# SETS IN CONTEXT AND RELATED ISSUES

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# BILKENT UNIVERSITY

## Department of Computer Engineering and Information Science

Technical Report BU-CEIS-94-10

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(Detailed Abstract)

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January 28, 1994

**Brief synopsis.** We formalize the notion of "context" for sets. We employ two predefined constructs for this—an extension predicate and a context sensitive membership relation. *Ext* returns the members of its first argument, with the context ($C$) specified as its second argument. The new context sensitive membership relation ($\in_C$) holds when the left hand side of the relation is in the extension of the right hand side. We show that the new membership relation is useful for a "commonsense set theory" à la Perlis and Zadrozny, and discuss related issues.

**Keywords:** commonsense set theory, context, Zermelo-Fraenkel set theory

**Suggested symposium topic:** knowledge representation

**NOTES:**

- Please direct all correspondence to the first author.
- Effective 1 January 1994, please dial (312) instead of (4) in the phone and fax numbers shown above.
- The results in this abstract have not been published before. They have not been submitted for publication elsewhere, including proceedings of other symposia and workshops.

## 1 Motivation

In the foundations of mathematics, the most popular approach is set theory. All of the mathematical objects can be constructed out of sets. In this view, mathematics deals only with the properties of sets, all of which can be deduced from a suitable list of axioms [5, 6].

In general, Zermelo-Fraenkel (ZF) is *the* basic axiomatization used heavily in mathematics [3]. Its origin and the underlying mathematical ideas for its axioms were extensively discussed in the literature [5, 10, 11]. The axioms are defined in first order logic and only the membership relation ($\in$) is considered to be a basic relation [9]. A fundamental axiom of ZF is *Extensionality*:

$$\forall x \forall y \forall z[(z \in x \leftrightarrow z \in y) \rightarrow x = y]$$

Basically, this axiom formalizes the notion of being a set: a set is a collection of elements, whose identity is completely determined by those elements. From a mathematical point of view, Extensionality is one of the least problematic axioms of ZF, but from a philosophical vantage point and from a commonsense perspective, things are not so simple. (The reader is referred to [7, 8, 12] for origins of a commonsense approach to set theory. Barwise [3] offers several arguments, illustrating the need for modeling context in set theory.)

First, a questionable thing is $\in$. This relation might be treated from assorted angles, i.e., we might think of graded membership, believed membership, etc. (These will be reviewed in the sequel.) Consider the barbers of Springfield (home-town of Bart Simpson). There might be three barbers working for money, and one barber who does not work for money (since he has another job) but serves the community by shaving senior citizens on Sundays. If we look at the situation from the commonsense perspective, there are four barbers in town, but from say, the mayor's point of view, there are only three (official, tax-paying, etc.) barbers. Thus, the context is crucial and we must have some knowledge about the situation we are facing [2].

The second objection has to the with Extensionality itself. Returning to our example, Springfield Fire Department and Springfield Barber-shop Quartet might have the same staff members [1]. Are these two sets to be regarded as equivalent? We hope not.

## 2 Commonsense set theory

We choose a graph representation to explain our proposal. In this representation, edges will represent the membership relation. We represent membership with labels on the edges. The situation in Springfield can be represented by the graph in Figure 1.

In Figure 1, we allowed individuals (urelements) as leaf nodes. The labels on the edges denote the situation in which the edge (i.e., the membership) is valid.

From the commonsense point of view:

$$Barbers^C = \{a, b, c, d\}$$

From the mayor's point of view:

$$Barbers^M = \{a, b, c\}$$

[1]Suggested by M. A. Jorgensen on July 20, 1993 in a discussion in the news group `sci.math`.

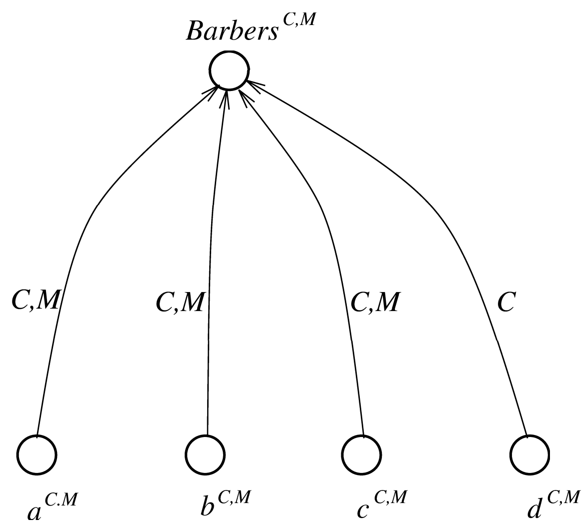

Figure 1: Barbers of Springfield

Once we adopt the above labeling schema, what are the basic building blocks of our set theory? These building blocks must include:

1. A set of previously defined situations (contexts) to be used as labels on the edges and the nodes. In the example above, this set is $\{C, M\}$, where $C$ represents the situation from the commonsense (people's) point of view, and $M$ represent the situation from the mayor's point of view.

2. A set of individuals. ($\{a, b, c, d\}$ is this set.)

3. The edge relation over the nodes.

We can define the equivalence of two sets relative to the situation (e.g., $Barbers^C$ and $Barbers^M$ are not compatible). However, since we have used the same node, they have to be the same set (although their extensions are different in each situation).

Consider now Figure 2. This example represents two different sets (i.e., the barbers of Springfield and the firemen of Springfield). That these two sets are not equal is due to the fact that they are represented by different nodes. But, from a mathematical point of view, they are equivalent since their extensions are equivalent. If we adopt Extensionality for a commonsense set theory, we cannot say that these sets are different. Therefore, the commonsense notion of equivalence must be different than the mathematical notion.

# 3 Related issues

In our study towards a Commonsense set theory, we have found three types of fuzziness. Each considers the fuzziness of the membership relation ($\in$) from different points of view.

## 3.1 Graded membership

In this view, the membership relation is considered as a continuous function from the universe of elements to the real interval $[0, 1]$. In classical set theory, this membership

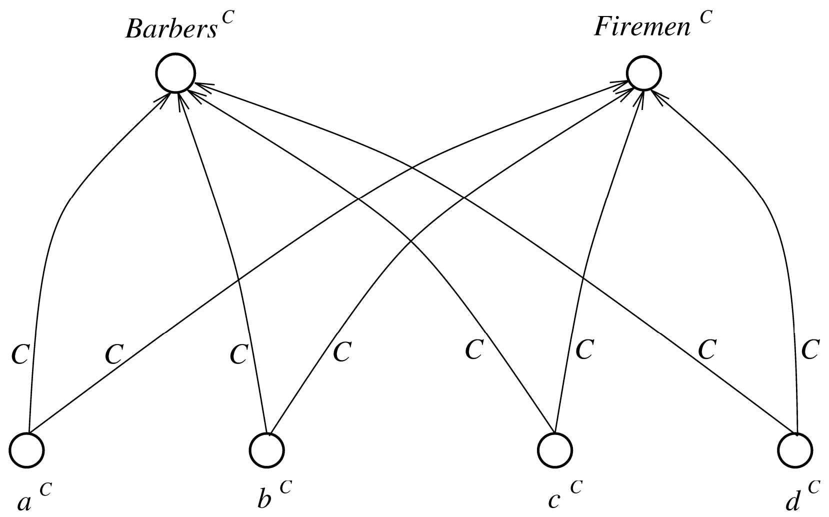

Figure 2: Set representation of Barbers and Firemen of Springfield

relation was a discrete one, that is a function from the universe of elements to the set $\{0,1\}$ where 0 represents the non-membership and one represents the membership. This approach forms the base of *fuzzy logic*, a superset of conventional (Boolean) logic that has been extended to handle the concept of partial truth — truth values between "completely true" and "completely false." It was first introduced by Lotfi Zadeh of University of California, Berkeley in the 1960's as a means to model uncertainty in natural language.

In classical set theory, a subset $A$ of a set $X$, can always be identified with a binary valued function $u_A : X \to \{0,1\}$, called its characteristic function. The movement from the discrete range (i.e., the set $\{0,1\}$) to a continuous range (i.e., the real interval $[0,1]$) in the characteristic function results in fuzzy sets. Thus, a fuzzy subset $F$ of $X$ is a function $f : X \to [0,1]$. The degree of truth of the statement $x \in F$ is determined by the value $f(x)$.

Fuzzy sets might be appropriate for a commonsense set theory. However, they have drawbacks for they say nothing about issues like cardinalities, well-ordering, circularity, etc. We might propose some *ad hoc* solutions for these issues, but we still require fuzzy inference mechanisms to use fuzzy sets in commonsense reasoning.

### 3.2 Believed membership

This kind of membership relations can be handled by considering membership relation with a modal operators B (for "believed") and K (for "known"). After this introduction of modal operators, the membership relation can easily be handled under the domain of modal logic as an ordinary relation.

Although such a uniform treatment via modal logic is a good feature, this approach have some practical/implementation difficulties. At first sight, we may say that this approach requires modal inference and belief revision mechanisms. Believed sets require revision, in other words, truth maintenance, when a new information is gathered from the environment or obtained by reasoning about previous knowledge. From a commonsense point of view, this seems demanding.

### 3.3 Point of view dependent membership

In this approach—which, we believe, is original with us to our best knowledge—the membership relation is indexed with a point of view parameter. The membership relation might be valid for some elements with a point of view parameter while it is not valid the same elements for another point of view parameter. We already introduced this approach in the preceding section, with an example set of Barbers of Springfield. In this example, there were three barbers working for money, and one barber who does not work for money (since he has another job) but occasionally serves the poor children. The fuzziness in this example was that from the commonsense perspective there were four barbers and from the mayor's point of view there were only three. We have used Figure 1 to represent the situation, in which, nodes represent the elements and the sets, edges represent the membership relation, and the labels on the edges and the superscripts on the nodes represent the context (point of view).

The labeling schema that we used in this example seems sufficiently general to deal with this type of fuzziness. This approach forms the basis for our model for a commonsense set theory. Its formalization will be discussed next.

## 4 A model for a commonsense set theory

In the explanation of our approach to a commonsense set theory, we have chosen a graph representation for sets and used the following examples:

- The set of barbers of Springfield. The fuzzy thing in this example was that the members of the set were different when we look from the viewpoint of people of the town and from the viewpoint of the mayor. We have used a single set name, but we have added a superscript to represent the views. Figure 1 corresponds to this case.

- The sets of barbers and the firemen of Springfield. In this example the fuzzy thing was that although the members of both sets are the same, the sets, as intension, were different. These sets are depicted in Figure 2.

To deal with the first case, we have introduced a labeling schema. In the second case, the fuzziness arises when we consider the sets together with their extensions. In other words, the classical Extensionality cannot be valid for our case. Instead, we are proposing the following mathematical foundation.

We trust that two predefined constructs are enough—an extension predicate and a context sensitive membership relation. $Ext$ returns the members of its first argument, with the context specified in its second argument. The new context sensitive membership relation ($\in_C$), holds when the left hand side of the relation is in the extension of the right hand side with its subscript (context). The relation between this new membership relation and $Ext$ is as follows:

$$a \in_C b \quad \text{iff} \quad a \in Ext(b, C)$$

where $\in$ represents the usual membership relation of sets. The following statements, which concern the previous example sets, might help illustrate the new formulation:

- $Ext(Barbers, M) = \{a, b, c\}$

- $Ext(Barbers, C) = \{a, b, c, d\}$
- $Ext(Firemen, C) = \{a, b, c, d\}$
- $Ext(Firemen, C) = Ext(Barbers, C)$ (Note that this statement does not imply that $Firemen^C = Barbers^C$.)
- $d \notin_M Barbers$ ; $d \notin Ext(Barbers, M)$
- $d \in_C Barbers$ ; $d \in Ext(Barbers, C)$
- $c \in_M Barbers$ ; $c \in Ext(Barbers, M)$
- $c \in_C Barbers$ ; $c \in Ext(Barbers, C)$

Now, we can begin to define the commonsense sets. The inverse of $Ext$ can be thought of as the set formation process. The word "inverse" should not be taken too seriously; the set formation process is in fact simply the drawing of the edges from some existing node to an existing or a newly created node, and labeling this edge. The process of set formation corresponds to a commonsense cognitive categorization mechanism.

In our universe we have two kind of sets — commonsense sets and classical sets. However, the second kind will not be used in the commonsense representations and reasoning, but used in some (somewhat low level) mathematical and logical operations. The good thing with commonsense sets is that they are still close to the classical sets, and we can use most of our previous knowledge and tools in this domain.

# 5 Related commonsense notions

## 5.1 Circularity

The need for representing circularity for commonsense reasoning has been widely discussed [4]. Unfortunately, ZFC does not permit it. Since there is considerable affinity between our representation and Aczel's [1] representation of hypersets, we can use his hyperset theory, which is an enrichment of ZFC (ZF with Choice). It is the collection of all the conventional axioms of ZFC modified to be consistent with the new universe involving atoms, except that the Axiom of Foundation is now replaced by a new Anti-Foundation Axiom invented by Aczel [1]. The sets in this theory are collections of urelements or other sets, whose hereditary membership relation can be depicted by graphs. These sets may be well-founded or non-well-founded, i.e., may have an infinite descending membership sequence [4].

In Aczel's conception—which inspired our representation—sets can be pictured by means of directed graphs in an unambiguous manner. In this representation, each nonterminal node represents the set which contains the entities represented by the nodes below it. The edges of the graph stand for the hereditary membership relation such that an edge from a node $n$ to a node $m$, denoted by $n \longrightarrow m$, means that $m$ is a member of $n$. The directed graphs which are used to picture hypersets have a specific node called a "point" and for every node $n$, there exists a path $n_0 \longrightarrow n_1 \longrightarrow \cdots \longrightarrow n$ from the point $n_0$ to $n$. Therefore, these graphs are called *accessible pointed graphs (apg)*. An apg is called *well-founded* if it has no infinite paths or cycles. The Anti-Foundation Axiom (AFA) states that every apg, well-founded or not, pictures a unique set.

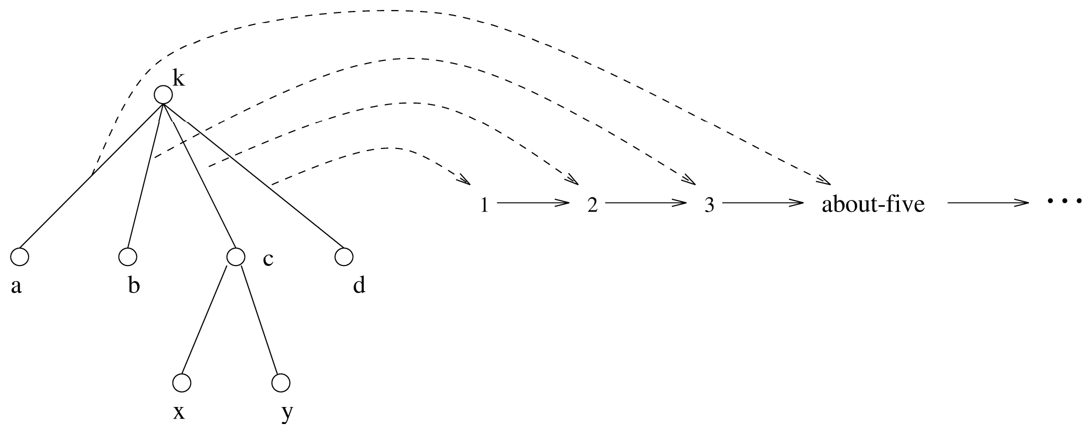


Figure 3: One-to-one order preservation

### 5.2 Cardinalities and well-orderings

In our commonsense set theory, we have used Zadrozny's approach to these issues. He thinks that these issues can be separately modeled in an existing set theory. In particular, he proposed a representation scheme based on Barwise's KPU [2] for cardinality functions, hence distinguishing reasoning about well-orderings from reasoning about cardinalities [12].

Zadrozny interprets sets as directed graphs and does not assume the FA. A graph in his conception is a triple $\langle V, SE, E\rangle$ where $V$ is a set of vertices, $SE$ is a set of edges, and $E$ is a function from a subset of $SE$ into $V \times V$. It is assumed that $x \in y$ if and only if there exists an edge between $x$ and $y$. He defines the edges corresponding to the members of a set as

$$EM(s) = \{e \in ES : \exists v[E(e) = (v, s)]\}.$$

In classical set theory, the cardinality of a finite set $s$ is a one-to-one function from a natural number $n$ onto a set, i.e., a function from a number onto the nodes of the graph of the set. However, Zadrozny defines the cardinality function as a one-to-one order preserving mapping (Figure 3) from the edges $EM(s)$ of a set $s$ into the numerals $Nums$ (an entity of numerals which is linked with sets by existence of a counting routine denoted by #, and which can take values like $1, 2, 3, 4$, or $1, 2, 3,$ *about-five*, or $1, 2, 3, many$). The last element of the range of the function is the cardinality. The cardinality of the four element set $k = \{a, b, \{x, y\}, d\}$ with three atoms and one two-atom set is *about–five*, i.e., the last element of $Nums$ which is the range of the mapping function from the edges of the set. (The cardinality might well be 4 if $Nums$ was defined as $1, 2, 3, 4$.) Zadrozny then proves two important theorems in which he shows that there exists a set $x$ with $n$ elements which does not have a well ordering and there exists a well ordering of type $n$, i.e., with $n$ elements, the elements of which do not form a set.

Following Zadrozny's footsteps, the natural numbers are represented with the graph of Figure 4. Intuitively, this set corresponds to a common sense counting mechanism. However, when people talk about large quantities, they make a generalization with the word "many." In Figure 4, this counting mechanism is tried to be shown. The labels on the edges and on the nodes correspond to the person who makes this counting. Depending on who this person is ($A$, $B$, or $C$) the notion of "many" changes drastically. For example,

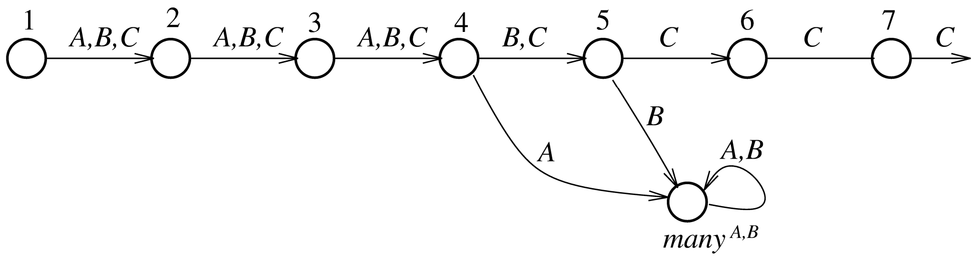


Figure 4: Natural Numbers à la Zadrozny

$A$ thinks that any quantity greater than 4 can be called "many" whereas for $C$ even 7 is not good enough to qualify as "many." Note also that once $A$ and $B$ reach "many" further operations with this notion results in itself, e.g., many apples plus many oranges gives many fruits.

### 5.3 Infinity

Infinity is quite an unclear concept in our daily life. In fact, we usually say "a lot" when we want to use the infinity. Infinity is usually used in a cardinality sense. We usually do not refer to infinities themselves (i.e., we need not compare the cardinalities $\aleph_0$ and $\aleph_1$). In our opinion, the place of the infinity is the end of the chain of natural numbers (cf. Figure 3). It must be represented with self-reference to itself, but indeed this reference will not be used since this node is inaccessible.

## 6 Conclusion

In order to obtain a commonsense set theory, we may begin with the above (hopefully original) view on membership. Extending (and streamlining) these ideas with other issues, we might get a useful commonsense set theory. In our opinion, the theory will grow in two different branches: purely mathematical and pure common sense. If we increase the contact points of these two branches, we will probably get a better commonsense set theory.

## Acknowledgments

The first author is grateful to Wlodek Zadrozny (IBM T. J. Watson Research Center) and Patrick Suppes (Stanford University) for general advice, moral support, and help with the references. He also acknowledges the partial support of the Scientific and Technical Research Council of Turkey under grant number TBAG–992.

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