Optimization and cost analysis of multi-stage and multi-channel filtering configurations.

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Abstract

The fractional Fourier domain multi-channel and multi-stage filtering configurations that have been recently proposed enable us to obtain either exact realizations or useful approximations of linear systems or matrix-vector products in many different applications. We discuss the solution and cost analysis for these configurations. It is shown that the problem can be reduced to a least squares problem which can be solved with fast iterative techniques.

1 Introduction

In many applications of digital and optical signal processing, it is desired to implement linear systems of the form $g(u) = \int H(u,u')f(u')\,du'$. Such systems take the form of a matrix-vector product when discretized: $g_k = \sum_{n=1}^{N} H_{kn} f_n$ or $\mathbf{g} = \mathbf{H}\mathbf{f}$. This may either represent a system which is inherently discrete or may constitute an approximation of a continuous system.

Linear shift-invariant systems are characterized by kernels of the special form $H(u,u') = h(u-u')$ or $H_{kn} = h_{k-n}$. These systems correspond to convolution in the time or space domain and multiplication with a filter function in the Fourier domain. Although the use of shift-invariant (convolution-type) systems are convenient in many applications, sometimes their use is inappropriate or at best a crude approximation.

In a variety of applications, greater flexibility and performance can be achieved at no additional cost, by filtering in fractional rather than ordinary Fourier domains (Fig. 1a) [3, 4, 5, 6, 7]. The ath order fractional Fourier transform $\mathcal{F}^a$ is the generalization of the ordinary Fourier transform, such that $a = 1$ corresponds to the ordinary Fourier transform and $a = 0$ corresponds to the identity operation [1, 3, 8]. Thus, when $a = 1$, the filtering scheme in Fig. 1a corresponds to ordinary Fourier domain filtering (shift-invariant or convolution-type systems). When $a = 0$, it corresponds to direct multiplication by $h(u)$ in the time domain. The costs of both digital and optical implementations of fractional Fourier domain filtering are the same as that of ordinary Fourier domain filtering [4, 9].

Further generalizations of the concept of fractional Fourier domain filtering have been suggested. These have been referred to as multi-stage (or repeated or serial) filtering in fractional Fourier domains, and multi-channel (or parallel) filtering in fractional Fourier domains, In the multi-stage system (Fig.1b) [3, 6, 10], the input is first transformed into the a_1th domain where it is multiplied by a filter $h_1(u)$. The result is then transformed back into the original domain. This process is repeated M times. In the more recently suggested multi-channel filter structure (Fig.1c) [11, 12], the inputs of all channels are identical and their outputs are added together. For each channel k, the input is first transformed to the a_kth domain where it is multiplied with a filter $h_k(u)$. The result is then transformed back to the original (time) domain.

In previous works ([3, 6, 10, 11, 12]), the matrices involved in each configuration were assumed to be square matrices of full rank. In this work a generalization of the formulation is carried out for arbitrary rectangular matrices of arbitrary dimensions and rank.

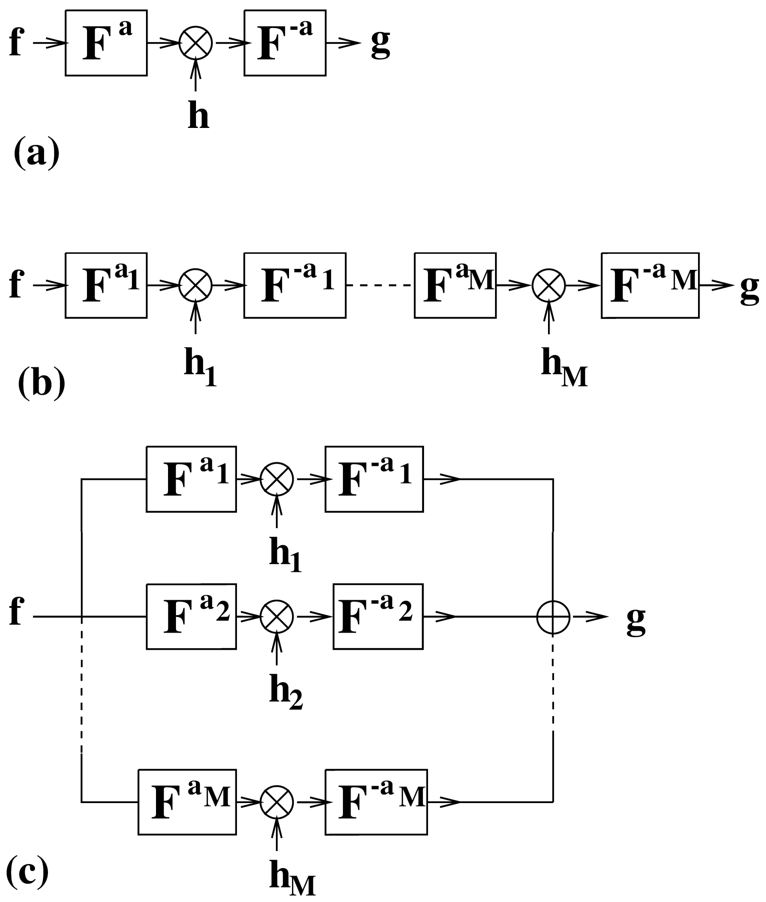

Figure 1: (a) ath order fractional Fourier domain filtering. (b) Multi-stage (serial) filtering. (c) Multi-channel (parallel) filtering.

2 Multi-stage and Multi-channel Filtering

In discrete-time notation, the outputs g_s and g_p of the serial and parallel configurations of Fig.1b and Fig.1c are related to the input f by the relations:

$$g_s = \left[\mathbf{F}^{-a_M}\mathbf{\Lambda}_M \dots \mathbf{F}^{a_2-a_1}\mathbf{\Lambda}_1\mathbf{F}^{a_1}\right] f, \tag{1}$$

$$g_p = \left[\sum_{k=1}^{M} \mathbf{F}^{-a_k}\mathbf{\Lambda}_k\mathbf{F}^{a_k}\right] f, \tag{2}$$

where $\mathbf{F}^{a_j}$ represents the a_jth order fractional Fourier transform matrix [13, 14], and $\mathbf{\Lambda}_j$ denote the diagonal matrix corresponding to multiplication by the filter function $h_k[j]$. The above may also be expressed as $g = \mathbf{T}f$ where $\mathbf{T}$ is the matrix representing the overall filtering configuration. In previous work ([3, 6, 10, 11]), $\mathbf{T}$ was assumed to be a square matrix of full rank. In this work we provide a generalization of both the formulation and cost analysis to arbitrary rectangular matrices $\mathbf{T}$ of dimension $N_{\text{out}} \times N_{\text{in}}$ of arbitrary rank R. In the multi-channel configuration, the dimensions of $\mathbf{F}^{-a_k}$, $\mathbf{\Lambda}_k$, and $\mathbf{F}^{a_k}$ become $N_{\text{out}} \times N_{\text{out}}$, $N_{\text{out}} \times N_{\text{in}}$, $N_{\text{in}} \times N_{\text{in}}$. To avoid confusion, in the rest of the paper we will use $\mathbf{F}_N^{a_k}$ to denote the a_kth order fractional Fourier transform matrix with dimensions $N \times N$. In the multi-stage case there exists a greater flexibility in choosing the dimensions of the intermediate filter matrices $\mathbf{\Lambda}_k$. A natural choice is to taper the dimensions of $\mathbf{\Lambda}_k$ gradually from N_{in} to N_{out} as k goes from 1 to M.

In a typical application we are given a linear system matrix $\mathbf{H}$ which we desire to implement (which may, for instance, be the optimal recovery operator of a signal restoration problem). Then, we seek the transform orders a_k and filters $h_k[j]$ such that the resulting matrix $\mathbf{T}$ (as given by Eqns. 1 and 2) is as close as possible to $\mathbf{H}$ according to some specified criteria, such as Froebenius norm: $\|\mathbf{T} - \mathbf{H}\|_{\text{F}}$. Alternatively, it is possible to take Eqn 1 or 2 as a constraint on the form of the linear matrix $\mathbf{H}$ to be employed in a specific application such as restoration, recovery, denoising, etc. Given a specific optimization criteria, such as minimum mean-square estimation error, we seek the optimal values of a_k and $h_k[j]$ such that the given criteria is optimized.

Before embarking on our analysis, we first note that each channel in multi-channel configuration may be more generally of the form $\mathbf{F}^{a'_k}\mathbf{\Lambda}_k\mathbf{F}^{a_k}$ where a_k and a'_k are arbitrary and do not necessarily satisfy $a'_k = -a_k$. In fact, there is no reason not to consider other parametric transforms with fast algorithms. This configuration can also be interpreted as a decomposition into operations which are shift-invariant in different fractional domains.

In the multi-channel case, regardless of which of these approaches we take, the problem of determining the optimal filter coefficients can be exactly solved since the overall kernel $\mathbf{T}$ depends linearly on the filter coefficients $h_k[j]$ as follows:

$$\mathbf{T} = \sum_{k=1}^{M} \sum_{j=1}^{N} h_k[j] \hat{\mathbf{T}}_{kj}, \tag{3}$$

where $N \equiv \min(N_{\text{in}}, N_{\text{out}})$. The dimensions of the matrices $\hat{\mathbf{T}}_{kj}$, indexed by kj, are the same as the dimensions of $\mathbf{T}$ ($N_{\text{out}} \times N_{\text{in}}$). These matrices play the role of a family of "basis matrices" which are used to construct the matrix $\mathcal{T}$. It can be shown that their elements $\hat{\mathcal{T}}_{kj}[m,n]$ are given by

$$\hat{\mathcal{T}}_{kj}[m,n] = \mathcal{F}_{N_{\text{out}}}^{-a_k}[m,j]\mathcal{F}_{N_{\text{in}}}^{a_k}[j,n], \tag{4}$$

where $\mathcal{F}_{N_{\text{out}}}^{-a_k}$ and $\mathcal{F}_{N_{\text{in}}}^{a_k}$ are fractional Fourier transform matrices of dimension N_{out} and N_{in} respectively.

The objective is to choose the NM filter coefficients $h_k[j]$ (N coefficients in each of M filters) so that the resulting linear system $\mathbf{T}$ is optimal according to some criteria. For instance, if we wish to minimize $\|\mathbf{T} - \mathbf{H}\|_{\text{F}}$, where $\mathbf{H}$ is a specified matrix, the problem can be exactly posed as a least-squares optimization problem leading to an associated set of normal equations or which can be solved with other standard techniques. To see this, it is necessary to first "vectorize" the above equations. Let $\underline{\mathbf{T}}$ denote the $N_{\text{out}}N_{\text{in}} \times 1$ vector obtained by stacking the columns of $\mathbf{T}$ on top of each other, and let $\underline{\hat{\mathbf{T}}}_{kj}$ denote the $N_{\text{out}}N_{\text{in}} \times 1$ vector obtained by stacking the columns of $\hat{\mathbf{T}}_{kj}$ on top of each other. Finally, let $\underline{\mathbf{h}}$ denote the $MN \times 1$ matrix obtained by stacking the M filters

$h_1[j]$, $h_2[j]$, ..., $h_N[j]$ on top of each other. With these conventions, we obtain

$$\underline{\mathbf{T}}[p] = \sum_{q=1}^{MN} \underline{\hat{\mathbf{T}}}_q[p]\underline{\mathbf{h}}[q] \qquad p = 1, 2, \ldots, N_{\text{out}}N_{\text{in}}, \tag{5}$$

where the indice q also follows a column ordering over the two indices kj. This equation can also be written in matrix form as

$$\underline{\mathbf{T}} = \left[\underline{\hat{\mathbf{T}}}_1\underline{\hat{\mathbf{T}}}_2 \ldots\ \underline{\hat{\mathbf{T}}}_{MN}\right]\underline{\mathbf{h}} \equiv \underline{\hat{\hat{\mathbf{T}}}}\,\underline{\mathbf{h}}, \tag{6}$$

where the new $N_{\text{out}}N_{\text{in}} \times MN$ matrix $\underline{\hat{\hat{\mathbf{T}}}}$ has been defined.

Now, we are finally able to state our problem in standard form as follows: Minimize the mean-square difference $\|\underline{\mathbf{T}} - \underline{\mathbf{H}}\|^2$ between the desired $\underline{\mathbf{H}}$ and $\underline{\mathbf{T}} = \underline{\hat{\hat{\mathbf{T}}}\mathbf{h}}$. This is a standard least squares problem and can be solved in a number of ways. The filter vector $\underline{\mathbf{h}}$ which minimizes $\|\underline{\mathbf{H}} - \underline{\hat{\hat{\mathbf{T}}}\mathbf{h}}\|^2$ is known to satisfy the so-called normal equations associated with the least squares problem:

$$\underline{\hat{\hat{\mathbf{T}}}}^{\mathrm{H}}\,\underline{\mathbf{H}} = \underline{\hat{\hat{\mathbf{T}}}}^{\mathrm{H}}\,\underline{\hat{\hat{\mathbf{T}}}}\,\underline{\mathbf{h}}, \tag{7}$$

where $\underline{\hat{\hat{\mathbf{T}}}}^{\mathrm{H}}$ is the Hermitian transpose of $\underline{\hat{\hat{\mathbf{T}}}}$.

In the multi-stage case, the overall kernel $\mathbf{T}$ depends nonlinearly on the filter coefficients $h_k[j]$, so that solution of the optimization problem arising in this case is much more difficult. (Nevertheless an iterative approach has been successfully applied to this problem [6, 10].)

The M-channel or M-stage filtering configuration has about MN degrees of freedom, as opposed to general linear systems which have $N_{\text{out}}N_{\text{in}}$ degrees of freedom and shift-invariant systems which have about N degrees of freedom. These configurations interpolate between general linear systems and shift-invariant systems both in terms of cost and flexibility. If we choose M to be small, cost and flexibility are both low; $M = 1$ corresponds to single-stage filtering. If we choose M larger, cost and flexibility are both higher; as M approaches N, the number of degrees of freedom approaches that of a general linear system. We show that exactly $M = N$ filters are necessary and sufficient to implement an arbitrary general linear system matrix exactly (with zero error) in the multi-channel case. Likewise, $M = N + 1$ filters are necessary and sufficient in the multi-stage case. In practice, most matrices are not wholly arbitrary and exhibit some kind of internal structure, although that structure may not be easy to identify or characterize. As will be illustrated by the examples, there are many applications in which acceptable or useful approximations to given linear systems are possible with small or moderate values of $M \ll N$, which as discussed next result in considerable cost savings.

Let the input of some general linear system be represented by N_{in} samples and the output by N_{out} samples. Digital implementation of general linear systems takes $O(N_{\text{out}}N_{\text{in}})$ time (the time to multiply the system matrix with the input vector). Direct optical implementations of general linear systems using matrix-vector product or multi-facet architectures require an optical system whose space-bandwidth product is $O(N_{\text{out}}N_{\text{in}})$ [2]. On the other hand, the digital implementation of shift-invariant systems takes $O(N_{\text{in}} \log N_{\text{in}} + N' + N_{\text{out}} \log N_{\text{out}}) \sim O(N \log N)$ time by using the fast Fourier transform, where $N \equiv \max(N_{\text{out}}, N_{\text{in}})$ and $N' \equiv \min(N_{\text{out}}, N_{\text{in}})$. Optical implementation of shift-invariant systems requires a pair of optical Fourier transformers whose space-bandwidth products are $O(N_{\rm in})$ and $O(N_{\rm out})$. The cost of filtering in a single fractional Fourier domain is the same as that of implementing shift-invariant systems (which correspond to filtering in the ordinary Fourier domain).

In the above paragraph, we implicitly assumed that all rows of the matrix representing the general linear system are linearly independent. Since the rank R of a matrix always satisfies $R \le N' \equiv \min(N_{\rm out}, N_{\rm in})$, this is possible only when $R = N_{\rm out} \le N_{\rm in}$. In the general case, the rank R corresponds to the number of linearly independent rows. Multiplying these linearly independent rows with the input takes $O(RN_{\rm in})$ time. Multiplication of other rows can be accomplished more easily since it is known that remaining $(N_{\rm out} - R)$ rows are known to be linear combinations of the other R rows. Since R coefficients are sufficient to characterize these rows, multiplying them with the input takes $O((N_{\rm out} - R)R)$ time. The total amount of time is thus

$$O(RN_{\rm in} + (N_{\rm out} - R)R) = O(R(N_{\rm in} + N_{\rm out}) - R^2). \quad (8)$$

We are not able to propose a simple scheme for exploiting rank information in optical implementation, so that we again take the cost of optical implementation as before.

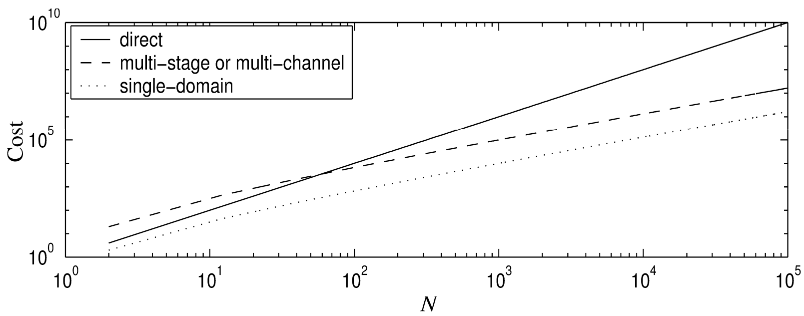

Figure 2: Cost of directly implementing a linear system compared with the cost of implementing multi-stage, multi-channel, and single-domain filtering configurations ($N_{\rm out} = N_{\rm in} = N$, $M = 10$).

We now turn our attention to multi-stage and multi-channel filtering configurations. These configurations consist of M single-domain filters. Thus the multi-channel configuration can be digitally implemented in

$$O\left(M(N_{\rm in}\log N_{\rm in} + N' + N_{\rm out}\log N_{\rm out})\right) \sim O(MN\log N) \quad (9)$$

time. Likewise, the multi-stage configuration can be digitally implemented in

$$\begin{aligned} O\left(N_{\rm in}\log N_{\rm in} + \sum_{k=1}^{M}[\min(N_{k-1}, N_k) + N_k\log N_k]\right) &\approx O\left(\sum_{k=0}^{M}\min(N_{k-1}, N_k) + N_k\log N_k\right) \\ &\approx O\left(\sum_{k=0}^{M} N_k\log N_k\right) \quad (10) \end{aligned}$$

time. Normally, the dimensions N_k of the intermediate stages would lie between $N_0 \equiv N_{\text{in}}$ and $N_M \equiv N_{\text{out}}$. It seems natural that $N_{\text{in}} = N_0 \leq N_1 \leq N_2 \leq \ldots N_{M-1} \leq N_M = N_{\text{out}}$ or $N_{\text{in}} = N_0 \geq N_1 \geq N_2 \geq \ldots N_{M-1} \geq N_M = N_{\text{out}}$ depending on whether $N_{\text{in}} \leq N_{\text{out}}$ or $N_{\text{in}} \geq N_{\text{out}}$. Therefore, the last expression is also $\sim O(MN \log N)$. We now consider the costs of optical implementation. The multi-channel configuration requires M pairs of fractional Fourier transformers whose space-bandwidth products are $O(N_{\text{in}})$ and $O(N_{\text{out}})$. The multi-stage configuration requires $M + 1$ fractional Fourier transformers whose space-bandwidth products are $O(N)$.

Figure 2 compares the time cost of directly implementing a linear system with that of implementing multi-stage or multi-channel configurations with a moderate number of filters.

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