Time-Order Signal Representations

Haldun M. Ozaktas
Bilkent University
Department of Electrical Engineering
TR-06533 Bilkent, Ankara, Turkey

M. Alper Kutay
Drexel University
Department of Electrical and Computer Engineering
Philadelphia, PA 19104, USA

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Bilkent University
Department of Computer Engineering
TR-O6533 Bilkent, Ankara, Turkey

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Abstract

Just like time-frequency and time-scale representations, time-order signal representations constitute an alternative way of displaying the content of a signal, with the potential to reveal features which may not be evident upon examination of its other representations and to lead to novel processing techniques. We derive many of their properties and their relations to other time-frequency representations. Their importance stems from the fact that the Radon transforms (integral projections) and slices of the Wigner distribution and the ambiguity function can be expressed in terms of products or convolutions of various scaled forms of the time-order representation and its two-dimensional Fourier transform.

1 Introduction

In this paper we discuss time-order (or space-order) signal representations. Just like time-frequency and time-scale (or space-frequency and space-scale) signal representations, these representations constitute an alternative way of displaying the content of a signal, with the potential to reveal features which may not be evident upon examination of its other representations. Similar to time-frequency and time-scale representations, these are redundant representations in that the information contained in a one-dimensional signal is displayed in two dimensions, with the potential to lead to novel processing techniques. These representations were first introduced in [9]. Here we will derive many of their properties and their relations to important time-frequency representations such as the Wigner distribution and the ambiguity function, showing that they are closely related to the projections and slices of these well-known time-frequency representations.

Time-order representations are based on the fractional Fourier transform (FRT), which in recent years has attracted a considerable amount of attention, leading to many applications in the areas of optics and signal processing. A comprehensive introduction to the FRT and historical references may be found in [1]. The transform has become popular in the optics and signal processing communities following the works of Ozaktas and Mendlovic [2, 3], Lohmann [4] and Almeida [5]. Some of the applications explored include optimal filtering in fractional Fourier domains [6], cost-efficient linear system synthesis and filtering [7, 8] and time-frequency analysis [5, 1]. Further references may be found in [1].

We will consider two variations of the time-order representation, the Cartesian time-order representation and the polar time-order representation. The Cartesian time-order representation is simply the fractional Fourier transform $f_a(u)$ of a function $f(u)$ interpreted as a two-dimensional function, with u the horizontal coordinate and a the vertical coordinate. The polar time-order representation is simply $f_{2\alpha/\pi}(\rho)$ interpreted as a polar two-dimensional function where ρ is the radial coordinate and $\alpha = a\pi/2$ is the angular coordinate. Both representations are complex valued.

2 Definitions and Properties

The ath order fractional Fourier transform of $f(u)$ is denoted by $f_a(u)$ and given by [1]

$$f_a(u) = A_\alpha \int_{-\infty}^{\infty} e^{i\pi(\cot\alpha\, u^2 - 2\csc\alpha\, uu' + \cot\alpha\, u'^2)} f(u')\, du', \tag{1}$$

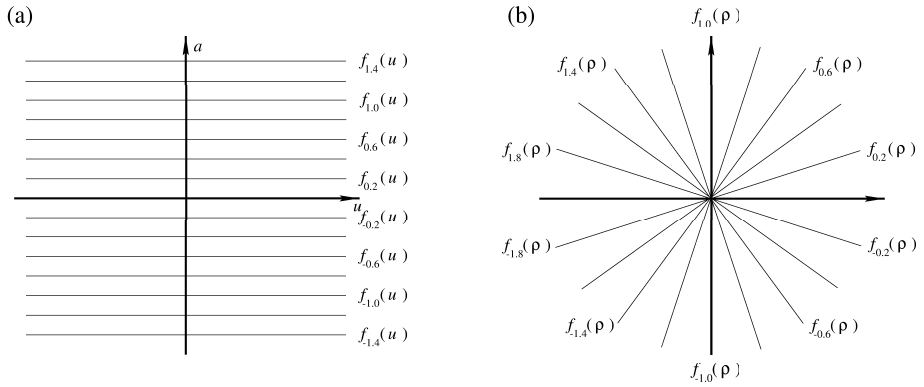

Figure 1: (a) The Cartesian time-order representation, (b) the polar time-order representation.

for $a \neq 2j$, and $f_a(u) = f(u)$ for $a = 4j$ and $f_a(u) = f(-u)$ for $a = 4j \pm 2$, where j is an integer, $\alpha = a\pi/2$, and $A_\alpha = \sqrt{1 - i\cot\alpha}$. The 0th transform of $f(u)$ is simply $f_0(u) = f(u)$ itself and the 1st transform is simply $f_1(u) = F(u)$, the ordinary Fourier transform. The a_2th transform of the a_1th transform is equal to the $(a_2 + a_1)$th transform, a property known as index additivity. The ath fractional Fourier domain makes an angle $\alpha = a\pi/2$ with the time (or space) domain in the time-frequency (or space-frequency) plane. This is confirmed by the fact that the Radon transform of the Wigner distribution of $f(u)$ onto this domain equals $|f_a(u)|^2$.

The Cartesian time-order representation is simply $f_a(u)$ interpreted as a two-dimensional function, with u the horizontal coordinate and a the vertical coordinate. As such, it is a display of all the fractional Fourier transforms of $f(u)$ next to each other. In other words, the representations of the signal f in all fractional domains are displayed simultaneously. Formally, we will denote the Cartesian time-order representation of a signal f by $T_f(u, a)$, so that we define

$$T_f(u, a) \equiv f_a(u). \tag{2}$$

Figure 1(a) illustrates the definition of the Cartesian time-order representation.

Although we will not elaborate, it is evident that all of the properties of the fractional Fourier transform can be interpreted as properties of the time-order representation. In particular, the following simple identities are sometimes useful in dealing with the representations of the products and convolutions of functions:

$$T_{h(u)f(u)}(u, a) = T_{H(u)*F(u)}(u, a - 1), \tag{3}$$

$$T_{h(u)*f(u)}(u, a) = T_{H(u)F(u)}(u, a - 1). \tag{4}$$

Of course, the Cartesian representation is periodic in a with period 4.

Figure 2 illustrates the Cartesian time-order representations of the signal $\exp(-\pi 2u^2)$ and the signal $\mathrm{rect}(u)$.

The polar time-order representation is simply $f_{2\alpha/\pi}(\rho)$ interpreted as a polar two-dimensional function where ρ is the radial coordinate and α is the angular coordinate. As such, it is a display

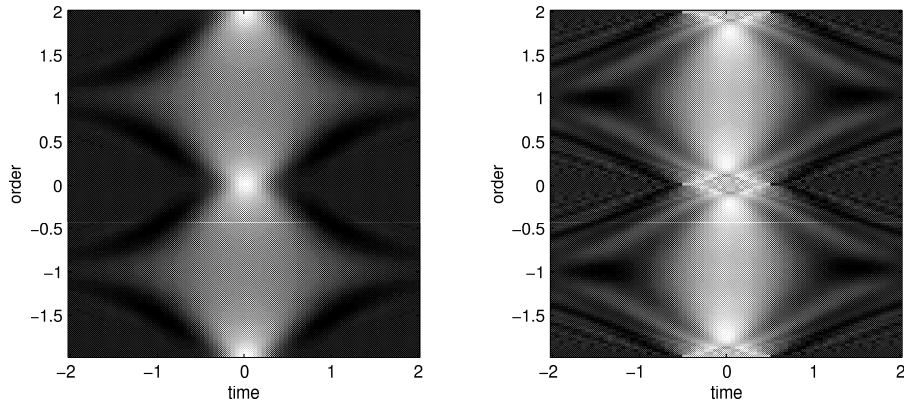

Figure 2: Magnitudes of the Cartesian time-order representations of $\exp(-\pi 2u^2)$ (left) and $\mathrm{rect}(u)$ (right).

of all the fractional Fourier transforms of $f(u)$ such that $f_a(\rho)$ lies along the radial line making angle $\alpha = a\pi/2$ with the horizontal u axis. As in the Cartesian representation, all transforms are displayed simultaneously. Formally, we will denote the polar time-order representation of a signal f by $T_f(\rho, \alpha)$, so that we define

$$T_f(\rho, \alpha) \equiv f_{2\alpha/\pi}(\rho). \tag{5}$$

Since $f_a(\rho)$ is periodic in a with period 4, $T_f(\rho, \alpha)$ is periodic in α with period 2π, as all polar functions must be. $T_f(\rho, \alpha)$ is defined for negative values of ρ as well. This does not pose any inconsistency since $f_{a\pm 2}(\rho) = f_a(-\rho)$, from which it also follows that $T_f(\rho, \alpha) = T_f(-\rho, \alpha \pm \pi)$. Figure 1(b) illustrates the definition of the polar time-order representation.

Figure 3 illustrates the polar time-order representations of the signal $\exp(-\pi 2u^2)$ and the signal $\mathrm{rect}(u)$.

The slices of the polar representation are simply equal to the fractional Fourier transforms:

$$\mathcal{S}_\alpha[T_f(\rho, \alpha)](u) = f_a(u), \qquad \alpha = a\pi/2. \tag{6}$$

The slice operator is defined as: $\mathcal{S}_\phi[T_f(\mu, \nu)](\mu') \equiv T_f(\mu' \cos\phi, \mu' \sin\phi)$. In particular, the slice at $\alpha = 0$ is the time-domain representation $f(u)$, and the slice at $\alpha = \pi/2$ is the frequency-domain representation $F(\mu)$. Other slices correspond to fractional transforms of other orders. We also know that the Radon transform of the Wigner distribution is given by [11, 1]

$$\mathcal{R}_\alpha[W_f(u, \mu)](\rho) = |f_{2\alpha/\pi}(\rho)|^2 = |T_f(\rho, \alpha)|^2, \tag{7}$$

where The Radon transform is defined as: $\mathcal{R}_\phi[W_f(u, \mu)](u') \equiv \int W_f(u' \cos\phi - v' \sin\phi, u' \sin\phi + v' \cos\phi)\, dv'$. Thus, the Radon transform of the Wigner distribution, interpreted as a polar function, corresponds to the absolute square of the time-order representation defined above. The relationship of time-order representations to the Wigner distribution and ambiguity function will be further discussed in section 3.

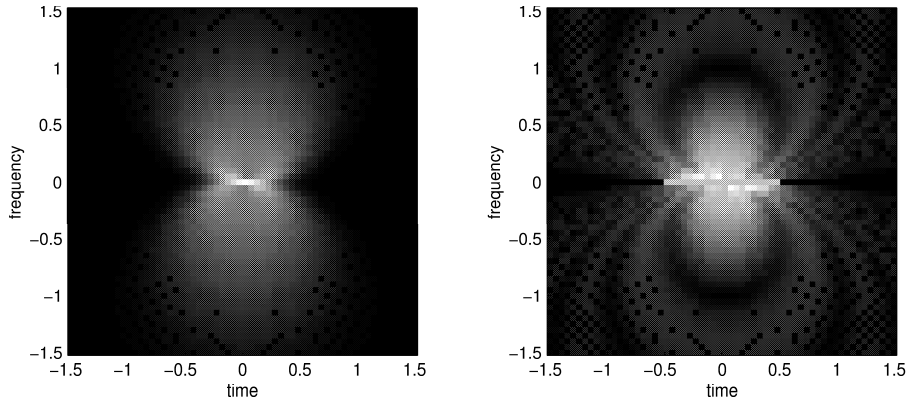

Figure 3: Magnitudes of the polar time-order representations of $\exp(-\pi 2u^2)$ (left) and $\mathrm{rect}(u)$ (right).

We now move on to discuss a number of properties of this representation. First, we note that obtaining the original function from the distribution is trivial: $f(u) = f_0(u) = T_f(u, 0)$. Obtaining the Fourier transform of the function or indeed any other fractional transform is likewise a direct consequence of the definition.

The time-order representation of the a'th fractional Fourier transform of a function is simply a rotated version of the time-order representation of the original function:

$$T_{f_{a'}}(\rho, \alpha) = T_f(\rho, \alpha + \alpha'), \tag{8}$$

where $\alpha' = a'\pi/2$. In particular, the time-order representation of the Fourier transform of a function is a $\pi/2$-rotated version of the original. Since the time-order representation is linear, the representation of any linear combination of functions is the same as the linear combination of their representations.

Various properties of the polar time-order representation follow immediately from properties of the fractional Fourier transform, and are presented in table 1.

3 Relationships with the Wigner distribution and the ambiguity function

We now return to the discussion of the relationship of time-order representations with other time-frequency representations, which we had postponed after equation 7. It is possible to show as a consequence of the projection-slice theorem that

$$\mathcal{S}_\alpha[A(\bar{u}, \bar{\mu})](\rho) \quad = \quad T_f(\rho, \alpha) * T_f^*(-\rho, \alpha) = f_{2\alpha/\pi}(\rho) * f^*_{2\alpha/\pi}(-\rho) \tag{9}$$

where $A_f(\bar{u}, \bar{\mu}) \equiv \int f(u' + \bar{u}/2) f^*(u' - \bar{u}/2) e^{-i2\pi\bar{\mu}u'} du'$ is the ambiguity function, and '$*$' denotes the convolution. We see that just as oblique projections of the Wigner distribution correspond to

$f(u)$	$T_f(\rho,\alpha)$
$f(-u)$	$T_f(-\rho,\alpha)$
$\lvert s\rvert^{-1}f(u/s)$	$K\,e^{\left[i\pi\rho^2\cot\alpha\left(1-\frac{\cos^2\alpha'}{\cos^2\alpha}\right)\right]}\,T_f\left(\frac{s\rho\sin\alpha'}{\sin\alpha},\alpha'\right)$
$f(u-u_0)$	$e^{i\pi\sin\alpha(u_0^2\cos\alpha-2\rho u_0)}\,T_f(\rho-u_0\cos\alpha,\alpha)$
$e^{i2\pi\mu_0 u}f(u)$	$e^{i\pi\cos\alpha(2\rho\mu_0-\mu_0^2\sin\alpha)}\,T_f(\rho-\mu_0\sin\alpha,\alpha)$
$u^n f(u)$	$[\cos\alpha\;\rho-\sin\alpha\;\frac{1}{i2\pi}\frac{d}{d\rho}]^n T_f(\rho,\alpha)$
$[\frac{1}{i2\pi}\frac{d}{du}]^n f(u)$	$[\sin\alpha\;\rho+\cos\alpha\;\frac{1}{i2\pi}\frac{d}{d\rho}]^n T_f(\rho,\alpha)$
$f(u)/u$	$-i\csc\alpha\,e^{i\pi\rho^2\cot\alpha}\int_{-\infty}^{2\pi\rho}T_f(\rho',\alpha)e^{-i\pi\rho'^2\cot\alpha}\,d\rho'$
$\int_\xi^u f(u')\,du'$	$\sec\alpha\,e^{-i\pi\rho^2\tan\alpha}\int_\xi^\rho T_f(\rho',\alpha)e^{i\pi\rho'^2\tan\alpha}\,d\rho'$
$f^*(u)$	$T_f^*(\rho,-\alpha)$
$f^*(-u)$	$T_f^*(-\rho,-\alpha)$
$[f(u)+f(-u)]/2$	$[T_f(\rho,\alpha)+T_f(-\rho,\alpha)]/2$
$[f(u)-f(-u)]/2$	$[T_f(\rho,\alpha)-T_f(-\rho,\alpha)]/2$

Table 1: Properties of the polar time-order representation. u_0, μ_0 and s are real but $s \neq 0, \pm\infty$. $\alpha' = \arctan(s^{-2}\tan\alpha)$ where α' is taken to be in the same quadrant as α. $K = \sqrt{(1-i\cot\alpha)/(1-is^2\cot\alpha)}$. Property 7 is not valid when α is an integer multiple of π and property 8 is not valid when α is an odd integer multiple of $\pi/2$.

the squared magnitudes of the fractional Fourier transforms of the function, the oblique slices of the ambiguity function correspond to the convolutions of the fractional Fourier transforms of the function.

Having discussed the Radon transform (projections) of the Wigner distribution and the slices of the ambiguity function, we now turn our attention to the slices of the Wigner distribution and the Radon transform of the ambiguity function. To proceed, we first write the Wigner distribution of $f_a(u)$:

$$W_{f_a}(u,\mu) = \int f_a(u+u'/2)f_a^*(u-u'/2)e^{-i2\pi\mu u'}\,du', \tag{10}$$

whose slice at the angle $\pi/2$ is easily obtained as

$$\mathcal{S}_{\pi/2}[W_{f_a}(u,\mu)](\rho) = \int f_a(u'/2)f_a^*(-u'/2)e^{-i2\pi\rho u'}\,du' = \mathcal{F}[f_a(u/2)f_a^*(-u/2)](\rho). \tag{11}$$

Now, we know that $W_{f_a}(u,\mu) = W_f(u\cos\alpha-\mu\sin\alpha, u\sin\alpha+\mu\cos\alpha)$ [3, 4], so that

$$W_{f_a}(0,\rho) = W_f(-\rho\sin\alpha,\rho\cos\alpha) = \mathcal{S}_{\alpha+\pi/2}[W_f(u,\mu)](\rho). \tag{12}$$

Combining equations 11 and 12 finally leads us to the desired expression for the slices of the Wigner distribution:

$$\begin{aligned}\mathcal{S}_\alpha[W_f(u,\mu)](\rho) &= \int f_{a-1}(u'/2)f_{a-1}^*(-u'/2)e^{-i2\pi\rho u'}\,du' \\ &= 4f_a(2\rho) * f_a^*(2\rho) = 4T_f(2\rho,\alpha) * T_f^*(2\rho,\alpha),\end{aligned} \tag{13}$$

$\mathcal{R}_\alpha[W_f(u,\mu)](\rho) = f_a(\rho) f_a^*(\rho) = T_f(\rho,\alpha) T_f^*(\rho,\alpha)$
$\mathcal{R}_\alpha[A_f(\bar{u},\bar{\mu})](\rho) = f_a(\rho/2) f_a^*(-\rho/2) = T_f(\rho/2,\alpha) T_f^*(-\rho/2,\alpha)$
$\mathcal{S}_\alpha[W_f(u,\mu)](\rho) = 2f_a(2\rho) * 2f_a^*(2\rho) = 2T_f(2\rho,\alpha) * 2T_f^*(2\rho,\alpha)$
$\mathcal{S}_\alpha[A_f(\bar{u},\bar{\mu})](\rho) = f_a(\rho) * f_a^*(-\rho) = T_f(\rho,\alpha) * T_f^*(-\rho,\alpha)$

Table 2: Radon transforms and slices of the Wigner distribution and the ambiguity function. The upper row can also be expressed as $|f_{2\alpha/\pi}(\rho)|^2 = |T_f(\rho,\alpha)|^2$.

where $a = 2\alpha/\pi$ as before. The slice of the Wigner distribution at angle α is equal to the convolution of $2T_f(2\rho,\alpha) = 2f_{2\alpha/\pi}(2\rho)$ with its conjugate. Since $T_f(2\rho,\alpha)$ is a function of polar coordinates, its slice at an angle α is simply $T_f(2\rho,\alpha)$ itself. Thus the slice of the Wigner distribution at a certain angle is equal to the convolution of the slice of the time-order representation with its conjugate.

Now, application of the projection-slice theorem allows us to arrive at the following result for the Radon transform of the ambiguity function [12]:

$$\mathcal{R}_\alpha[A_f(\bar{u},\bar{\mu})](\rho) = f_{2\alpha/\pi}(\rho/2) f_{2\alpha/\pi}^*(-\rho/2) = T_f(\rho/2,\alpha) T_f^*(-\rho/2,\alpha). \tag{14}$$

The special case of $\alpha = 0$ yields $\mathcal{R}_0[A_f(\bar{u},\bar{\mu})](\rho) = \int A_f(\rho,\bar{\mu})$ $d\bar{\mu} = f(\rho/2) f^*(-\rho/2)$, which could have also been derived directly from the definition of the ambiguity function.

Table 2 summarizes the Radon transforms and slices of the Wigner distribution and ambiguity function. We note that although not shown, the results of these operations can also be expressed in terms of the fractional Fourier transform. For both the Wigner distribution and the ambiguity function, the Radon transform is of product form and the slice is of convolution form. The essential difference between the Wigner distribution and the ambiguity function lies in the scaling of ρ by 2 or 1/2 on the right hand side.

We already know the slice of $T_f(\rho,\alpha)$ to be simply given by $f_{2\alpha/\pi}(\rho)$, by definition. Now, we embark on deriving the Radon transform of $T_f(\rho,\alpha)$. A polar-to-rectangular coordinate conversion allows us to write $T_f(\rho,\alpha)$ in Cartesian coordinates as follows:

$$T_{f,\mathrm{rect}}(u,\mu) \equiv T_f(\rho,\alpha) \quad \rho = \sqrt{u^2+\mu^2}\,, \quad \alpha = \arctan(\mu/u). \tag{15}$$

The Radon transform of $T_f(\rho,\alpha)$ at an angle ϕ is

$$\mathcal{R}_\phi[T_f(\rho,\alpha)](\varrho) = \int T_{f,\mathrm{rect}}(\varrho\cos\phi - \mu'\sin\phi, \varrho\sin\phi + \mu'\cos\phi)\, d\mu', \tag{16}$$

$$= \int T_f\left[\sqrt{\varrho^2+\mu'^2}, \arctan\left(\frac{\varrho\sin\phi + \mu'\cos\phi}{\varrho\cos\phi - \mu'\sin\phi}\right)\right] d\mu'. \tag{17}$$

Introducing the following change of integration variable from μ' to θ: $\mu' = \varrho\tan\theta$, we can write the above integral as

$$\mathcal{R}_\phi[T_f(\rho,\alpha)](\varrho) = \int_{-\pi/2}^{\pi/2} T_f(\varrho\sec\theta, \phi+\theta)\varrho\sec^2\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} f_{2(\phi+\theta)/\pi}(\varrho \sec\theta)\, \varrho \sec^2\theta \, d\theta, \tag{18}$$

which will be our final expression for the Radon transform of $T_f(\rho, \alpha)$.

We will denote the two-dimensional ordinary Fourier transform of $T_{f,\mathrm{rect}}(u, \mu)$ as $\tilde{T}_{f,\mathrm{rect}}(\bar{\mu}, \bar{u}) \equiv \int\int T_{f,\mathrm{rect}}(u, \mu) e^{-i2\pi(\bar{\mu}u + \bar{u}\mu)}\, du\, d\mu$. The same Fourier transform relation can also be expressed in polar coordinates $\bar{\rho} = \sqrt{\bar{u}^2 + \bar{\mu}^2}$, $\bar{\alpha} = \arctan(\bar{\mu}/\bar{u})$ as

$$\tilde{T}_f(\bar{\rho}, \bar{\alpha}) = \int_0^{2\pi}\int_0^{\infty} T_f(\rho, \alpha) e^{-i2\pi\bar{\rho}\rho\cos(\bar{\alpha}-\alpha)} \rho \, d\rho \, d\alpha \tag{19}$$

where $\tilde{T}_f(\bar{\rho}, \bar{\alpha})$ denotes the two-dimensional Fourier transform $\tilde{T}_{f,\mathrm{rect}}(\bar{\mu}, \bar{u})$ in polar coordinates. The slice of $\tilde{T}_f(\bar{\rho}, \bar{\alpha})$ at an angle ϕ is simply

$$\mathcal{S}_\phi[\tilde{T}_f(\bar{\rho}, \bar{\alpha})](\varrho) = \tilde{T}_f(\varrho, \phi). \tag{20}$$

Now, application of the projection-slice theorem allows us to write

$$\begin{aligned} \mathcal{S}_\phi[\tilde{T}_f(\bar{\rho}, \bar{\alpha})](\varrho) = \tilde{T}_f(\varrho, \phi) = \mathcal{F}\left[\mathcal{R}_\phi[T_f(\rho, \alpha)]\right](\varrho) &= \int \left[\int_{-\pi/2}^{\pi/2} T_f(\varrho' \sec\theta, \phi + \theta)\ \varrho' \sec^2\theta \, d\theta\right] e^{-i2\pi\varrho\varrho'}\, d\varrho' \\ &= \frac{i}{2\pi}\int_{-\pi/2}^{\pi/2} f'_{2(\phi+\theta)/\pi+1}(\varrho\cos\theta)\ \sec\theta \, d\theta, \end{aligned} \tag{21}$$

where $f'_{2(\phi+\theta)/\pi+1}(z)$ denotes the derivative $df_{2(\phi+\theta)/\pi+1}(z)/dz$.

Having obtained its slice, what remains is to write an expression for the Radon transform of $\tilde{T}_f(\bar{\rho}, \bar{\alpha})$, which follows without much difficulty from the projection-slice theorem:

$$\begin{aligned} \mathcal{R}_\phi[\tilde{T}_f(\bar{\rho}, \bar{\alpha})](\varrho) = \mathcal{F}^{-1}\mathcal{S}_{\phi+\pi}[T_f(\rho, \alpha)](\varrho) &= \mathcal{F}^{-1} f_{\phi+\pi}(\varrho) \\ &= f_{\phi+\pi/2}(\varrho) = T_f(\varrho, \phi + \pi/2). \end{aligned} \tag{22}$$

Thus, we have now completed a set of four expressions for the Radon transforms and slices of the polar time-order representation $T_f(\rho, \alpha)$ and its two-dimensional Fourier transform $\tilde{T}_f(\bar{\rho}, \bar{\alpha})$. The slice of $T_f(\rho, \alpha)$ at a certain angle is simply equal to the fractional Fourier transform $f_a(\rho)$ by definition (with $\alpha = a\pi/2$). The Radon transform of $\tilde{T}_f(\bar{\rho}, \bar{\alpha})$ at an angle ϕ, we have just seen to be given by $f_{b+1}(\rho)$ or $T_f(\rho, \phi+\pi/2)$, a ninety-degree rotated version of $T_f(\rho, \alpha)$ (with $\phi = b\pi/2$). (The remaining two relations are more complicated and are given by equations 18 and 21.) We already knew that the time-frequency representation whose projections are equal to $|f_a(u)|^2$ is the Wigner distribution. We now see that the time-frequency representation whose projections are equal to $f_a(u)$ is the two-dimensional Fourier transform of the polar time-order representation (within a rotation). Table 3 summarizes the Radon transforms and slices of the time-order representation and its two-dimensional Fourier transform. We see that the results of these operations can be expressed in terms of the fractional Fourier transform.

Looking back, we see that we have derived a total of eight expressions for the Radon transforms and slices of the Wigner distribution and its two-dimensional Fourier transform (the ambiguity

$\mathcal{R}_\phi[T_f(\rho,\alpha)](\varrho) = \int_{-\pi/2}^{\pi/2} f_{2(\phi+\theta)/\pi}(\varrho\sec\theta)\ \varrho\sec^2\theta\, d\theta = \int_{-\pi/2}^{\pi/2} T_f(\varrho\sec\theta, \phi+\theta)\ \varrho\sec^2\theta\, d\theta$
$\mathcal{R}_\phi[\tilde{T}_f(\bar{\rho},\bar{\alpha})](\varrho) = f_{\phi+\pi/2}(\varrho) = T_f(\varrho, \phi+\pi/2)$
$\mathcal{S}_\phi[T_f(\rho,\alpha)](\varrho) = f_{2\phi/\pi}(\varrho) = T_f(\varrho,\phi)$
$\mathcal{S}_\phi[\tilde{T}_f(\bar{\rho},\bar{\alpha})](\varrho) = \frac{i}{2\pi}\int_{-\pi/2}^{\pi/2} f'_{2(\phi+\theta)/\pi+1}(\varrho\cos\theta)\ \sec\theta\, d\theta = \frac{i}{2\pi}\int_{-\pi/2}^{\pi/2} T'_f(\varrho\cos\theta, \phi+\theta+\pi/2)\ \sec\theta\, d\theta$

Table 3: Radon transforms and slices of the time-order representation and its two-dimensional Fourier transform. $T'_f(\rho,\alpha)$ denotes the derivative $dT_f(\rho,\alpha)/d\rho$.

function), and the Radon transforms and slices of the polar time-order representation and its two-dimensional Fourier transform.

The polar time-order representation is a linear time-frequency representation, unlike the Wigner distribution which is a quadratic time-frequency representation. We note that the Wigner distribution can be interpreted as giving the distribution of signal energy over time and frequency [10]. As such it is an example of an *energetic* time-frequency representation. In contrast, the ambiguity function was seen to have qualities reminiscent of correlation, and it was an example of a *correlative* time-frequency representation. We also know that the Wigner distribution and ambiguity function constitute a two-dimensional Fourier transform pair. The polar time-order representation and its two-dimensional Fourier transform fall into neither category and do not belong to Cohen's class of shift-invariant representations. Their importance stems from the fact that the Radon transforms (integral projections) and slices of the Wigner distribution and the ambiguity function can be expressed in terms of products or convolutions of various scaled forms of the time-order representation and its two-dimensional Fourier transform.

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