CS473- Algorithms I

Lecture 4

The Divide-and-Conquer Design Paradigm

The Divide-and-Conquer Design Paradigm

1. ***Divide*** **the problem (instance) into subproblems.**
2. ***Conquer*** **the subproblems by solving them recursively.**
3. ***Combine*** **subproblem solutions.**

Example: Merge Sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear- time merge.

$$T(n) = 2\,T(n/2) + O(n)$$

 # subproblems (2)

subproblem size (n/2)

work dividing and combining (O(n))

Master Theorem (reprise)

$$T(n) = aT(n/b) + f(n)$$

CASE 1: $f(n) = O(n^{\log_b a - \varepsilon}) \Rightarrow T(n) = \Theta(n^{\log_b a})$.

CASE 2: $f(n) = \Theta(n^{\log_b a} \lg^k n) \Rightarrow T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

CASE 3: $f(n) = \Omega(n^{\log_b a + \varepsilon})$ and $af(n/b) \leq cf(n)$

$\Rightarrow T(n) = \Theta(f(n))$.

Merge Sort: $a = 2$, $b = 2 \Rightarrow n^{\log_b a} = n$

$\Rightarrow$ Case 2 ($k = 0$) $\Rightarrow T(n) = \Theta(n \lg n)$.

Binary Search

Find an element in a sorted array:

1. Divide: Check middle element.

2. Conquer: Recursively search 1 subarray.

3. Combine: Trivial.

Example: Find **9**

3	**5**	**7**	**8**	**9**	**12**	**15**

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Recurrence for Binary Search

$$T(n) = 1\, T(n/2) + \Theta(1)$$

 # subproblems → 1

subproblem size → $n/2$

work dividing and combining → $\Theta(1)$

$n^{\log_b a} = n^{\log_2 1} = n^0 = 1$ => CASE 2 ($k = 0$)

=> $T(n) = \Theta(\lg n)$.

Powering a Number

- **Problem:** Compute a^n, where n is in N.
- **Naive algorithm:** $\Theta(n)$
- **Divide-and-conquer algorithm:**

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd} \end{cases}$$

$$T(n) = T(n/2) + \Theta(1) \quad \Rightarrow \quad T(n) = \Theta(\lg n).$$

Matrix Multiplication

Input : $A = [a_{ij}]$, $B = [b_{ij}]$.
Output: $C = [c_{ij}] = A \cdot B$.

$i, j = 1, 2, \ldots, n.$

$$\begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{pmatrix}$$

$$c_{ij} = \sum_{1 \leq k \leq n} a_{ik} . b_{kj}$$

Standard Algorithm

for $i \leftarrow 1$ to n

do for $j \leftarrow 1$ to n

do $c_{ij} \leftarrow 0$

for $k \leftarrow 1$ to n

do $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$

Running time = $\Theta(n^3)$

Divide-and-Conquer Algorithm

IDEA: n x n matrix = 2 x 2 matrix of (n/2) x (n/2) submatrices

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$C \quad = \quad A \quad \cdot \quad B$$

$$\left.\begin{aligned} c_{11} &= a_{11} b_{11} + a_{12} b_{21} \\ c_{12} &= a_{11} b_{12} + a_{12} b_{22} \\ c_{21} &= a_{21} b_{11} + a_{22} b_{21} \\ c_{22} &= a_{21} b_{21} + a_{22} b_{22} \end{aligned}\right\}$$

8 mults of (n/2)x(n/2) submatrices

4 adds of (n/2)x(n/2) submatrices

Analysis of D&C Algorithm

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

 # submatrices — **submatrix size** — **work dividing submatrices**

$$n^{\log_b^a} = n^{\log_2^8} = n^3 \quad \Rightarrow \text{ CASE 1 } \Rightarrow \; T(n) = \Theta(n^3).$$

No better than the ordinary algorithm!

Strassen's Idea

- Multiply 2x2 matrices with only 7 recursive mults

$$P_1 = a_{11} \times (b_{12} - b_{22})$$
$$P_2 = (a_{11} + a_{12}) \times b_{22}$$
$$P_3 = (a_{21} + a_{22}) \times b_{11}$$
$$P_4 = a_{22} \times (b_{21} - b_{11})$$
$$P_5 = (a_{11} + a_{22}) \times (b_{11} + b_{22})$$
$$P_6 = (a_{11} - a_{22}) \times (b_{21} + b_{22})$$
$$P_7 = (a_{11} - a_{21}) \times (b_{11} + b_{12})$$

$$c_{11} = P_5 + P_4 - P_2 + P_6$$
$$c_{12} = P_1 + P_2$$
$$c_{21} = P_3 + P_4$$
$$c_{22} = P_5 + P_1 - P_3 - P_7$$

7 mults 18 adds/subs.

Does not rely on commutativity of mult.

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- Multiply 2x2 matrices with only 7 recursive mults

$$P_1 = a_{11} \times (b_{12} - b_{22})$$
$$P_2 = (a_{11} + a_{12}) \times b_{22}$$
$$P_3 = (a_{21} + a_{22}) \times b_{11}$$
$$P_4 = a_{22} \times (b_{21} - b_{11})$$
$$P_5 = (a_{11} + a_{22}) \times (b_{11} + b_{22})$$
$$P_6 = (a_{11} - a_{22}) \times (b_{21} + b_{22})$$
$$P_7 = (a_{11} - a_{21}) \times (b_{11} + b_{12})$$

$$c_{12} = P_1 + P_2$$
$$= a_{11}(b_{12} - b_{22}) + (a_{11} + a_{12})b_{22}$$
$$= a_{11}b_{12} - a_{11}b_{22} + a_{11}b_{22} + a_{12}b_{22}$$
$$= a_{11}b_{12} + a_{12}b_{22}$$

Strassen's Algorithm

1. ***Divide:*** Partition A and B into (n/2) x (n/2) submatrices. Form terms to be multiplied using + and –.
2. ***Conquer:*** Perform 7 multiplications of (n/2) x (n/2) submatrices recursively.
3. ***Combine:*** Form C using + and – on (n/2) x (n/2) submatrices.

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

Analysis of Strassen

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

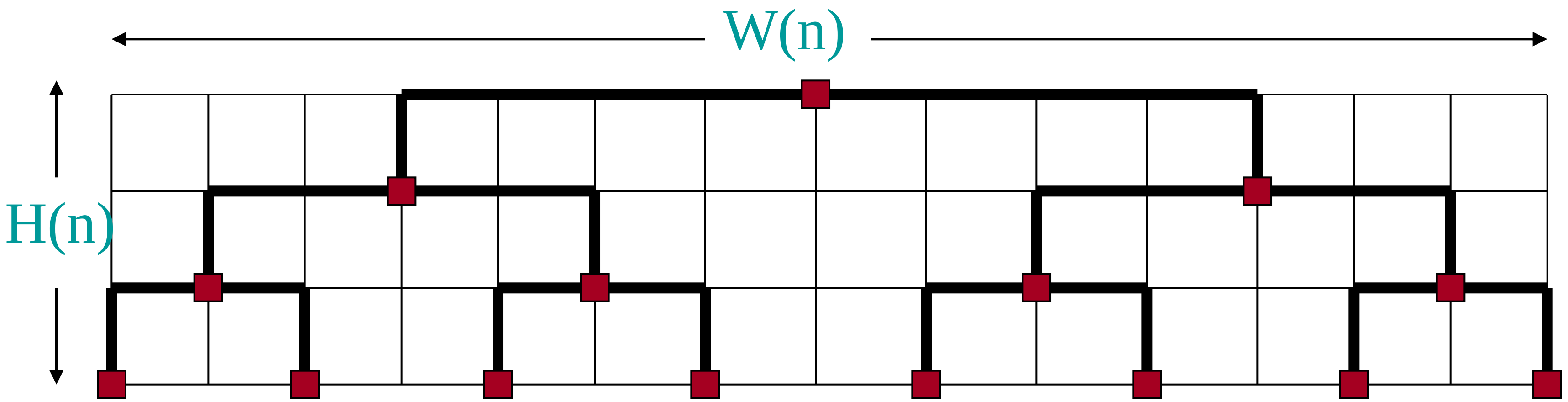
$n^{\log_b a} = n^{\log_2 7} = n^{2.81} \Rightarrow$ CASE 1 $\Rightarrow T(n) = \Theta(n^{\lg 7})$.

- The number 2.81 may not seemmuch smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithmon today's machines for $n \geq 30$ or so.

Best to date (of theoretical interest only) : $\Theta(n^{2.376\ldots})$.

VLSI Layout

- Problem: Embed a complete binary tree with *n* leavesin a grid using minimal area.

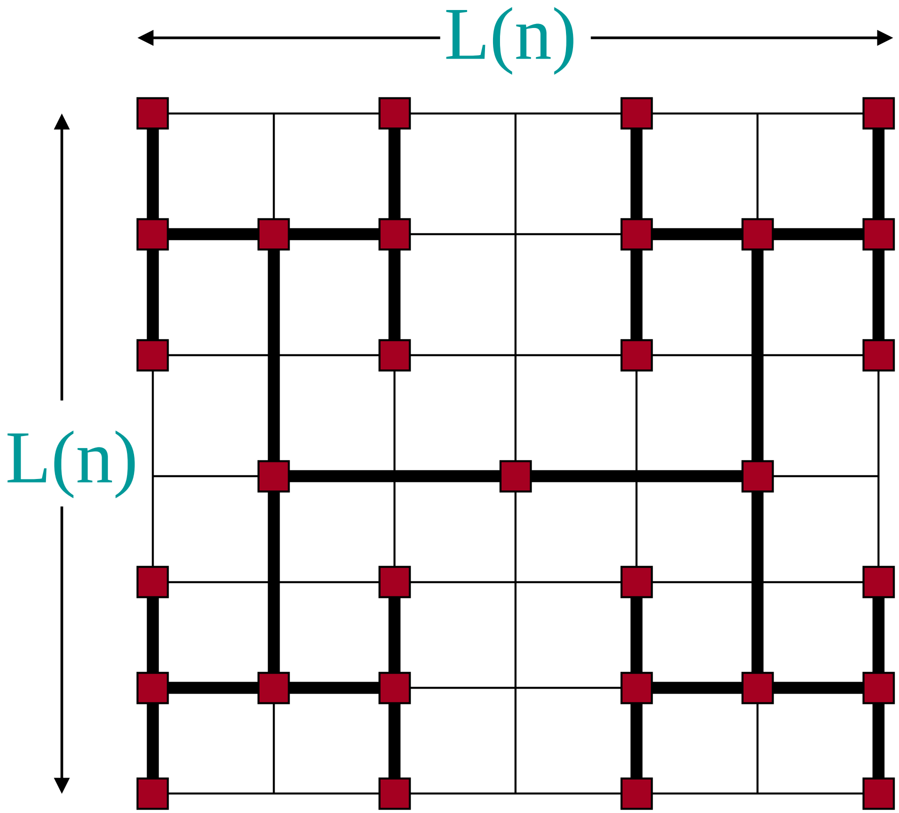


$H(n) = H(n/2) + \Theta(1)$
$= \Theta(\lg n)$

$W(n) = 2\,W(n/2) + \Theta(1)$
$= \Theta(n)$

Area $= \Theta(n\lg(n))$

H-tree Embedding



$$L(n) = 2\,L(n/4) + \Theta(1)$$
$$= \Theta(\sqrt{n})$$

$$\text{Area} = \Theta(n)$$

Conclusion

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- Can lead to more efficient algorithms