CS473 - Algorithms I

Lecture 9
Sorting in Linear Time

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How Fast Can We Sort?

- The algorithms we have seen so far:
  - Based on comparison of elements
  - We only care about the relative ordering between the elements (not the actual values)
  - The smallest worst-case runtime we have seen so far: O(nlgn)
  - Is O(nlgn) the best we can do?

- ***Comparison sorts***: Only use comparisons to determine the relative order of elements.

Decision Trees for Comparison Sorts

- Represent a sorting algorithm abstractly in terms of a decision tree
  - A binary tree that represents the comparisons between elements in the sorting algorithm
  - Control, data movement, and other aspects are ignored

- One decision tree corresponds to one sorting algorithm and one value of n (input size)

Reminder: Insertion Sort (from Lecture 1)

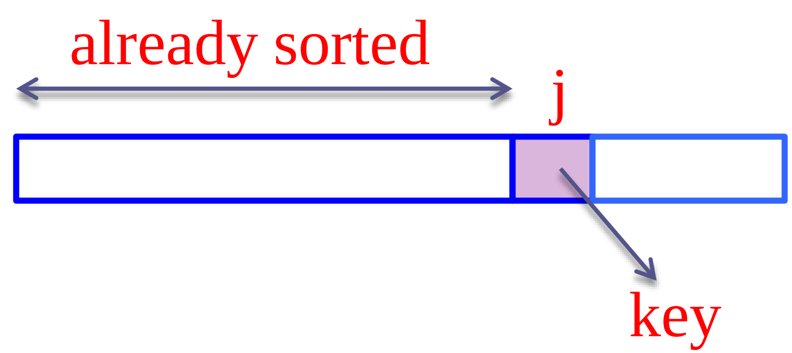
Insertion-Sort (A)

```
for j ← 2 to n do
    key ← A[j];
    i ← j - 1;
    while i > 0 and A[i] > key
   do
        A[i+1] ← A[i];
        i ← i - 1;
    endwhile
    A[i+1] ← key;
endfor
```

Iterate over array elts j

Loop invariant:

The subarray A[1..j-1] is always sorted



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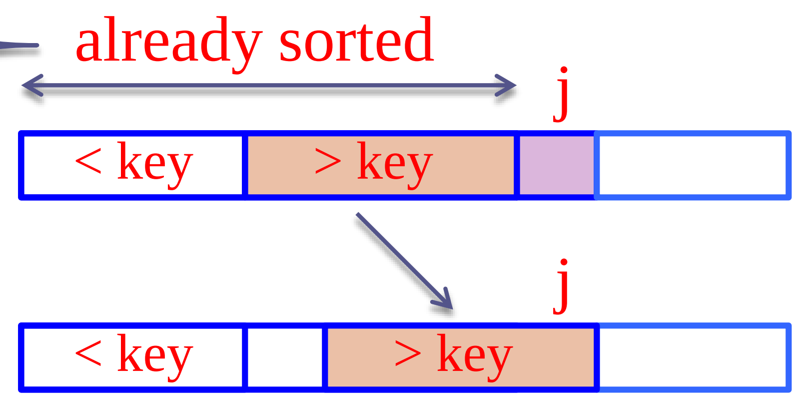
    do
        A[i+1] ← A[i];
        i ← i - 1;
    endwhile
    A[i+1] ← key;
endfor
```

Shift right the entries
in A[1..j-1] that are > key



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    while i > 0 and A[i] > key
    do
        A[i+1] ← A[i];
        i ← i - 1;
    endwhile
    A[i+1] ← key;
endfor
```
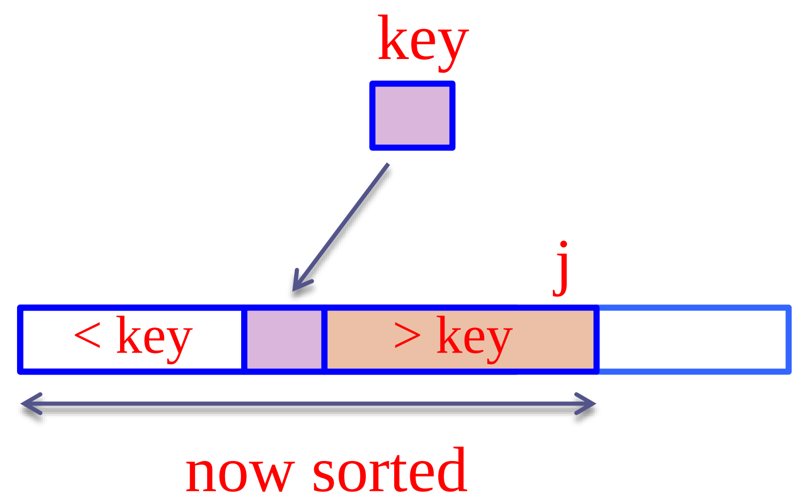
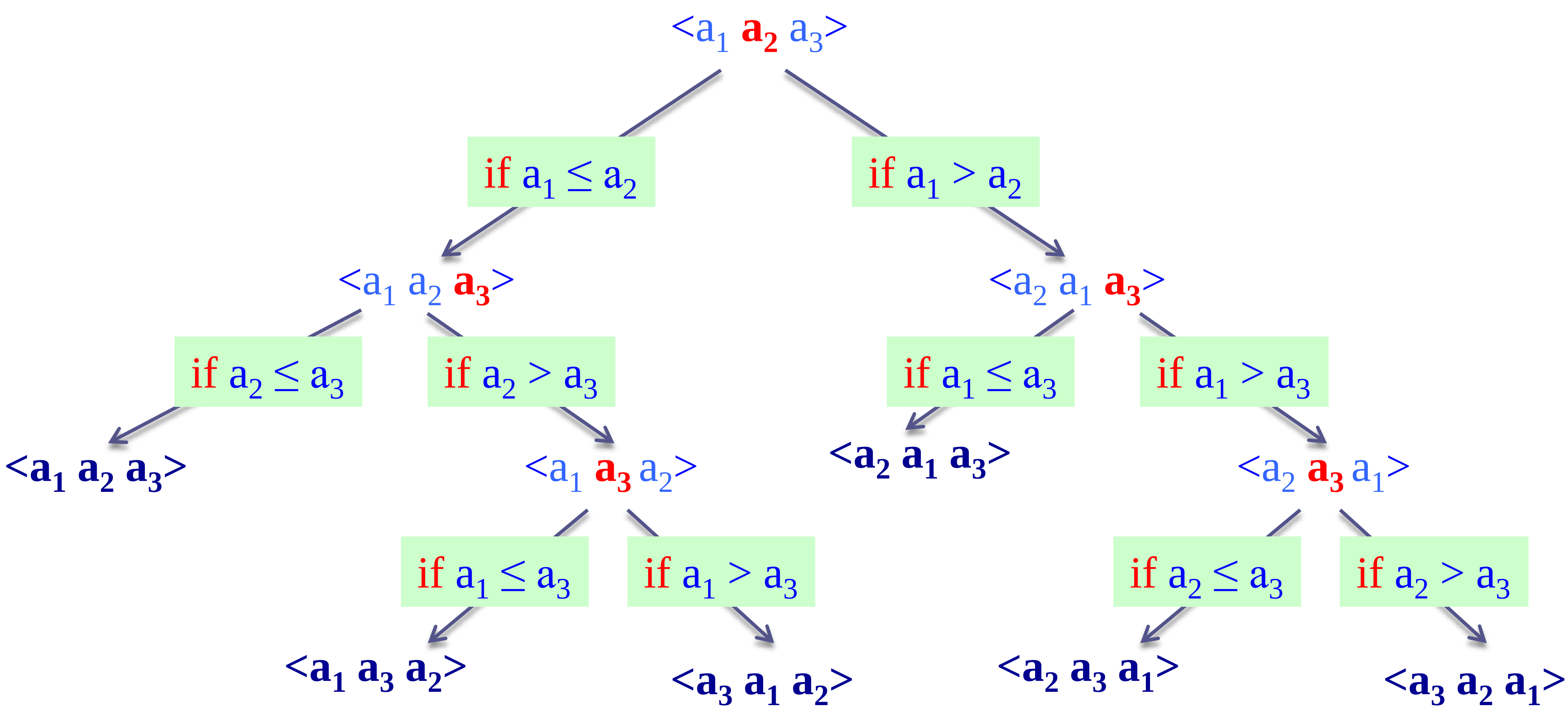



Insert key to the correct location

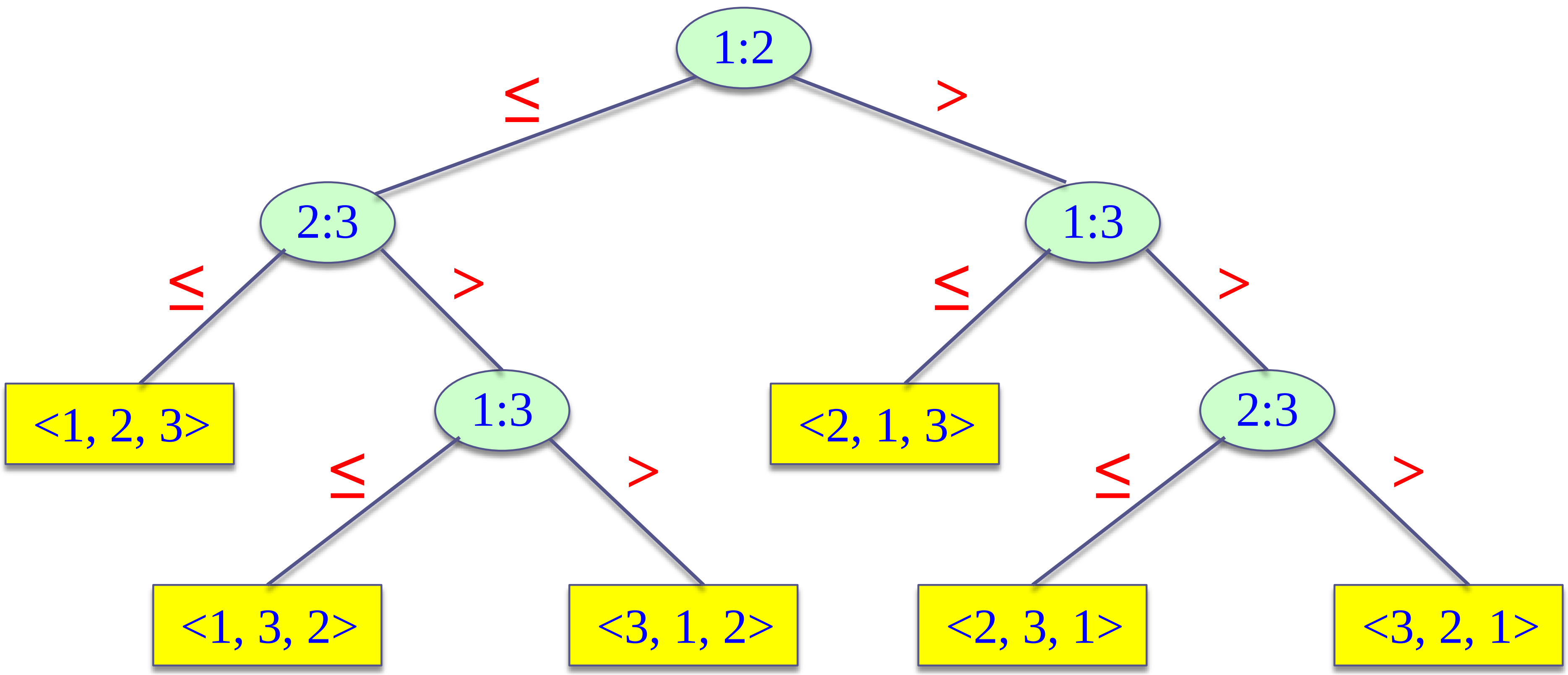
End of iter j: A[1..j] is sorted

Different Outcomes for Insertion Sort and n=3

Input: $<a_1, a_2, a_3>$



Decision Tree for Insertion Sort and n=3

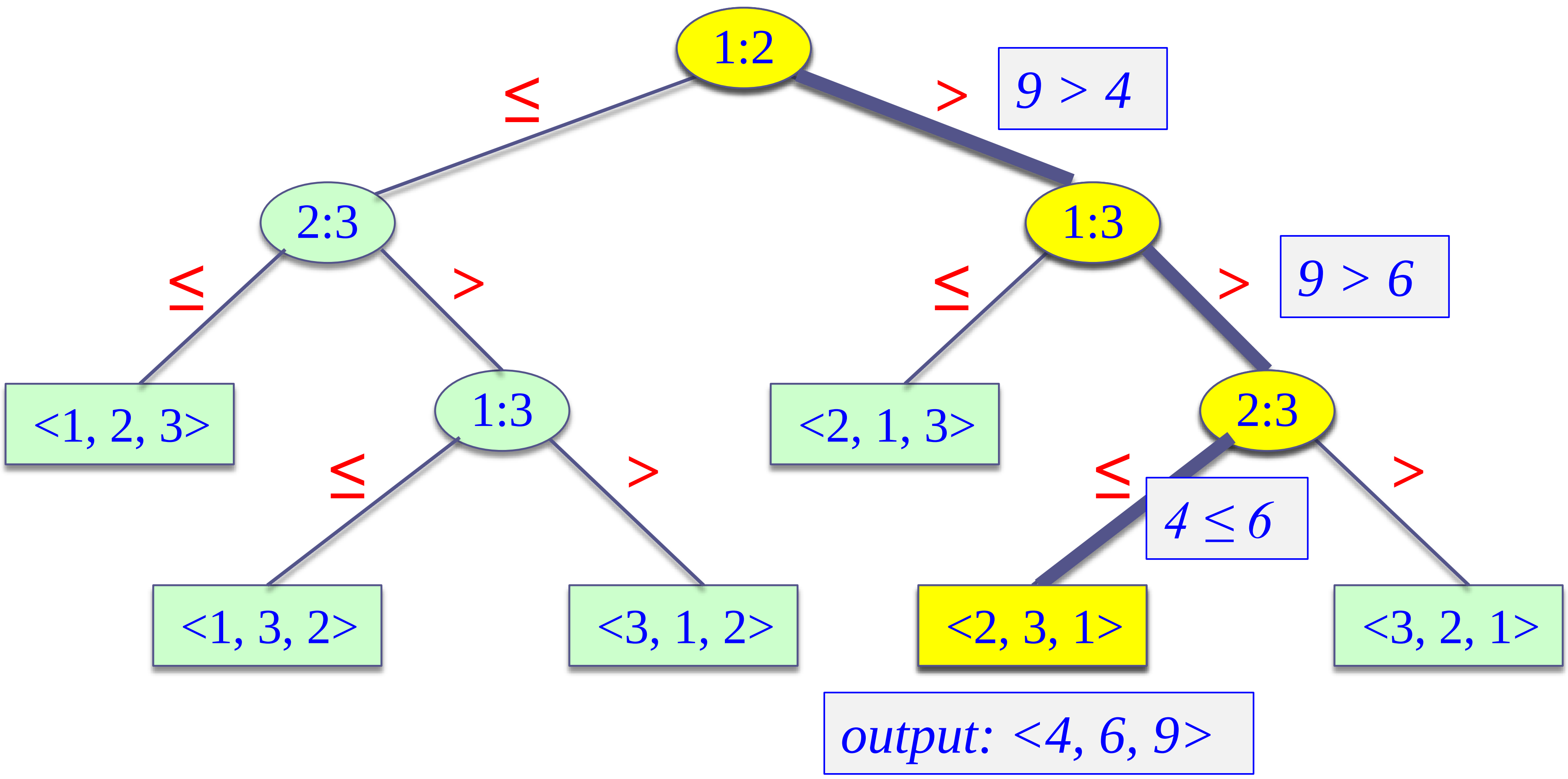


Decision Tree Model for Comparison Sorts

- *Internal node (i:j)*: Comparison between elements a_i and a_j
- *Leaf node*: An output of the sorting algorithm
- *Path from root to a leaf:* The execution of the sorting algorithm for a given input
- All possible executions are captured by the decision tree
- All possible outcomes (permutations) are in the leaf nodes

Decision Tree for Insertion Sort and n=3

Input: <9, 4, 6>



Decision Tree Model

- *A decision tree can model the execution of any comparison sort:*
  - One tree for each input size n
  - View the algorithm as splitting whenever it compares two elements
  - The tree contains the comparisons along all possible instruction traces

The running time of the algorithm = the length of the path taken

Worst case running time = height of the tree

Lower Bound for Comparison Sorts

- Let n be the number of elements in the input array.
- What is the min number of leaves in the decision tree?

  $n!$ *(because there are n! permutations of the input array, and all possible outputs must be captured in the leaves)*
- What is the max number of leaves in a binary tree of height h?

  2^h
- So, we must have:

$$\boxed{2^h \geq n!}$$

Lower Bound for Decision Tree Sorting

Theorem: Any comparison sort algorithm requires $\Omega(n\lg n)$ comparisons in the worst case.

Proof: We'll prove that any decision tree corresponding to a comparison sort algorithm must have height $\Omega(n\lg n)$

$$2^h \geq n! \quad \text{(from previous slide)}$$

$$h \geq \lg(n!)$$

$$\geq \lg((n/e)^n) \quad \text{(Stirling's approximation)}$$

$$= n\lg n - n\lg e$$

$$= \Omega(n\lg n)$$

Lower Bound for Decision Tree Sorting

Corollary: Heapsort and merge sort are asymptotically optimal comparison sorts.

Proof: The $O(n\lg n)$ upper bounds on the runtimes for heapsort and merge sort match the $\Omega(n\lg n)$ worst-case lower bound from the previous theorem.

Sorting in Linear Time

Counting sort: No comparisons between elements

Input: A[1 .. n], where $A[j] \in \{1, 2, \ldots, k\}$

Output: B[1 .. n], sorted

Auxiliary storage: C[1 .. k]

Counting Sort

```
for i ← 1 to k do
    C[i] ← 0
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|

for i ← 2 to k do
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|

for j ← n downto 1 do
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1
```

A:

4	1	3	4	3

B:

C:

1	2	3	4

Counting Sort

for i ← 1 **to** k **do**
    C[i] ← 0
for j ← 1 **to** n **do**
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|

for i ← 2 **to** k **do**
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|

for j ← n **downto** 1 **do**
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1

Step 1: Initialize all counts to 0

A:	4	1	3	4	3

B:					

	1	2	3	4
C:	0	0	0	0

Counting Sort

```
for i ← 1 to k do
    C[i] ← 0
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|

for i ← 2 to k do
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|

for j ← n downto 1 do
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1
```

Step 2: Count the number of occurrences of each value in the input array

	j				
A:	4	1	3	4	3

B:					

	1	2	3	4
C:	1	0	2	2

Counting Sort

for i ← 1 **to** k **do**
    C[i] ← 0
for j ← 1 **to** n **do**
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|

for i ← 2 **to** k **do**
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|

for j ← n **downto** 1 **do**
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1

Step 3: Compute the number of elements less than or equal to each value

A:

4	1	3	4	3

B:

i

C:

1	2	3	4
1	1	3	5

Counting Sort

for i ← 1 **to** k **do**
    C[i] ← 0
for j ← 1 **to** n **do**
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|

for i ← 2 **to** k **do**
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|

for j ← n **downto** 1 **do**
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1

Step 4: Populate the output array

There are C[3] = 3 elts that are ≤ 3

A: (j at position 5)

				j
4	1	3	4	3

B:

1	2	3	4	5

C:

1	2	3	4
1	1	2	5

Counting Sort

for i ← 1 **to** k **do**
    C[i] ← 0
for j ← 1 **to** n **do**
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|

for i ← 2 **to** k **do**
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|

for j ← n **downto** 1 **do**
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1

Step 4: Populate the output array

There are C[4] = 5 elts that are ≤ 4

A: (j at position 4)

			j	
4	1	3	4	3

B:

1	2	3	4	5
		3		

C:

1	2	3	4
1	1	2	4

Counting Sort

```
for i ← 1 to k do
    C[i] ← 0
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|

for i ← 2 to k do
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|

for j ← n downto 1 do
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1
```

Step 4: Populate the output array

There are C[3] = 2 elts that are ≤ 3

			j		
A:	4	1	3	4	3

	1	2	3	4	5
B:			3		4

	1	2	3	4
C:	1	1	1	4

Counting Sort

```
for i ← 1 to k do
    C[i] ← 0
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|

for i ← 2 to k do
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|

for j ← n downto 1 do
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1
```

Step 4: Populate the output array

There are C[1] = 1 elts that are ≤ 1

		j			
A:	4	1	3	4	3

	1	2	3	4	5
B:		3	3		4

	1	2	3	4
C:	0	1	1	4

Counting Sort

```
for i ← 1 to k do
    C[i] ← 0
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|

for i ← 2 to k do
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|

for j ← n downto 1 do
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1
```

Step 4: Populate the output array

There are C[4] =4 elts that are ≤ 4

j

A:

4	1	3	4	3

B:

1	2	3	4	5
1	3	3		4

C:

1	2	3	4
0	1	1	3

Counting Sort

for $i \leftarrow 1$ **to** k **do**
    $C[i] \leftarrow 0$
for $j \leftarrow 1$ **to** n **do**
    $C[A[j]] \leftarrow C[A[j]] + 1$
// $C[i] = |\{key = i\}|$

for $i \leftarrow 2$ **to** k **do**
    $C[i] \leftarrow C[i] + C[i-1]$
// $C[i] = |\{key \leq i\}|$

for $j \leftarrow n$ **downto** 1 **do**
    $B[C[A[j]]] \leftarrow A[j]$
    $C[A[j]] \leftarrow C[A[j]] - 1$

After Count Sort:

A:

4	1	3	4	3

B:

1	2	3	4	5
1	3	3	4	4

C:

1	2	3	4
0	1	1	2

Counting Sort: Runtime Analysis

```
for i ← 1 to k do
    C[i] ← 0
```
$\Theta(k)$

```
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|
```
$\Theta(n)$

```
for i ← 2 to k do
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|
```
$\Theta(k)$

```
for j ← n downto 1 do
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1
```
$\Theta(n)$

Total runtime: $\Theta(n+k)$

n: size of the input array
k: the range of input values

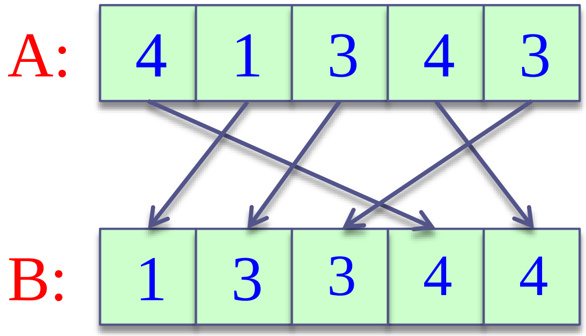
Counting Sort: Runtime

- Runtime is $\Theta(n+k)$
- If $k = O(n)$, then counting sort takes $\Theta(n)$

- *Question*: We proved a lower bound of $\Theta(n\lg n)$ before! Where is the fallacy?
- *Answer*:
  - $\Theta(n\lg n)$ lower bound is for comparison-based sorting
  - Counting sort is not a comparison sort
  - In fact, not a single comparison between elements occurs!

Stable Sorting

- Counting sort is a ***stable sort***: It preserves the input order among equal elements.
  - i.e. The numbers with the same value appear in the output array in the same order as they do in the input array.



Exercise: Which other sorting algorithms have this property?

Radix Sort

- *Origin*: Herman Hollerith's card-sorting machine for the 1890 US Census.
- *Basic idea*: Digit-by-digit sorting

- Two variations:
  - Sort from MSD to LSD (bad idea)
  - Sort from LSD to MSD (good idea)
  - *LSD/MSD: Least/most significant digit*

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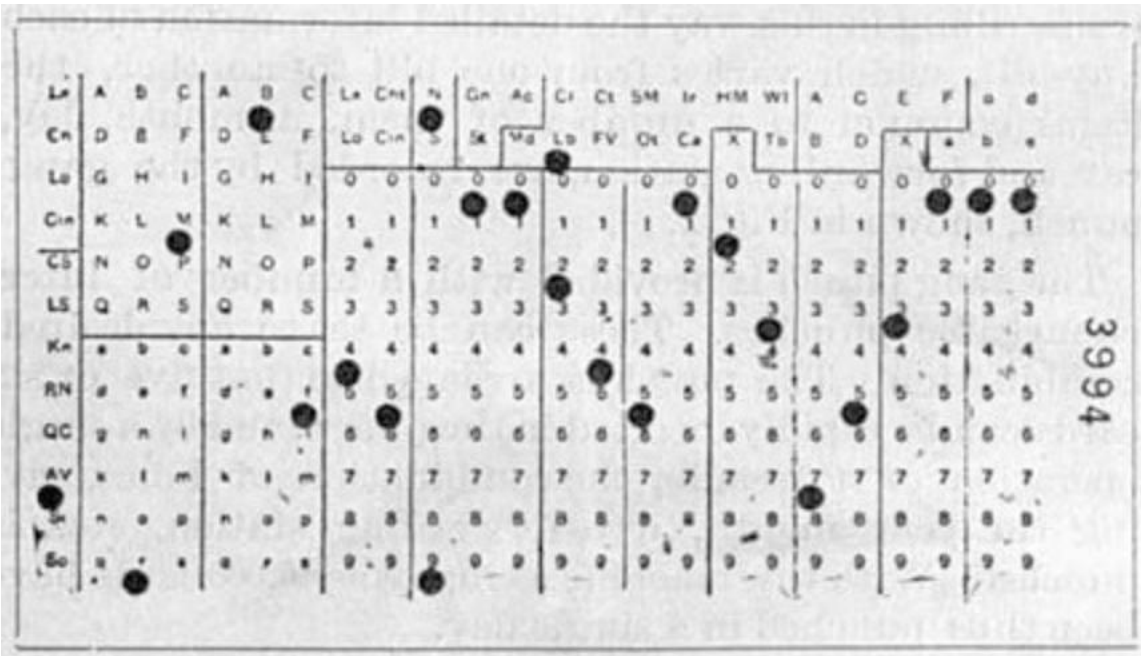
Herman Hollerith (1860-1929)

- The 1880 U.S. Census took almost 10 years to process.
- While a lecturer at MIT, Hollerith prototyped punched-card technology.
- His machines, including a "card sorter," allowed the 1890 census total to be reported in 6 weeks.
- He founded the Tabulating Machine Company in 1911, which merged with other companies in 1924 to form International Business Machines (IBM).

Hollerith Punched Card



- 12 rows and 24 columns
- coded for age, state of residency, gender, etc.

Punched card: A piece of stiff paper that contains digital information represented by the presence or absence of holes.

"Modern" IBM card

- One character per column

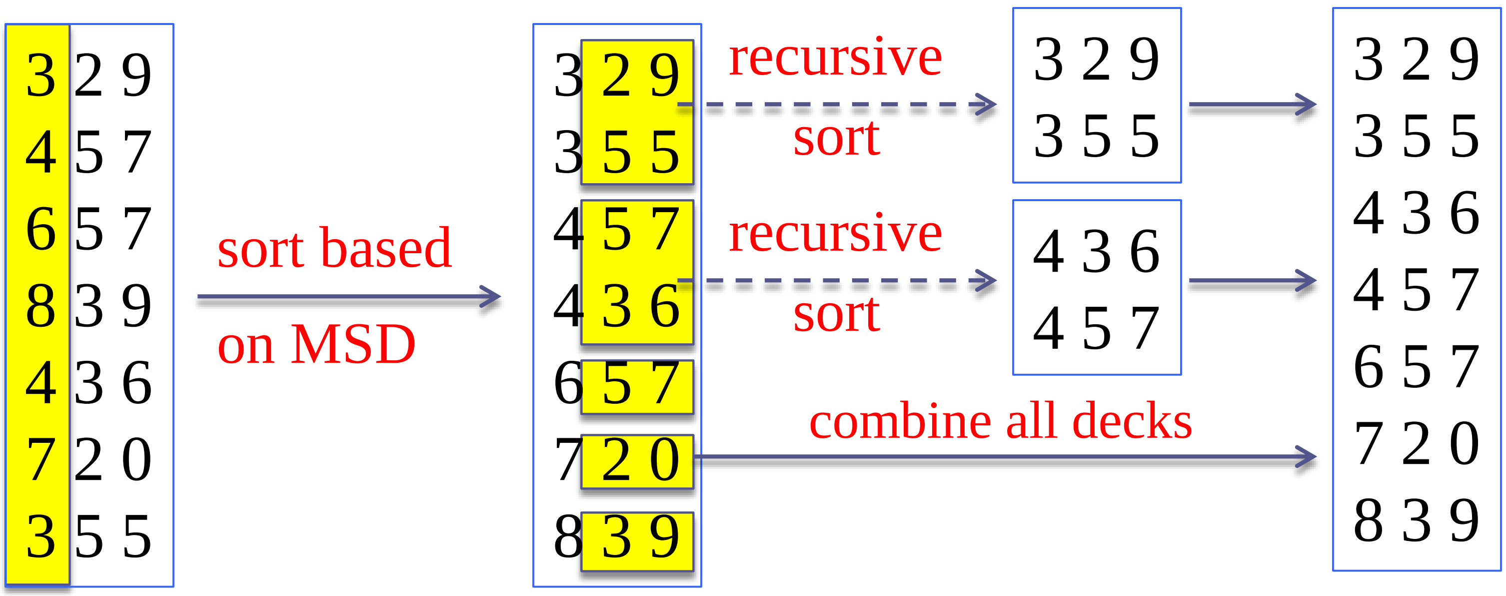
So, that's why text windows have 80 columns!

Hollerith Tabulating Machine and Sorter

- Mechanically sorts the cards based on the hole locations.
- Sorting performed for one column at a time
- Human operator needed to load/retrieve/move cards at each stage

Hollerith's MSD-First Radix Sort

- Sort starting from the most significant digit (MSD)
- Then, sort each of the resulting bins recursively
- At the end, combine the decks in order



Hollerith's MSD-First Radix Sort

- To sort a subset of cards recursively:
  - All the other cards need to be removed from the machine, because the machine can handle only one sorting problem at a time.
  - The human operator needs to keep track of the intermediate card piles

3 2 9
3 5 5
4 5 7
4 3 6
6 5 7
7 2 0
8 3 9

to sort these two cards recursively, remove all the other cards from the machine

3 2 9
3 5 5

intermediate piles

457, 436, 657, 720, 839

Hollerith's MSD-First Radix Sort

- MSD-first sorting may require:

  -- very large number of sorting passes

  -- very large number of intermediate card piles to maintain

- S(d): # of passes needed to sort d-digit numbers (worst-case)
- Recurrence:

$$S(d) = 10\,S(d-1) + 1 \quad \text{with } S(1) = 1$$

Reminder: Recursive call made to each subset with the same most significant digit (MSD)

Hollerith's MSD-First Radix Sort

Recurrence: $S(d) = 10S(d-1) + 1$

$$
\begin{aligned}
S(d) &= 10\ S(d-1) + 1 \\
&= 10\ (10\ S(d-2) + 1) + 1 \\
&= 10\ (10\ (10\ S(d-3) + 1) + 1) + 1 \\
&= 10^i\ S(d-i) + 10^{i-1} + 10^{i-2} + \ldots + 10^1 + 10^0
\end{aligned}
$$

Iteration terminates when $i = d-1$ with $S(d-(d-1)) = S(1) = 1$

$$S(d) = \sum_{i=0}^{d-1} 10^i = \frac{10^d - 1}{10 - 1} = \frac{1}{9}(10^d - 1) \Rightarrow \boxed{S(d) = \frac{1}{9}(10^d - 1)}$$

Hollerith's MSD-First Radix Sort

$P(d)$: # of intermediate card piles maintained (worst-case)

Reminder: Each routing pass generates 9 intermediate piles except the sorting passes on least significant digits (LSDs)

There are 10^{d-1} sorting calls to LSDs

$$P(d) = 9\,(S(d) - 10^{d-1}) = 9\,((10^d - 1)/9 - 10^{d-1})$$

$$= (10^d - 1 - 9 \,.\, 10^{d-1}) = 10^{d-1} - 1$$

$$\boxed{P(d) = 10^{d-1} - 1}$$

Alternative solution: Solve the recurrence:

$$P(d) = 10P(d-1) + 9$$
$$P(1) = 0$$

Hollerith's MSD-First Radix Sort

- Example: To sort 3 digit numbers, in the worst case:

  $S(d) = (1/9)\ (10^3-1) = 111$ sorting passes needed

  $P(d) = 10^{d-1}-1 = 99$ intermediate card piles generated

- MSD-first approach has more recursive calls and intermediate storage requirement
  - Expensive for a "tabulating machine" to sort punched cards
  - Overhead of recursive calls in a modern computer

LSD-First Radix Sort

- Least significant digit (LSD)-first radix sort seems to be a folk invention originated by machine operators.
- It is the counter-intuitive, but the better algorithm.
- Basic algorithm:

    Sort numbers on their LSD first

    Stable sorting needed!!!

    Combine the cards into a single deck in order

    Continue this sorting process for the other digits

    from the LSD to MSD

> - Requires only d sorting passes
> - No intermediate card pile generated

LSD-first Radix Sort: Example

Step 1: Sort 1st digit

3 2 9	7 2 0
4 5 7	3 5 5
6 5 7	4 3 6
8 3 9	4 5 7
4 3 6	6 5 7
7 2 0	3 2 9
3 5 5	8 3 9

Step 2: Sort 2nd digit

7 2 0	7 2 0
3 5 5	3 2 9
4 3 6	4 3 6
4 5 7	8 3 9
6 5 7	3 5 5
3 2 9	4 5 7
8 3 9	6 5 7

Step 3: Sort 3rd digit

7 2 0	3 2 9
3 2 9	3 5 5
4 3 6	4 3 6
8 3 9	4 5 7
3 5 5	6 5 7
4 5 7	7 2 0
6 5 7	8 3 9

Correctness of Radix Sort (LSD-first)

Proof by induction: *Base case*: d=1 is correct (trivial)

Inductive hyp: Assume the first d-1 digits are sorted correctly

Prove that all d digits are sorted correctly after sorting digit d

7	2	0
3	2	9
4	3	6
8	3	9
3	5	5
4	5	7
6	5	7

sort based on digit d

3	2	9
3	5	5
4	3	6
4	5	7
6	5	7
7	2	0
8	3	9

last 2 digits sorted due to ind. hyp.

Two numbers that differ in digit d are correctly sorted (e.g. 355 and 657)

Two numbers equal in digit d are put in the same order as the input

➔ correct order

Radix Sort: Runtime

- Use counting-sort to sort each digit

  Reminder: Counting sort complexity: $\Theta(n+k)$

  n: size of input array

  k: the range of the values

- Radix sort runtime: $\Theta(d(n+k))$

  d: # of digits

- How to choose the d and k?

Radix Sort: Runtime – Example 1

- We have flexibility in choosing d and k
- Assume we are trying to sort 32-bit words
  - We can define each digit to be 4 bits
  - Then, the range for each digit $k = 2^4 = 16$

    So, counting sort will take $\Theta(n+16)$
  - The number of digits $d = 32/4 = 8$
  - Radix sort runtime: $\Theta(8(n+16)) = \Theta(n)$

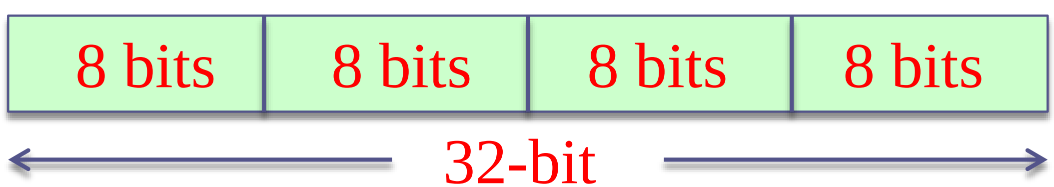
4 bits	4 bits	4 bits	4 bits	4 bits	4 bits	4 bits	4 bits

32-bit

Radix Sort: Runtime – Example 2

- We have flexibility in choosing d and k
- Assume we are trying to sort 32-bit words
  - Or, we can define each digit to be 8 bits
  - Then, the range for each digit $k = 2^8 = 256$

    So, counting sort will take $\Theta(n+256)$
  - The number of digits $d = 32/8 = 4$
  - Radix sort runtime: $\Theta(4(n+256)) = \Theta(n)$



Radix Sort: Runtime

- Assume we are trying to sort b-bit words
  - Define each digit to be r bits
  - Then, the range for each digit $k = 2^r$

    So, counting sort will take $\Theta(n+2^r)$
  - The number of digits $d = b/r$

Radix sort runtime: $$T(n,b) = \Theta\left(\frac{b}{r}\left(n+2^r\right)\right)$$

$$\overbrace{\boxed{\text{r bits}}\boxed{\text{r bits}}\boxed{\text{r bits}}\boxed{\text{r bits}}}^{\text{b/r digits}}$$

$\longleftarrow$ b-bit $\longrightarrow$

Radix Sort: Runtime Analysis

$$T(n,b) = \Theta\left(\frac{b}{r}\left(n+2^r\right)\right)$$

Minimize T(n, b) by differentiating and setting to 0

Or, intuitively:

We want to balance the terms (b/r) and $(n + 2^r)$

Choose $r \approx \lg n$

If we choose $r \ll \lg n$ ➔ $(n + 2^r)$ term doesn't improve

If we choose $r \gg \lg n$ ➔ $(n + 2^r)$ increases exponentially

Radix Sort: Runtime Analysis

$$T(n,b)=\Theta\left(\frac{b}{r}\left(n+2^{r}\right)\right)$$

Choose $r = \lg n$ → $T(n, b) = \Theta(bn/\lg n)$

For numbers in the range from 0 to $n^d - 1$, we have:

The number of bits $b = \lg(n^d) = d \lg n$

→ Radix sort runs in $\Theta(dn)$

Radix Sort: Conclusions

Choose $r = \lg n$ → $T(n, b) = \Theta(bn/\lg n)$

- Example: Compare radix sort with merge sort/heapsort

  1 million (2^{20}) 32-bit numbers ($n = 2^{20}$, $b = 32$)

  Radix sort: $\lceil 32/20 \rceil = 2$ passes

  Merge sort/heap sort: $\lg n = 20$ passes

- Downsides:

  Radix sort has little locality of reference (more cache misses)

  The version that uses counting sort is not in-place

- On modern processors, a well-tuned quicksort implementation typically runs faster.