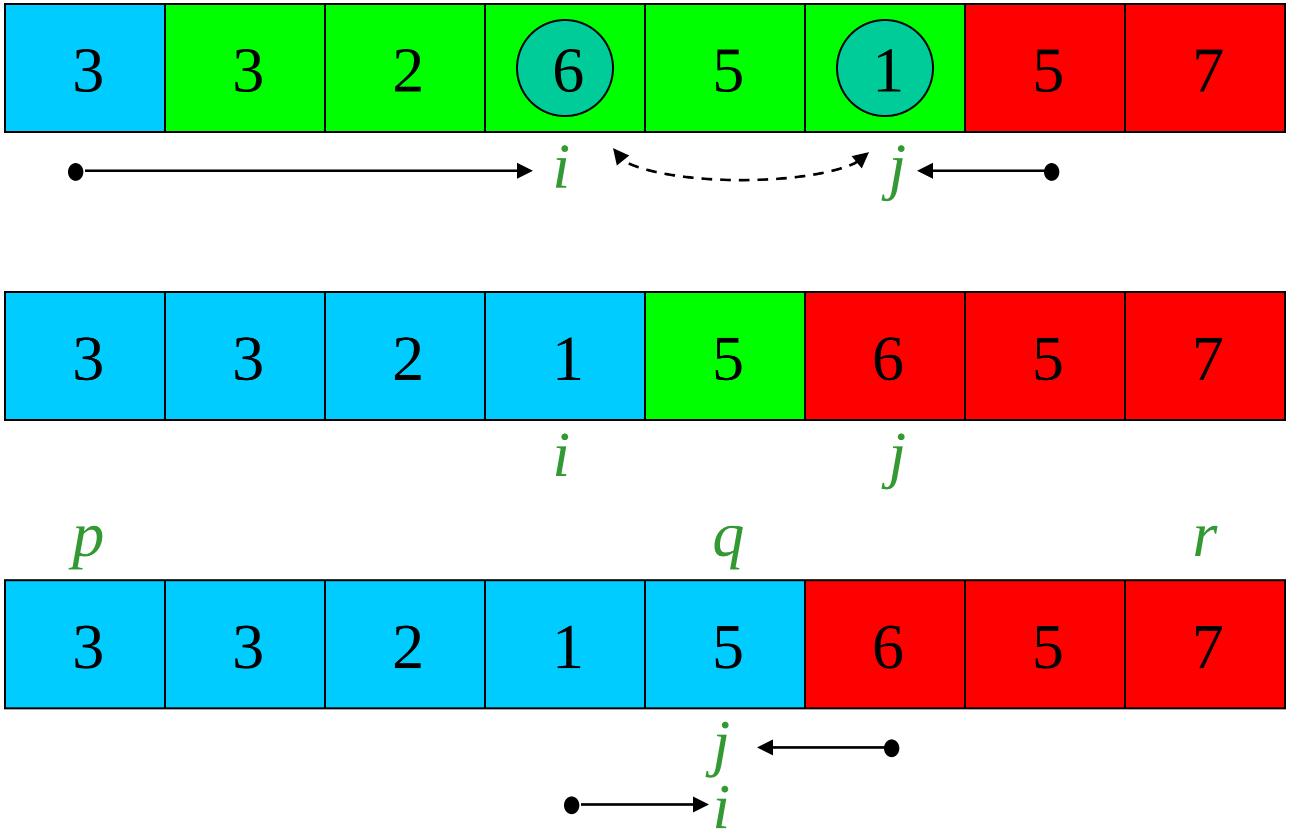


5	3	2	6	5	1	3	7

5	3	2	6	5	1	3	7

3	3	2	6	5	1	5	7

Hoare’s Algorithm: Example 2 (pivot = 5)



3	3	2	6	5	1	5	7

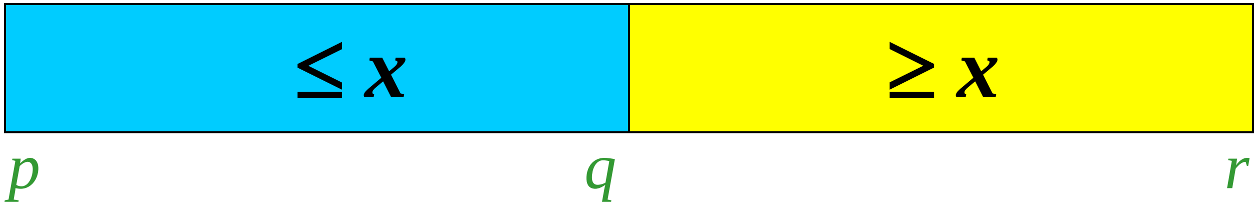
3	3	2	1	5	6	5	7

3	3	2	1	5	6	5	7

Termination: $i = j = 5$

```
QUICKSORT (A, $p$, $r$)
    if $p < r$ then
        $q \leftarrow$ H-PARTITION(A, $p$, $r$)
        QUICKSORT(A, $p$, $q$)
        QUICKSORT(A, $q$ +1, $r$)
```

Initial invocation: QUICKSORT(A, 1, n)



Question

```
H-PARTITION (A, p, r)
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```

```
QUICKSORT (A, p, r)
    if $p < r$ then
        $q \leftarrow$ H-PARTITION(A, p, r)
        QUICKSORT(A, p, q)
        QUICKSORT(A, q +1, r)
```

Q: What happens if we select pivot to be A[r] instead of A[p] in H-PARTITION?

✖ *a)* *QUICKSORT* will still work correctly.

✖ *b)* *QUICKSORT* may return incorrect results for some inputs.

✔ *c)* *QUICKSORT* may not terminate for some inputs.

Hoare's Partitioning Algorithm: Pivot Selection

```
H-PARTITION (A, p, r)
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    $j \leftarrow r + 1$
    while true do
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        if $i < j$ then exchange $A[i] \leftrightarrow A[j]$
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```

```
QUICKSORT (A, p, r)
    if $p < r$ then
        $q \leftarrow$ H-PARTITION(A, p, r)
        QUICKSORT(A, p, q)
        QUICKSORT(A, q +1, r)
```

If A[r] is chosen as the pivot:

Consider the example where A[r] is the largest element in the array:

5	3	6	4	3	7

End of H-PARTITION: **i = j = r**

In QUICKSORT: **q = r**

So, recursive call to:

QUICKSORT (A, p, q=r)

➔ **infinite loop**

Correctness analysis needed!

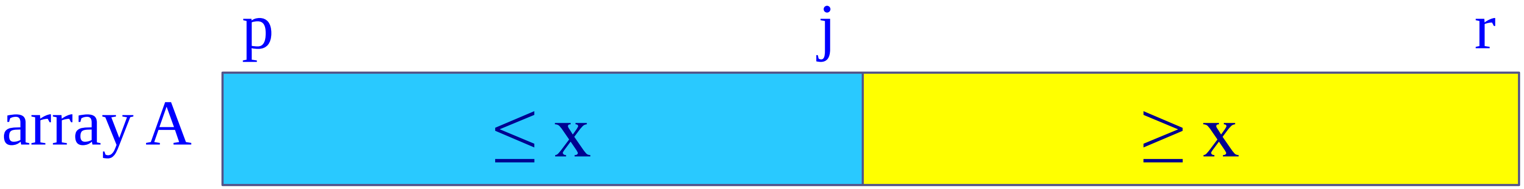
Correctness of Hoare's Algorithm

We need to prove 3 claims to show correctness:

a) Indices i & j never reference A outside the interval $A[p..r]$

b) Split is always non-trivial; i.e., $j \neq r$ at termination

c) Every element in $A[p..j] \leq$ every element in $A[j+1..r]$ at termination



Correctness of Hoare's Algorithm

Notations:

k: # of times the while-loop iterates until termination

i_m: the value of index i at the end of iteration m

j_m: the value of index j at the end of iteration m

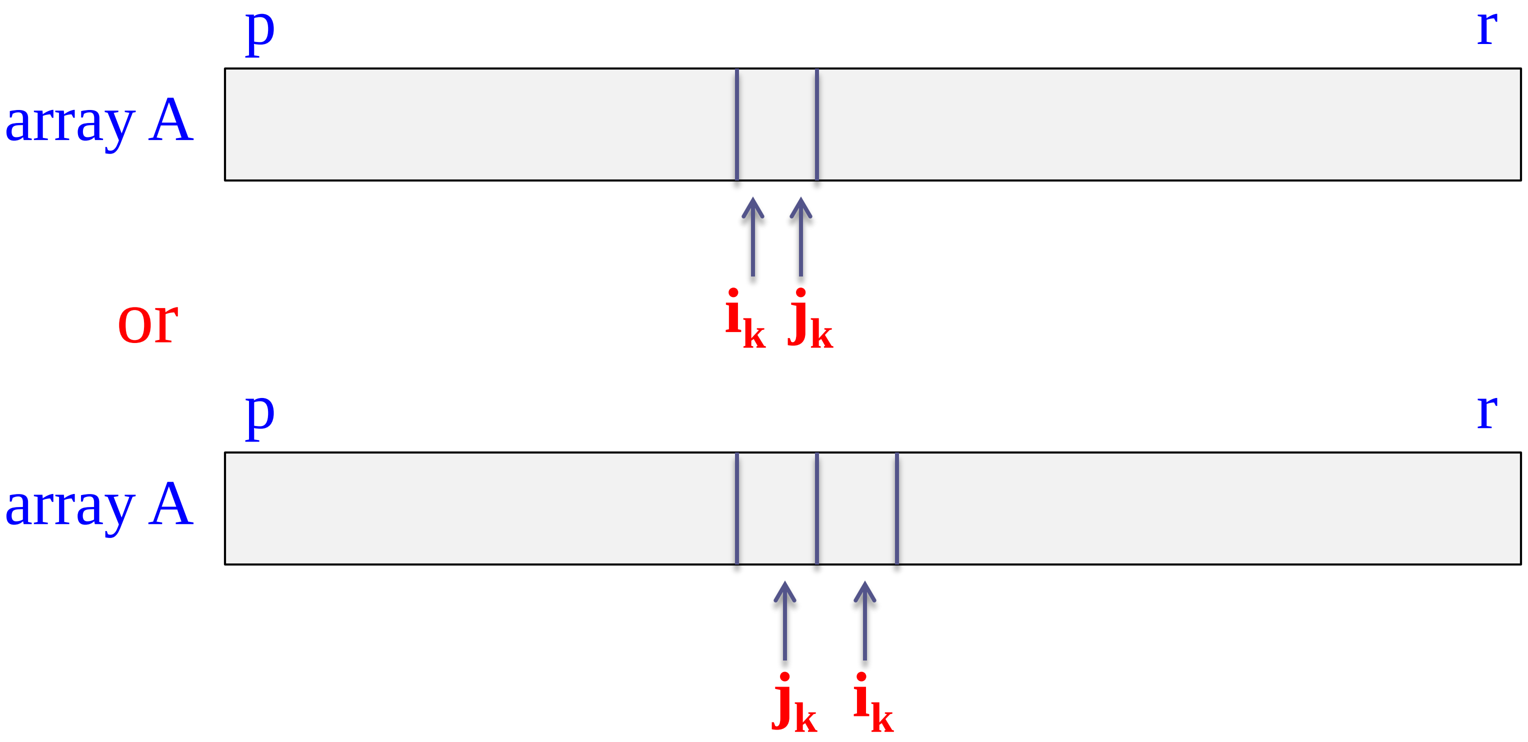
x: the value of the pivot element

Note: We always have $i_1 = p$ and $p \leq j_1 \leq r$

because $x = A[p]$

Correctness of Hoare's Algorithm

Lemma 1: Either $i_k = j_k$ or $i_k = j_k + 1$ at termination



Correctness of Hoare's Algorithm

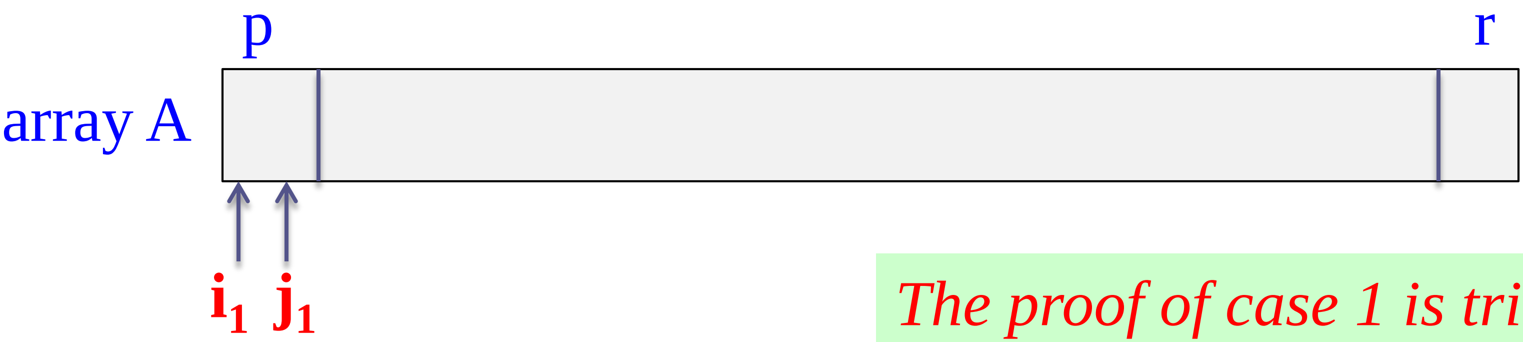
Proof of Lemma 1:

The algorithm terminates when $i \geq j$ (the else condition).

So, it is sufficient to prove that $\mathbf{i_k - j_k \leq 1}$

There are 2 cases to consider:

Case 1: $k = 1$, i.e. the algorithm terminates in a single iteration



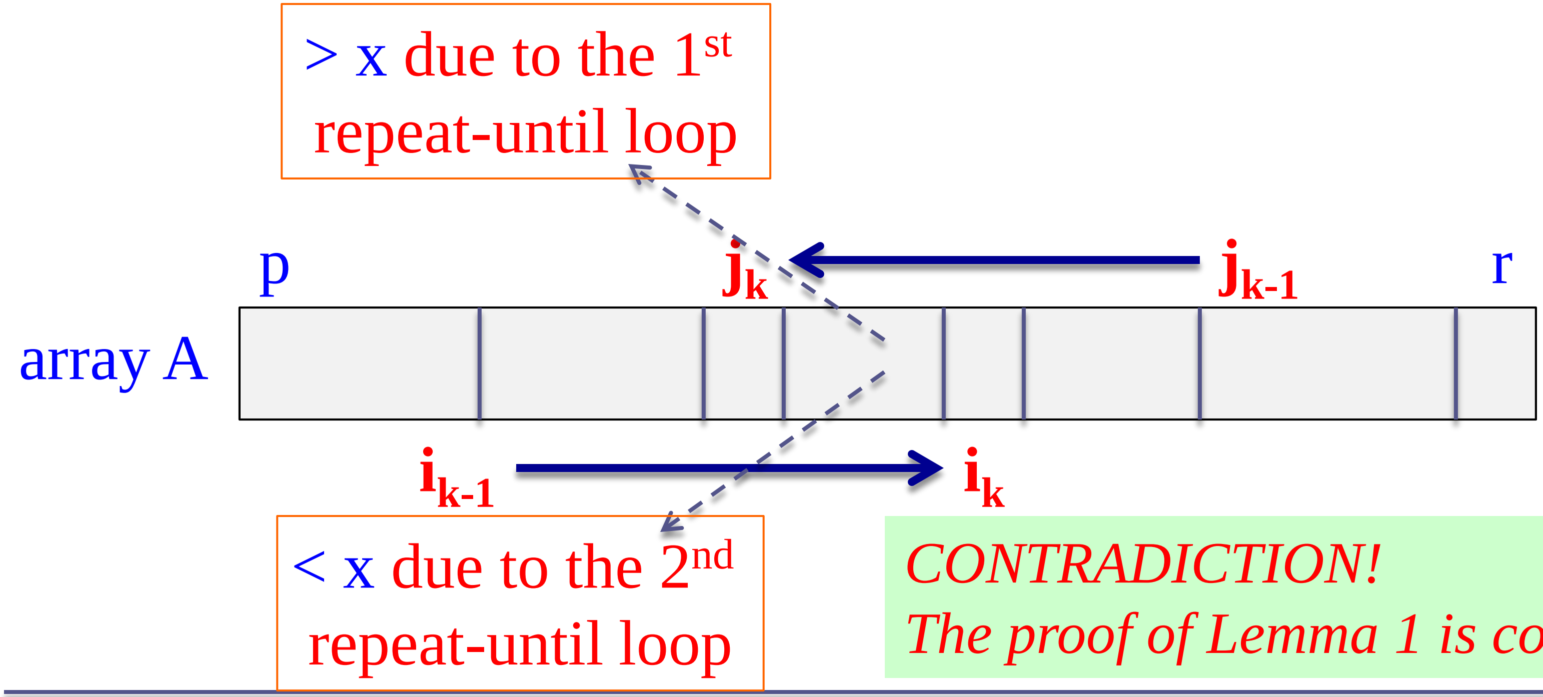
The proof of case 1 is trivial

Correctness of Hoare's Algorithm

Proof of Lemma 1 (cont'd):

Case 2: $k > 1$, i.e. the alg. does not terminate in a single iter.

By contradiction, assume there is a run with $i_k - j_k > 1$



> x due to the 1st repeat-until loop
p
j_k
j_{k-1}
r
array A
i_{k-1}
i_k
< x due to the 2nd repeat-until loop

CONTRADICTION!

The proof of Lemma 1 is complete!

Correctness of Hoare's Algorithm

Original correctness claims:

(a) Indices *i* & *j* never reference A outside the interval A[*p*…*r*]

(b) Split is always non-trivial; i.e., $j \neq r$ at termination

Proof:

For k = 1: Trivial because $i_1 = j_1 = p$ *(see Case 1 in proof of Lemma 2)*

For k > 1:

$i_k > p$ and $j_k < r$ *(due to the repeat-until loops moving indices)*

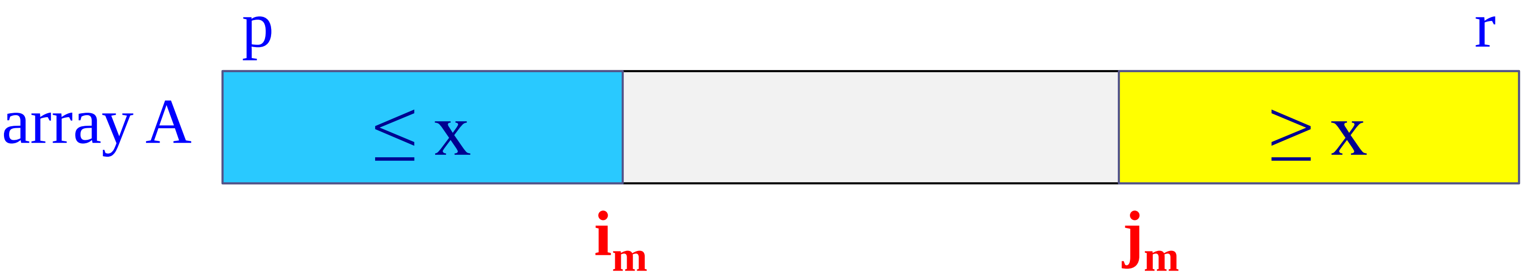
$i_k \leq r$ and $j_k \geq p$ *(due to Lemma 1 and the statement above)*

➔ The proof of claims (a) and (b) complete

Correctness of Hoare's Algorithm

Lemma 2: At the end of iteration m, where $m < k$ (*i.e.* *m is not the last iteration*), we must have:

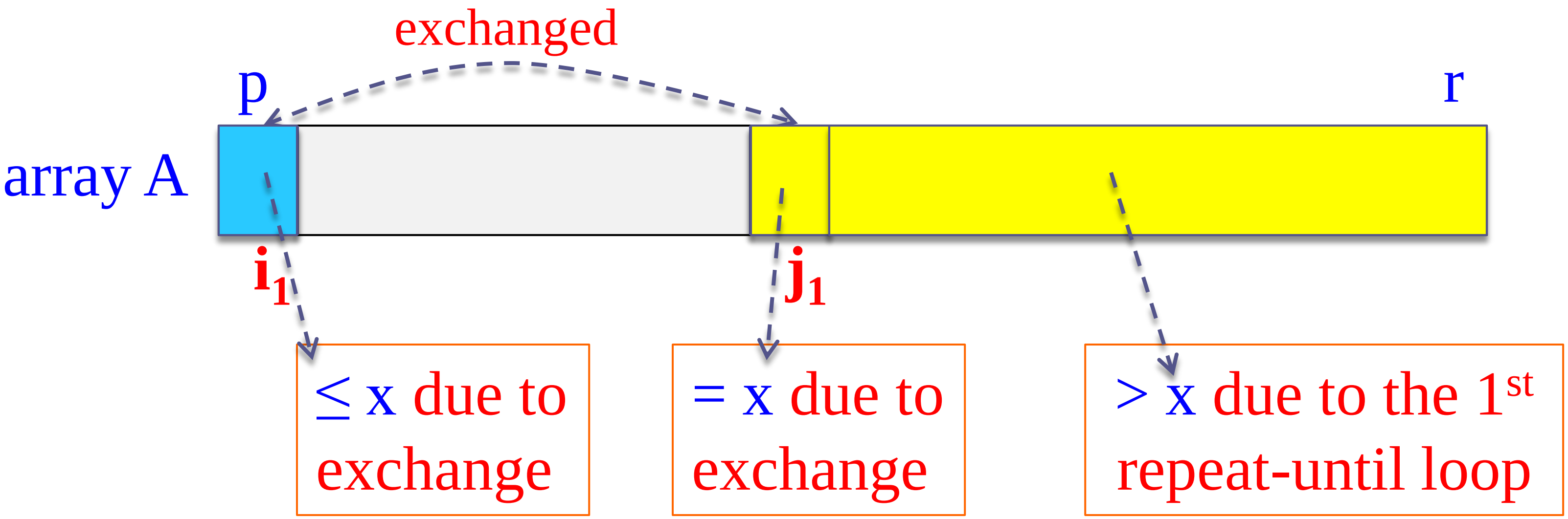
$$A[p..i_m] \leq x \quad \text{and} \quad A[j_m .. r] \geq x$$



Correctness of Hoare's Algorithm

Proof of Lemma 2:

Base case: m=1 and k > 1 *(i.e. the alg. does not terminate in the first iter.)*



Proof of base case complete!

Correctness of Hoare's Algorithm

Proof of Lemma 2(cont'd):

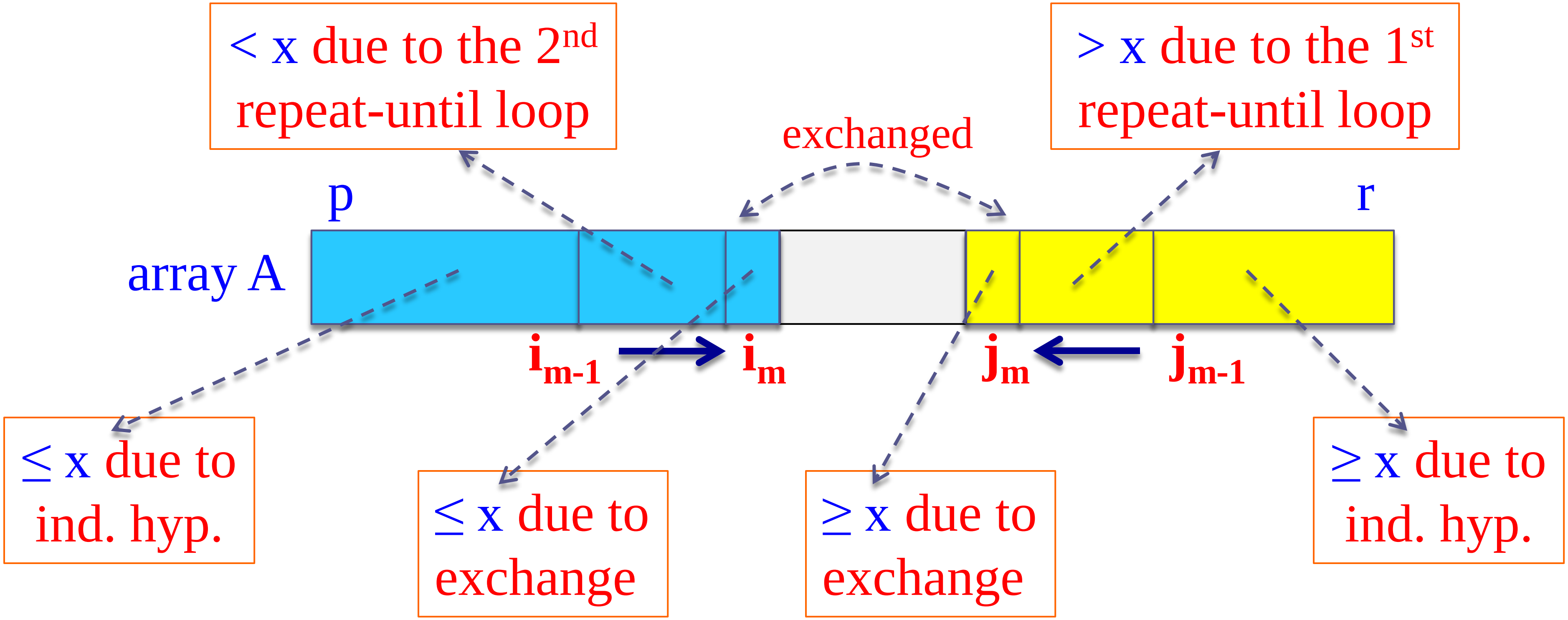
Inductive hypothesis: At the end of iteration $m-1$, where $m < k$ *(i.e. m is not the last iteration)*, we must have:

$$A[p..i_{m-1}] \le x \quad \text{and} \quad A[j_{m-1} .. r] \ge x$$

General case: The lemma holds for m, where $m < k$

Correctness of Hoare's Algorithm

For $1 < m < k$, at the end of iteration m, we have:



Proof of Lemma 2 complete!

Correctness of Hoare's Algorithm

Original correctness claim:

(c) Every element in A[$p \ldots j$] ≤ every element in A[$j + 1 \ldots r$] at termination

Proof of claim (c)

There are 3 cases to consider:

Case 1: $k = 1$, *i.e. the algorithm terminates in a single iteration*

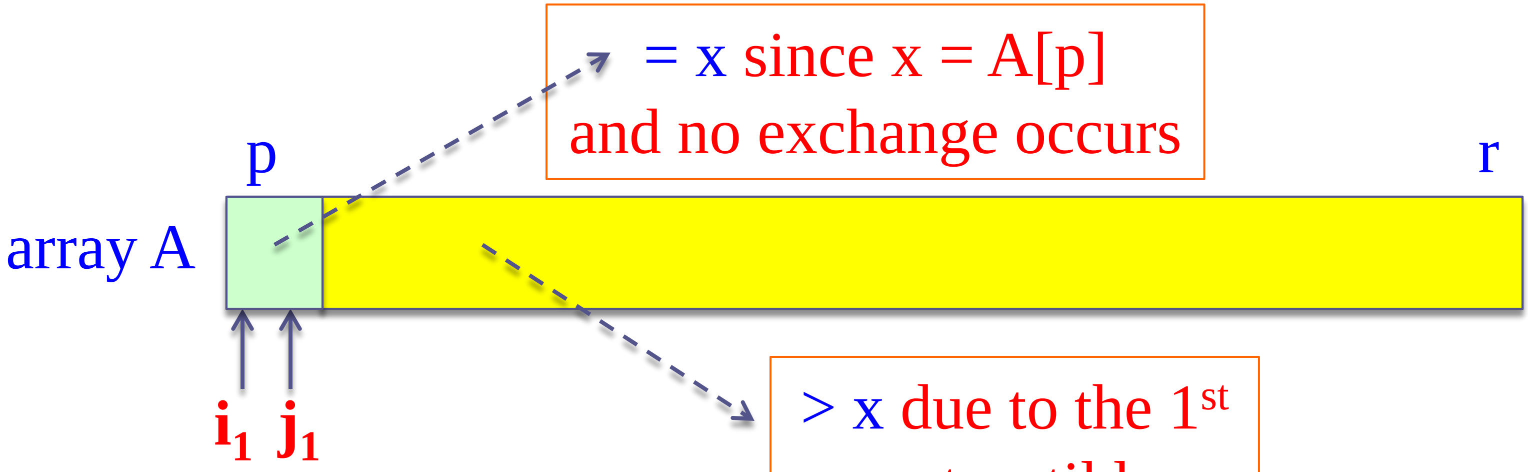
Case 2: $k > 1$ and $i_k = j_k$

Case 3: $k > 1$ and $i_k = j_k + 1$

Correctness of Hoare's Algorithm

Proof of claim (c):

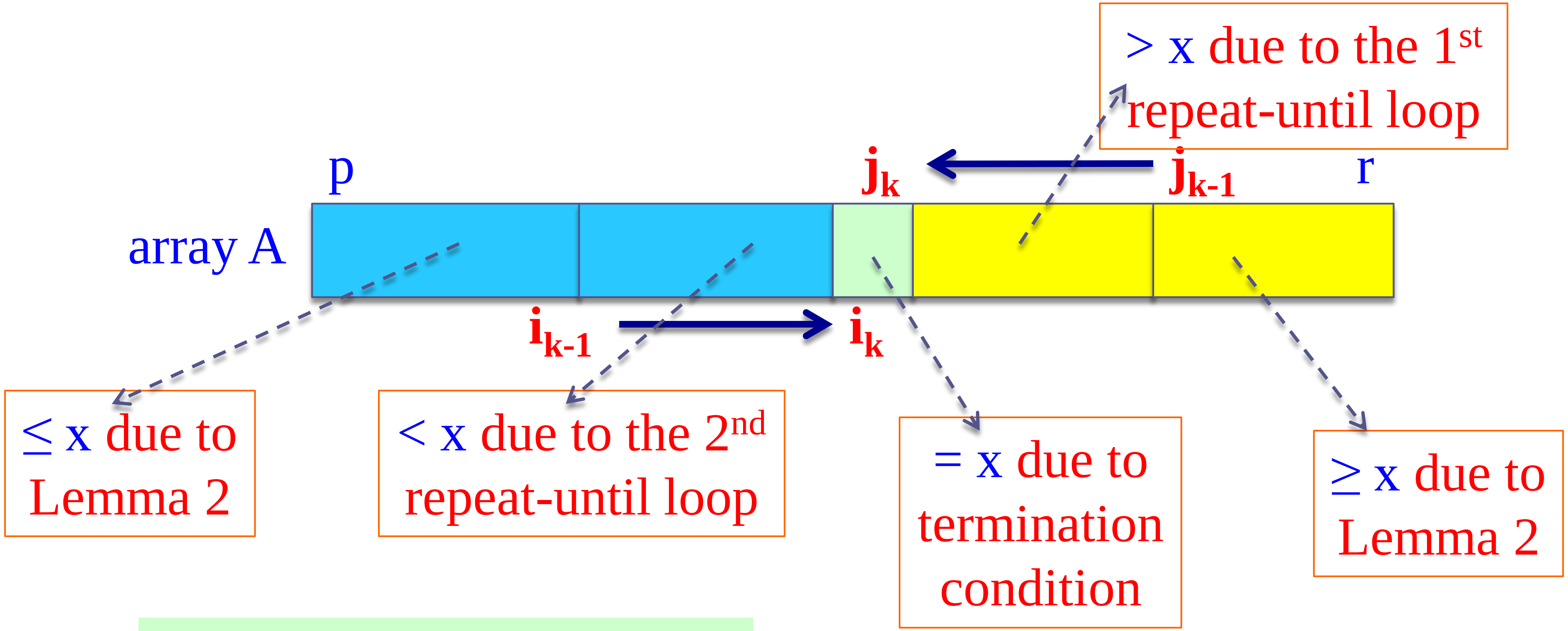
Case 1: $k = 1$, *i.e. the algorithm terminates in a single iteration*



Proof of case 1 complete!

Correctness of Hoare's Algorithm

Proof of claim (c) (cont'd): Case 2: $k > 1$ and $i_k = j_k$



> x due to the 1st repeat-until loop

≤ x due to Lemma 2

< x due to the 2nd repeat-until loop

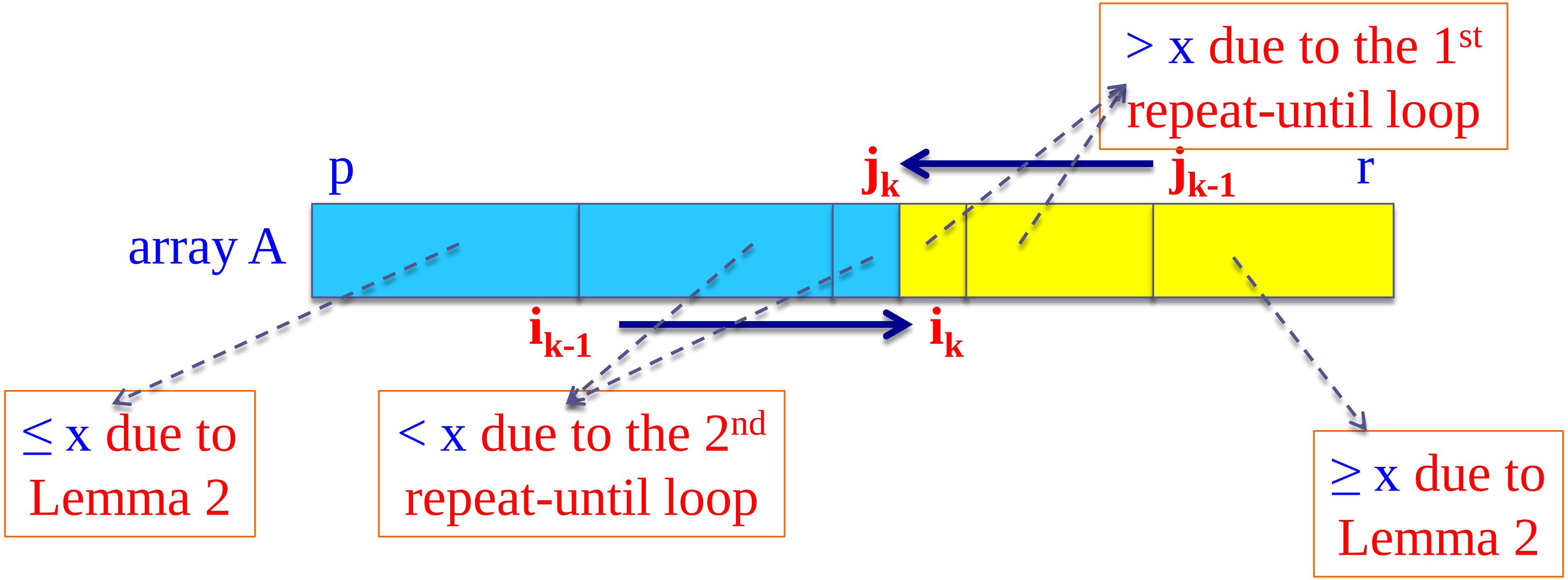
= x due to termination condition

≥ x due to Lemma 2

Proof of Case 2 complete!

Correctness of Hoare's Algorithm

Proof of claim (c) (cont'd): Case 3: $k > 1$ and $i_k = j_k+1$



Proof of Case 3 complete!

Correctness proof complete!

Lomuto's Partitioning Algorithm

1. Choose a pivot element: pivot = x = A[r]
2. Grow two regions:

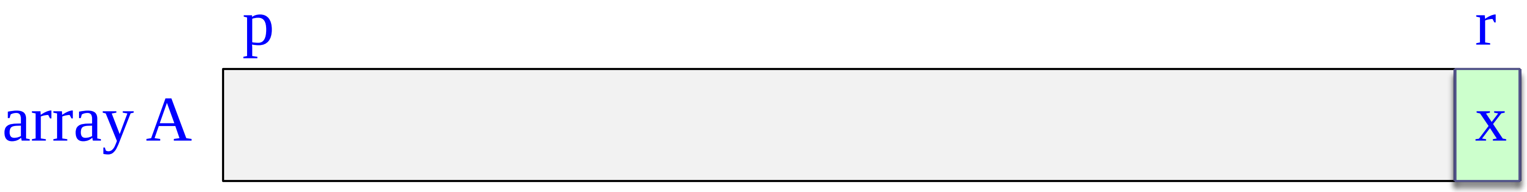
   from left to right: A[p..i]

   from left to right: A[i+1..j]

   such that:

   every element in A[p…i] ≤ pivot

   every element in A[i+1…j] > pivot



Lomuto's Partitioning Algorithm

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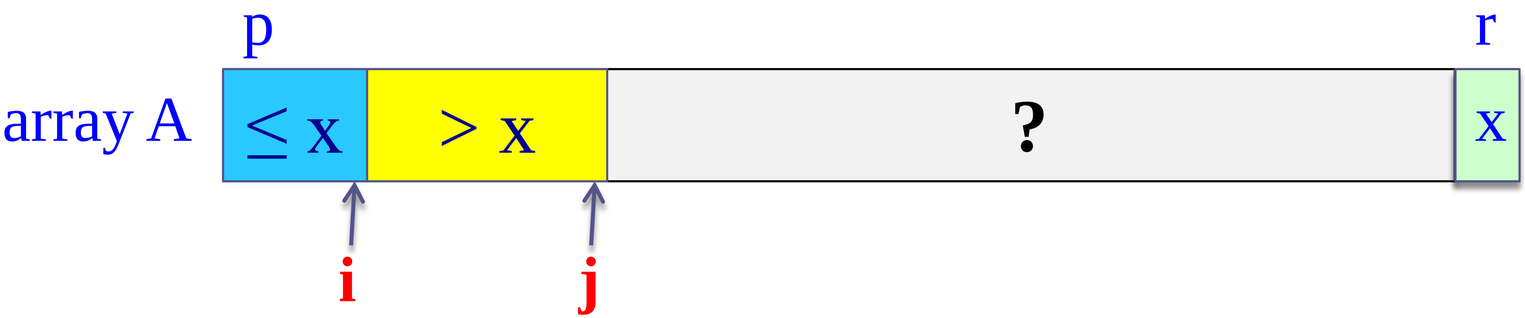

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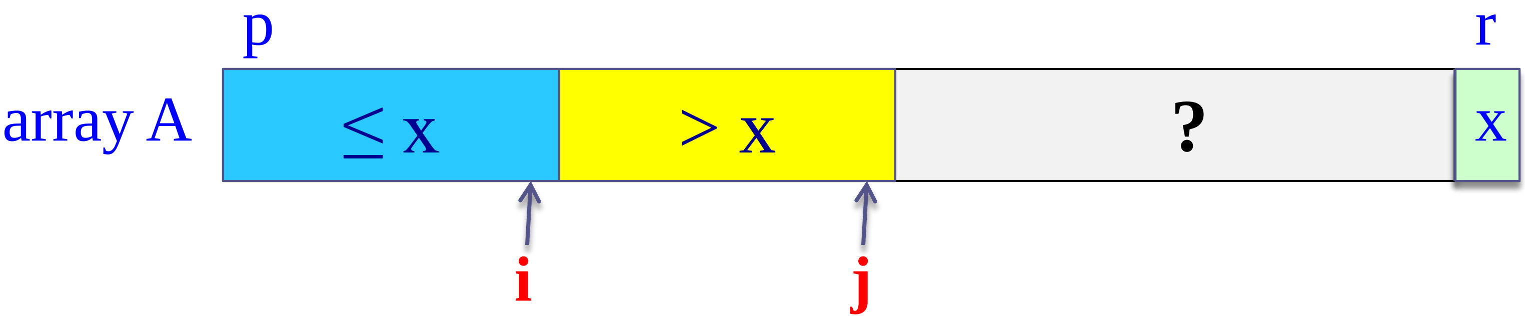

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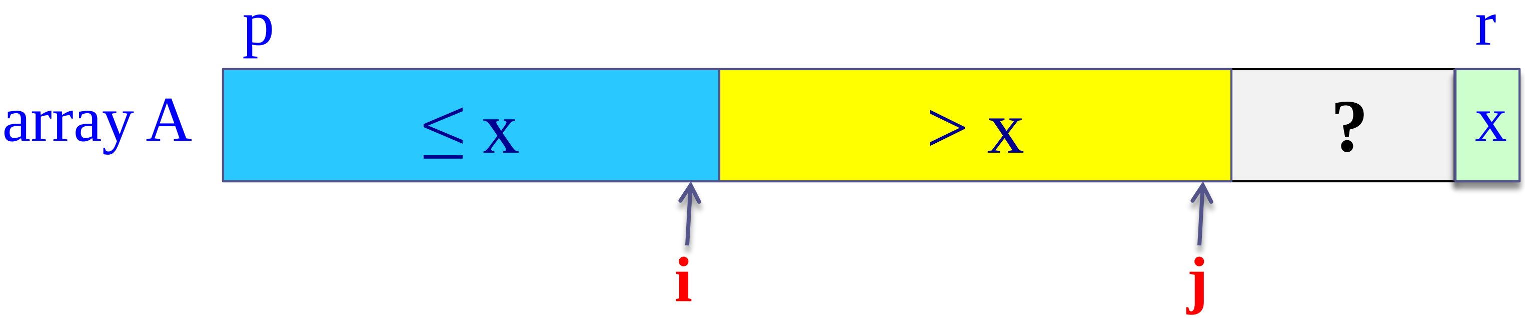

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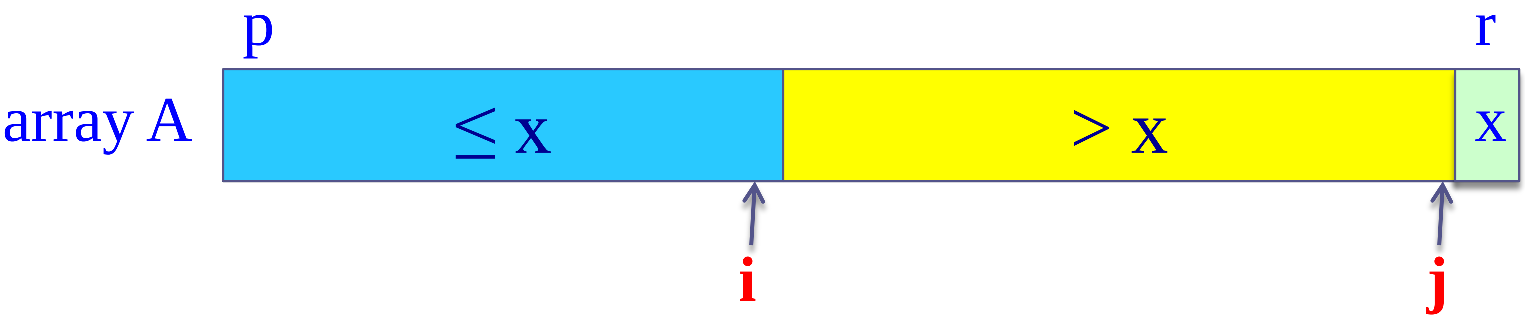

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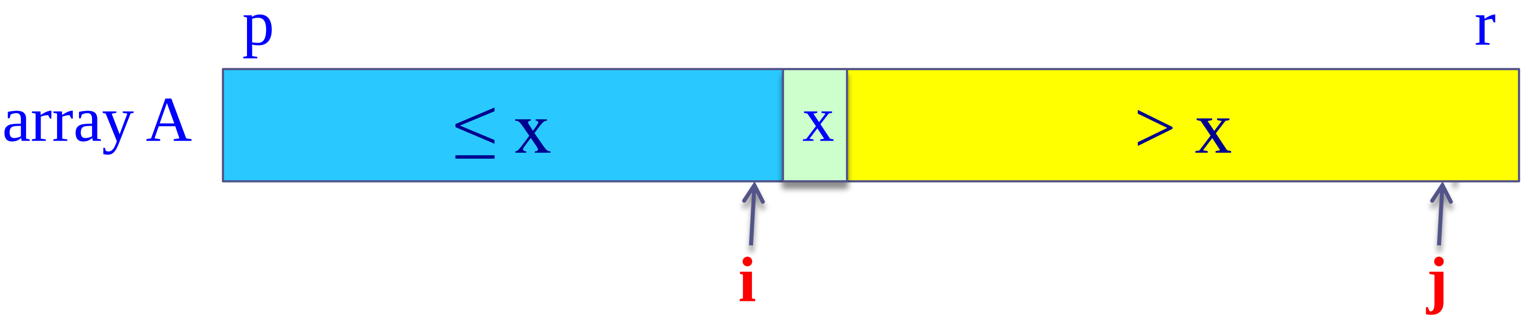

Lomuto's Partitioning Algorithm

```
L-PARTITION (A, p, r)
    $pivot \leftarrow A[r]$
    $i \leftarrow p - 1$
    for $j \leftarrow p$ to $r - 1$ do
        if $A[j] \leq pivot$ then
            $i \leftarrow i + 1$
            exchange $A[i] \leftrightarrow A[j]$
    exchange $A[i+1] \leftrightarrow A[r]$
    return $i + 1$
```

	p							r
array A	7	8	2	6	5	1	3	4

pivot = 4

Lomuto's Partitioning Algorithm

```
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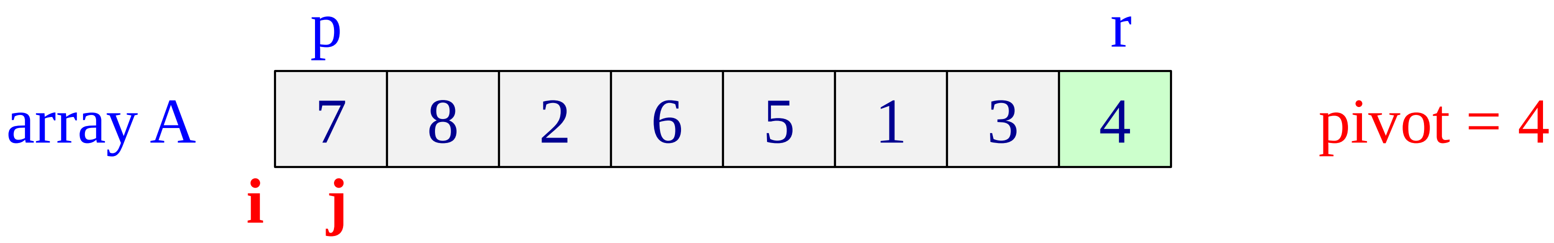

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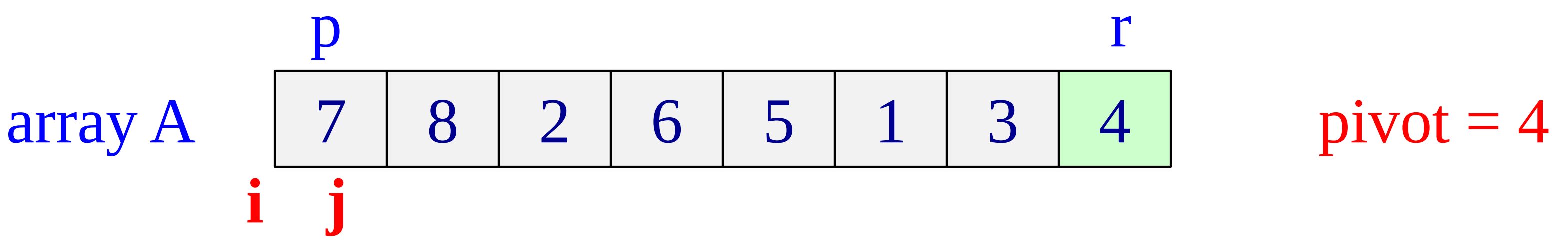

array A

p							r
7	8	2	6	5	1	3	4

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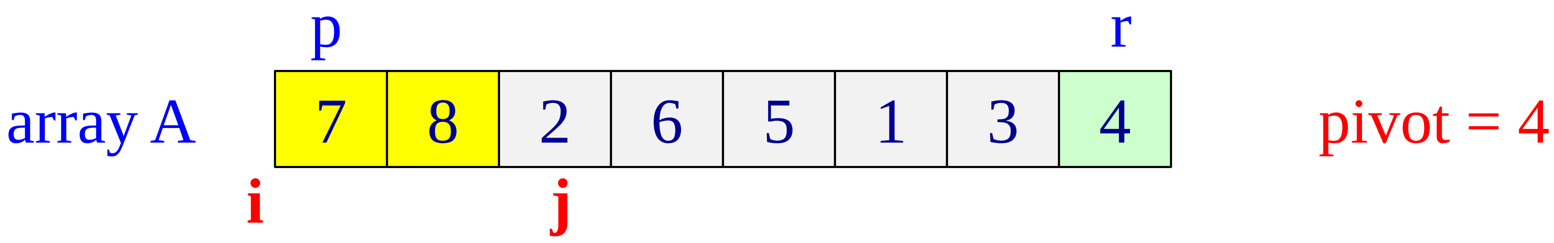

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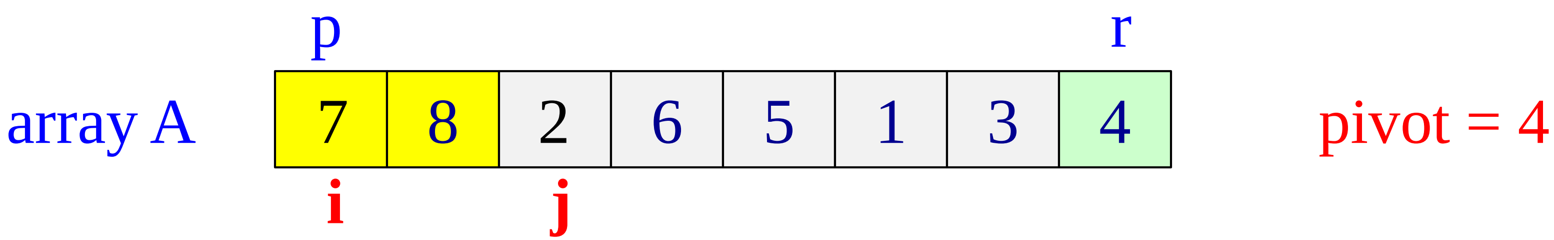

	p							r	
array A	7	8	2	6	5	1	3	4	pivot = 4
	i		j						

Lomuto's Partitioning Algorithm

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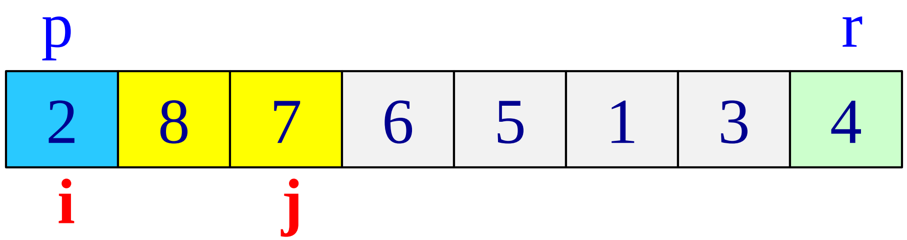

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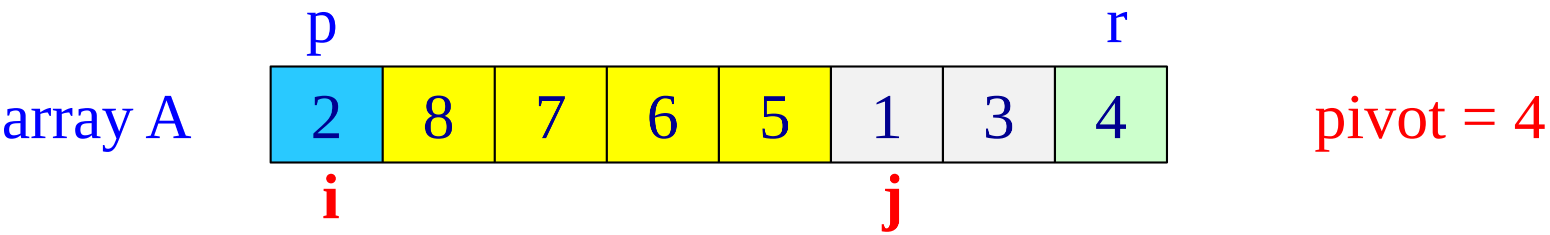

	p						r	
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	i				j			

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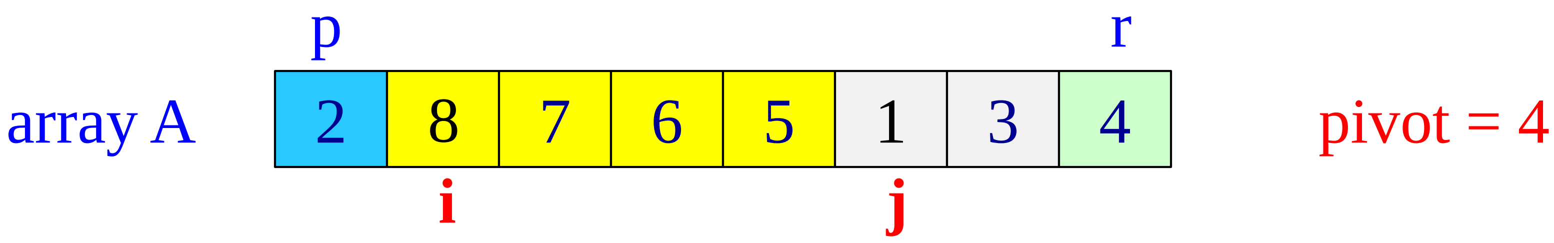

p							r
2	8	7	6	5	1	3	4
	i				j		

array A

pivot = 4

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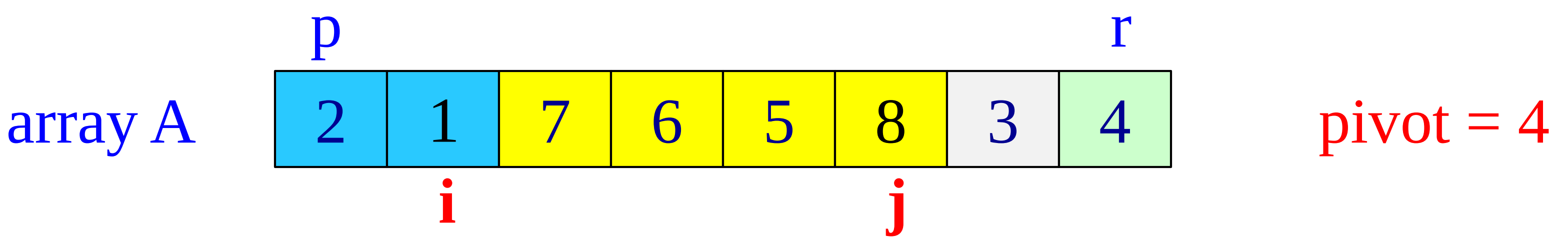

	p							r
array A	2	1	7	6	5	8	3	4
		i				j		

pivot = 4

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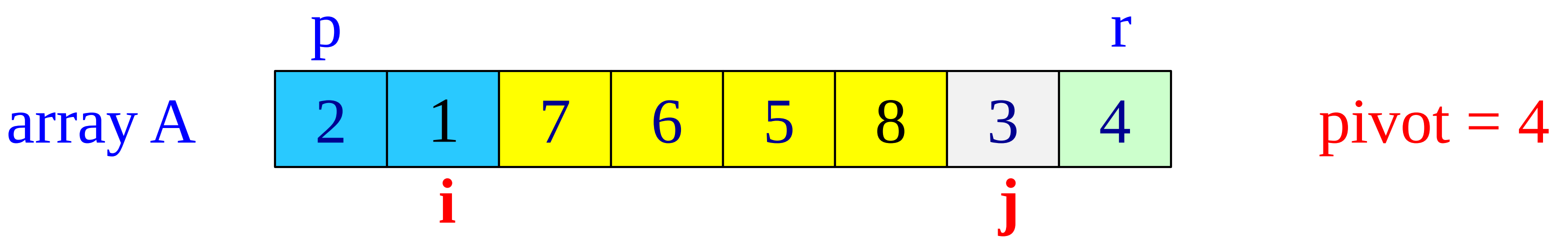

	p							r	
array A	2	1	7	6	5	8	3	4	pivot = 4
		i					j		

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array A

p							r
2	1	7	6	5	8	3	4
		i				j	

pivot = 4

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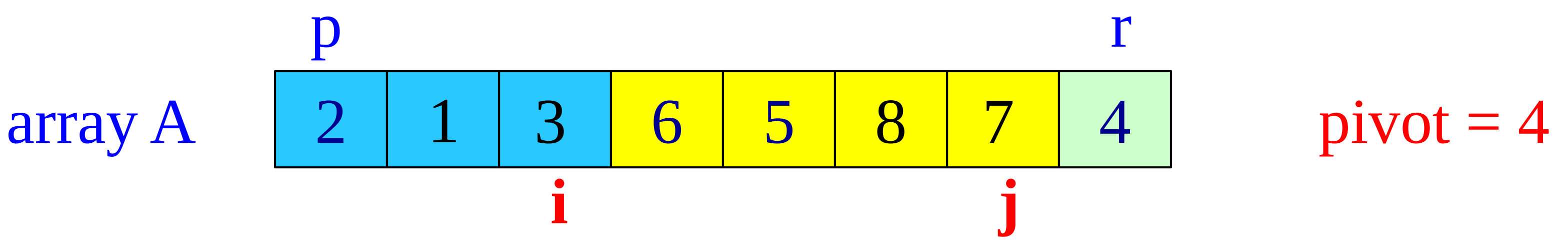

array A

p							r
2	1	3	6	5	8	7	4
		i				j	

pivot = 4

Lomuto’s Partitioning Algorithm

```
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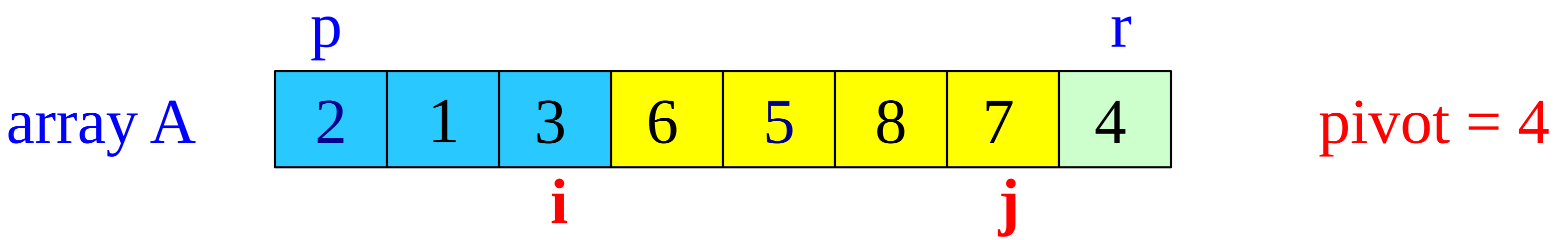

Lomuto's Partitioning Algorithm

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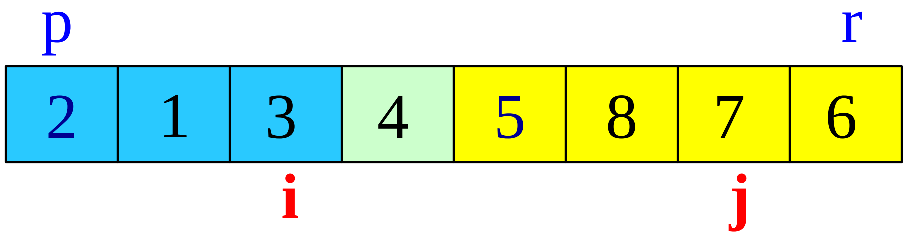
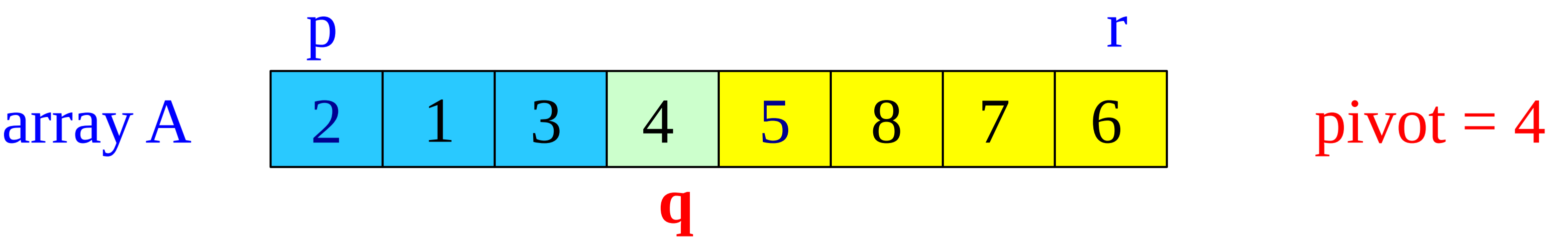


	p							r
array A	2	1	3	4	5	8	7	6
			i				j	

pivot = 4

Lomuto’s Partitioning Algorithm

```
L-PARTITION (A, p, r)
    pivot ← A[r]
    i ← p – 1
    for j ← p to r – 1 do
        if A[j] ≤ pivot then
            i ← i + 1
            exchange A[i] ↔ A[j]
    exchange A[i + 1] ↔ A[r]
    return i + 1
```



p r

array A

2	1	3	4	5	8	7	6

q

pivot = 4

Lomuto’s Partitioning Algorithm

```
L-PARTITION (A, p, r)
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    exchange $A[i + 1] \leftrightarrow A[r]$
    return $i + 1$
```
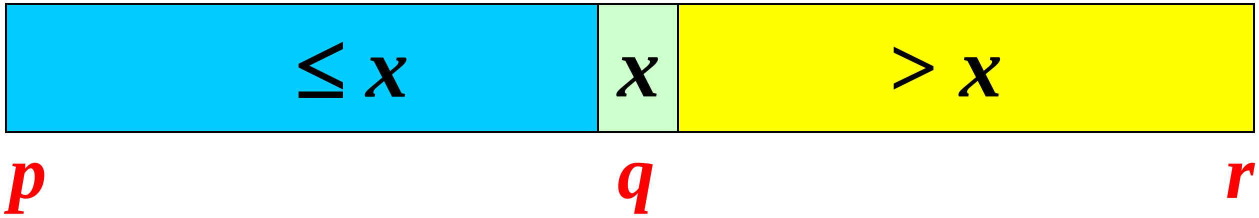

What is the runtime of L-PARTITION? **$\Theta(n)$**

```
QUICKSORT (A, $p$, $r$)
    if $p < r$ then
        $q \leftarrow$ L-PARTITION(A, $p$, $r$)
        QUICKSORT(A, $p$, $q - 1$)
        QUICKSORT(A, $q + 1$, $r$)
```

Initial invocation: QUICKSORT(A, 1, n)



$\leq x$

x

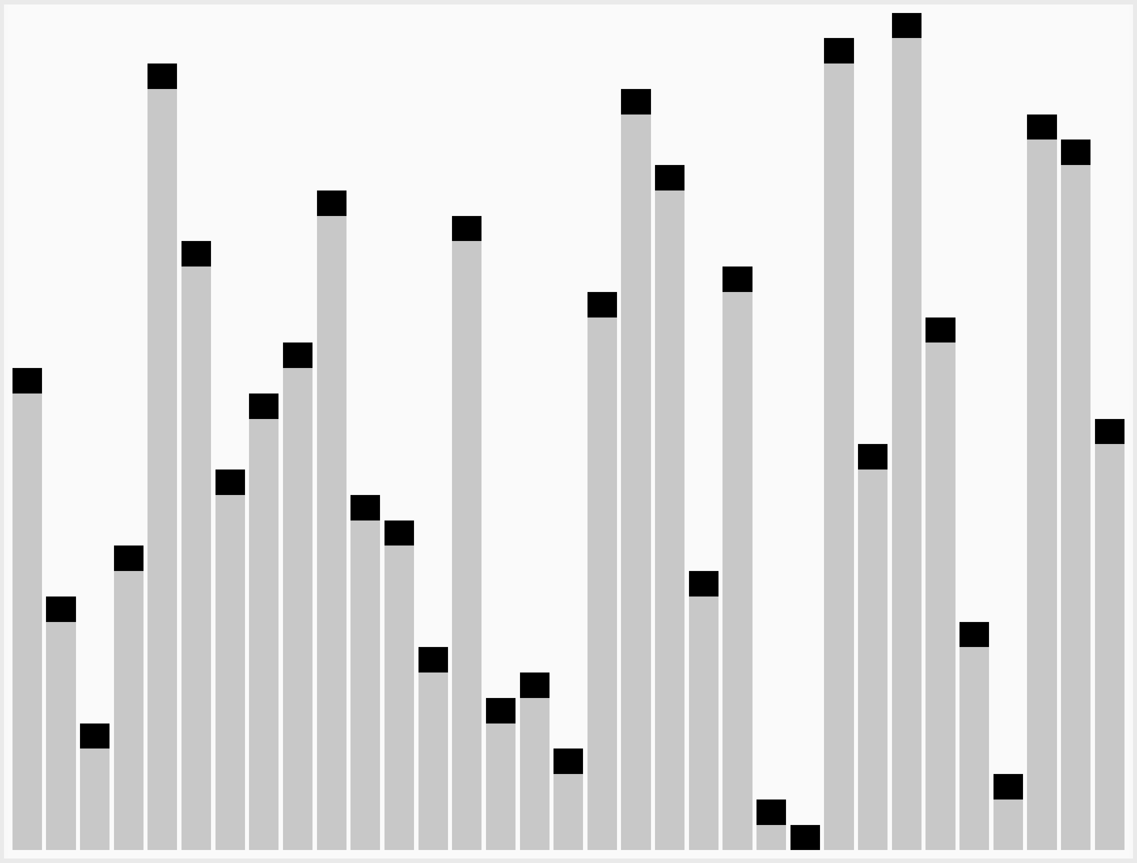
$> x$

p

q

r

Quicksort Animation



from Wikimedia Commons

Comparison of Hoare's & Lomuto's Algorithms

Notation: $n = r - p + 1$ & $pivot = \mathrm{A}[p]$ (Hoare)

& $pivot = \mathrm{A}[r]$ (Lomuto)

- # of element exchanges: $e(n)$
  - Hoare: $0 \le e(n) \le \left\lfloor \frac{n}{2} \right\rfloor$
    - Best: $k = 1$ with $i_1 = j_1 = p$ (i.e., $\mathrm{A}[\,p+1 \ldots r\,] > pivot$)
    - Worst: $\mathrm{A}[\,p+1 \ldots p + \left\lfloor \frac{n}{2} \right\rfloor - 1] \ge pivot \ge \mathrm{A}[\,p + \left\lceil \frac{n}{2} \right\rceil \ldots r\,]$
  - Lomuto: $1 \le e(n) \le n$
    - Best: $\mathrm{A}[\,p \ldots r - 1\,] > pivot$
    - Worst: $\mathrm{A}[\,p \ldots r - 1\,] \le pivot$

Comparison of Hoare's & Lomuto's Algorithms

- # of element comparisons: $c_e(n)$
  - Hoare: $n + 1 \leq c_e(n) \leq n + 2$
    - Best: $i_k = j_k$
    - Worst: $i_k = j_k + 1$
  - Lomuto: $c_e(n) = n - 1$

- # of index comparisons: $c_i(n)$
  - Hoare: $1 \leq c_i(n) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \quad (c_i(n) = e(n) + 1)$
  - Lomuto: $c_i(n) = n - 1$

Comparison of Hoare's & Lomuto's Algorithms

- # of index increment/decrement operations: $a(n)$
  - Hoare: $n + 1 \leq a(n) \leq n + 2 \quad (a(n) = c_e(n))$
  - Lomuto: $n \leq a(n) \leq 2n - 1 \quad (a(n) = e(n) + (n - 1))$

- Hoare's algorithm is in general faster

- Hoare behaves better when pivot is repeated in A[$p \ldots r$]
  - Hoare: Evenly distributes them between left & right regions
  - Lomuto: Puts all of them to the left region