# QUASI-BIRTH-AND-DEATH PROCESSES WITH LEVEL-GEOMETRIC DISTRIBUTION

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**Abstract.** A special class of homogeneous continuous-time quasi-birth-and-death (QBD) Markov Chains (MCs) which possess level-geometric (LG) stationary distribution are considered. Assuming that the stationary vector is partitioned by levels into subvectors, in an LG distribution all stationary subvectors beyond a finite level number are multiples of each other. Specifically, each pair of stationary subvectors that belong to consecutive levels are related by the same scalar, hence the term level-geometric. Necessary and sufficient conditions are specified for the existence of such a distribution and the results are elaborated on three examples.

**Key Words.** Markov chains, QBD processes, geometric distributions

**AMS(MOS) subject classification.** 60J27, 65F50, 65H10, 65F05, 65F10, 65F15

**1. Introduction.** The continuous-time Markov process on the countable state space $\mathcal{S} = \{(l, i) \ : \ l \geq 0, \ 1 \leq i \leq m\}$ with block tridiagonal infinitesimal generator matrix

$$Q = \begin{pmatrix} B_0 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{1}$$

having blocks of order $m$ is called a homogeneous continuous-time quasi-birth-and-death (QBD) Markov chain (MC). The row sums of $Q$ are zero meaning $(B_0 + A_0)e = 0$ and $(A_0 + A_1 + A_2)e = 0$, where $e$ is a column vector of 1's with appropriate length. The matrices $A_0$ and $A_2$ are nonnegative, and the matrices $B_0$ and $A_1$ have nonnegative off-diagonal elements and strictly negative diagonals. The first component, $l$, of the state descriptor vector denotes the level and its second component, $i$, the phase. In homogeneous QBD MCs, the elements of the $(m \times m)$ matrices $B_0$, $A_0$, $A_1$, and $A_2$ do not depend on level number.

Neuts has done substantial work in the area of matrix analytic methods for such processes and has written two books [11], [12]. An informative resource that discusses the developments in the area since then is the recent book of Latouche and Ramaswami [9]. The most significant application area of these methods at present is the performance evaluation of communication systems.

We assume that the homogeneous continuous-time QBD MC at hand is irreducible and positive recurrent meaning its steady state probability distribution vector, $\pi$ (see [13]), exists. Throughout the paper, we adhere to the convention that probability vectors are row vectors. Being a stationary distribution, $\pi$ satisfies $\pi Q = 0$ and $\pi e = 1$. Now, let $\pi$ be partitioned by levels into subvectors $\pi_l$, $l \geq 0$, where $\pi_l$ is of length $m$. Then $\pi$ also satisfies the matrix-geometric property [9, p. 142]:

$$\pi_{l+1} = \pi_l R \quad \text{for} \quad l \geq 0, \tag{2}$$

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where the matrix $R$ of order $m$ records the rate of visit to level $(l+1)$ per unit of time spent in level $l$. Fortunately, the elements of $R$ for homogeneous QBD MCs do not depend on level number. Quadratically convergent algorithms for solving QBD MCs appear in [8], [3], [1].

In this paper, we consider a special class of homogeneous continuous-time QBD MCs which possess, what we call, level-geometric (LG) stationary distribution. A level-geometric distribution is one that satisfies

$$\pi_{l+1} = \alpha \pi_l \quad \text{for} \quad l \geq L, \tag{3}$$

where $\alpha \in (0,1)$ and $L$ is a finite nonnegative integer. Note that an LG distribution with $L = 0$ is a product-form solution. An LG distribution can be expressed alternatively as

$$\pi_{L+k} = (1-\alpha)\alpha^k a \quad \text{for} \quad k \geq 0, \tag{4}$$

where $a$ is a positive probability vector of length $m$, with $ae = 1$ when $L = 0$. In an LG distribution, the level is independent of the phase for level numbers greater than or equal to $L$, and the marginal probability distribution of the levels are given by $\pi_{L+k}e = (1-\alpha)\alpha^k ae$ [9, pp. 295–299] for $k \geq 0$. Throughout the paper, we refer to an LG distribution for which $L$ is the smallest possible nonnegative integer that satisfies equation (3) as an LG distribution with parameter $L$. Our motivation is to come up with a solution method for this special class of QBD MCs that does not require $R$ to be computed. We remark that if $S_\epsilon$ is the number of iterations required to reach an accuracy of $\epsilon$ by the successive substitution algorithm [5, p. 160], then the computation of $R$ with quadratically convergent algorithms takes about $O(\log_2 S_\epsilon)$ iterations (hence, the term quadratically convergent) each of which has a time complexity of $O(m^3)$ floating-point operations. The results that we develop can be extended to the homogeneous discrete-time case without difficulty.

In section 2, we provide background information on the solution of QBD MCs with special structure. In section 3, we give three examples of QBD MCs with LG stationary distribution. In section 4, we specify conditions related to such a distribution and show how it can be computed when it exists. In section 5, we reconsider the three examples of section 3 under the light of the new results introduced in section 4. In section 6, we conclude.

**2. Background material.** In this section, an overview of some concepts discussed in [9] and other remarks are given. Wherever something has been taken from [9], the appropriate reference to the corresponding page(s) is placed.

Due to the fixed pattern of transitions among levels and within each level, it is not difficult to check the irreducibility of $Q$. The next remark is about checking the positive recurrence of $Q$ when $Q$ and $A = A_0 + A_1 + A_2$ are both irreducible. When $Q$ is irreducible, but $A$ has multiple irreducible classes, one can resort to the theorem in [9, p. 160]. Note that $A$ is an infinitesimal generator matrix.

REMARK 1. *If $Q$ and $A$ are irreducible, then $Q$ is positive recurrent if and only if $\pi_A(A_0 - A_2)e < 0$, where $\pi_A$ satisfies $\pi_A A = 0$ and $\pi_A e = 1$ [9, p. 158].*

Throughout the paper, we assume that the homogeneous continuous-time QBD MC at hand is irreducible and positive recurrent. Now, let $\rho(R)$ denote the spectral radius of $R$ (i.e., $\rho(R) = \max\{|\lambda| \mid \lambda \in \lambda(R)\}$, where $\lambda(R) = \{\lambda \mid Rv = \lambda v, v \neq 0\}$ is its spectrum). Then, $\rho(R) < 1$ [9, p. 133].

The next remark specifies necessary and sufficient conditions for the existence of an LG distribution with parameter $L = 0$.

REMARK 2. *The stationary distribution of $Q$ is LG with parameter $L = 0$ if and only if $a(A_0 + \alpha A_1 + \alpha^2 A_2) = 0$ and $a(B_0 + \alpha A_2) = 0$, where $\alpha = \rho(R)$ and $ae = 1$ [9, pp. 297–298].*

This remark, although very concise and to the point, has two shortcomings. First, it does not indicate how to check for an LG distribution with parameter $L \geq 1$. Second, it requires the solution of a nonlinear system of equations.

The following two remarks indicate the improvement that is obtained in the solution when $A_2$ and/or $A_0$ are rank-1 matrices.

REMARK 3. *When $A_2$ is of rank-1, $R = -A_0(A_1 + A_0 e b^T)^{-1}$, where $A_2 = cb^T$ and $b^T e = 1$ [9, p. 197]. Furthermore, $\pi_0$ can be computed up to a multiplicative constant using $\pi_0(B_0 + A_0 e b^T) = 0$ [9, p. 236].*

Hence, it is relatively simple to compute the stationary distribution when $A_2$ is of rank-1.

REMARK 4. *When $A_0$ is of rank-1, $R = c\xi^T$, where $A_0 = cb^T$, $b^T e = 1$, $\xi^T = -b^T(A_1 + \alpha A_2)^{-1}$, and $\alpha = \xi^T c$ with $\alpha = \rho(R)$ [9, p. 198]. The stationary subvectors satisfy $\pi_0 = \pi_1 C_0$, where $C_0 = -A_2 B_0^{-1}$, and $\pi_l = \pi_{l+1} C_1$ for $l \geq 1$, where $C_1 = -A_2(A_1 + A_2 e b^T)^{-1}$ [9, p. 236].*

COROLLARY 1. *When $A_0$ is of rank-1, $R$ is also of rank-1, and $R^2 = \alpha R$ thereby implying $\pi_{l+1} = \alpha \pi_l$ for $l \geq 1$. Hence, $Q$ has an LG distribution with parameter $L \leq 1$.*

The next section elaborates these results on three examples.

## 3. Examples.

The following examples all have LG distributions, and they aid in understanding the concepts introduced in section 2 and the concepts to be developed in section 4.

### 3.1. Example 1.

The first example we consider is a system of two independent queues, where queue 1 is M/M/1 and queue 2 is M/M/1/$m-1$. Queue $i \in \{1, 2\}$ has a Poisson arrival process with rate $\lambda_i$ and an exponential service distribution with rate $\mu_i$. This system corresponds to a QBD process with level representing the length of queue 1, which is unbounded, and phase representing the length of queue 2, which can range between 0 and $(m-1)$. We assume $\lambda_1 < \mu_1$. Letting $d = \lambda_1 + \lambda_2 + \mu_1 + \mu_2$, we have $A_0 = \lambda_1 I$, $A_2 = \mu_1 I$,

$$A_1 = \begin{pmatrix} -(d-\mu_2) & \lambda_2 & & & \\ \mu_2 & -d & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_2 & -d & \lambda_2 \\ & & & \mu_2 & -(d-\lambda_2) \end{pmatrix},$$

and

$$B_0 = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_2 & & & \\ \mu_2 & -(d-\mu_1) & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_2 & -(d-\mu_1) & \lambda_2 \\ & & & \mu_2 & -(\lambda_1+\mu_2) \end{pmatrix}.$$

$Q$ is irreducible, and from Remark 1 we have

$$A = A_0 + A_1 + A_2 = \begin{pmatrix} -\lambda_2 & \lambda_2 & & & \\ \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 \\ & & & \mu_2 & -\mu_2 \end{pmatrix},$$

which is irreducible, and $\pi_A$ is the truncated geometric distribution with parameter $\lambda_2/\mu_2$ [5, p. 84]. Hence, $\pi_A(A_0 - A_2)e = \lambda_1 - \mu_1 < 0$ and $Q$ is positive recurrent. For this example, $\alpha = \lambda_1/\mu_1$, $a_k = \nu^k(1-\nu)/(1-\nu^m)$, $0 \le k \le m-1$, and $L = 0$, where $\nu = \lambda_2/\mu_2$, turn out to be the parameters in equation (4) that specifies an LG distribution.

Note that the QBD MC in this example is lumpable [7, p. 124], and the lumped chain represents queue 1.

**3.2. Example 2.** The second example we consider is the continuous-time equivalent of the discrete-time QBD process discussed in [8, pp. 668–669]. The model has 2 phases at each level (i.e., $m = 2$). Assuming that $0 < p < 1$, the process moves from state $(l, 1)$, $l \ge 1$, to $(l, 2)$ with rate $p$, and to $(l-1, 1)$ with rate $(1-p)$. The process moves from state $(l, 2)$, $l \ge 0$, to $(l, 1)$ with rate $2p$, and to $(l+1, 2)$ with rate $(1-2p)$. Finally, the process moves from state $(0, 1)$ to $(0, 2)$ with rate 1. All diagonal elements of $Q$ are $-1$. Hence, we have

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1-2p \end{pmatrix}, \; A_1 = \begin{pmatrix} -1 & p \\ 2p & -1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 1-p & 0 \\ 0 & 0 \end{pmatrix}, \; B_0 = \begin{pmatrix} -1 & 1 \\ 2p & -1 \end{pmatrix}.$$

$Q$ is irreducible, and from Remark 1 we have

$$A = A_0 + A_1 + A_2 = \begin{pmatrix} -p & p \\ 2p & -2p \end{pmatrix},$$

which is irreducible, and $\pi_A = (2/3 \;\; 1/3)$. Hence, $\pi_A(A_0 - A_2)e = -1/3 < 0$ and $Q$ is positive recurrent. For this example, $\alpha = (1-2p)/(1-p)$, $a = (1/2 \;\; 1/2)$, and $L = 0$ turn out to be the parameters in equation (4) that specifies an LG distribution. Direct substitution in $\pi Q = 0$ and $\pi e = 1$ confirms this solution.

In this example, Remark 3 applies with $c = (1-p)e_1$ and $b = e_1$, where $e_i$ is the $i$th principal axis vector. Hence, $R = (1-2p)e_2^T e/(1-p)$, and $\rho(R) = \alpha$ as expected. Furthermore, $\pi_0 = (1-\alpha)(1/2 \;\; 1/2)$. Note that in this example, Remark 4 applies as well. The rate matrix is of rank-1 and $\xi = e/(1-p)$. In section 5, we will argue why this example has an LG distribution with parameter $L = 0$, and not $L = 1$. Finally, we remark that this example is also used as a test case in [1].

**3.3. Example 3.** The third example we consider is the $\mathrm{E}_m/\mathrm{M}/1$ FCFS queue which has an exponential service distribution with rate $\mu$ and an $m$-phase Erlang arrival process with rate $m\lambda$ in each phase [9, pp. 206–208]. The expected interarrival time and the expected service time of this queue are respectively $1/\lambda$ and $1/\mu$. We assume $\lambda < \mu$. The queue corresponds to a QBD process with level representing the queue length (including any in service) and phase representing the state of the Erlang arrival process. Letting $d = m\lambda + \mu$, we have the $(m \times m)$ matrices $A_0 = m\lambda e_m e_1^T$, $A_2 = \mu I$,

$$A_1 = \begin{pmatrix} -d & m\lambda & & \\ & \ddots & \ddots & \\ & & -d & m\lambda \\ & & & -d \end{pmatrix}, \quad B_0 = \begin{pmatrix} -m\lambda & m\lambda & & \\ & \ddots & \ddots & \\ & & -m\lambda & m\lambda \\ & & & -m\lambda \end{pmatrix}.$$

$Q$ is irreducible, and from Remark 1 we have

$$A = A_0 + A_1 + A_2 = \begin{pmatrix} -m\lambda & m\lambda & & \\ & \ddots & \ddots & \\ & & -m\lambda & m\lambda \\ m\lambda & & & -m\lambda \end{pmatrix},$$

which is irreducible, and $\pi_A = e^T/m$. Hence, $\pi_A(A_0 - A_2)e = \lambda - \mu < 0$ and $Q$ is positive recurrent. Although, the $\mathrm{E}_m/\mathrm{M}/1$ FCFS queue does not have an explicit solution, it can be shown by following the formulae in [6, p. 323] that its stationary distribution has an LG distribution with parameter $L = 1$.

In this example, Remark 4 applies with $c = m\lambda e_m$ and $b = e_1$, implying $R$ is of rank-1, $C_0 = -A_2 B_0^{-1}$, and $C_1 = -A_2(A_1 + \mu e e_1^T)^{-1}$.

The next section builds on the results in section 2 with the aim of coming up with a solution method to compute an LG distribution when it exists.

**4. Checking for and computing the LG distribution.** The negated infinitesimal generator $-Q$ is known to be a singular M-matrix (see [2]). From the values in $B_0$ and our assumption of irreducibility for $Q$, it follows that $-B_0$ is a nonsingular M-matrix and $B_0^{-1}$ exists [10, p. 626]. The next remark is essential in formulating the results in this section.

REMARK 5. *If $D_l^T$, $l \geq 0$, denotes the diagonal block at level $l$ after $l$ steps of block Gaussian-elimination (GE) on $Q^T$, then $D_0 = B_0$ and $D_{l+1} = A_1 - A_2 D_l^{-1} A_0$ for $l \geq 0$ since $-D_l$ is a nonsingular M-matrix, therefore invertible and $-D_l^{-1} \geq 0$. Furthermore, $\pi_l = \pi_{l+1} C_l$, where $C_l = -A_2 D_l^{-1} \geq 0$, for $l \geq 0$.*

In other words, since $Q^T$ is a block tridiagonal matrix, block GE on $Q^T \pi^T = 0$ yields $Z^T \pi^T = 0$ (or equivalently $\pi Z = 0$), where

$$Z = \begin{pmatrix} D_0 & & & \\ A_2 & D_1 & & \\ & A_2 & D_2 & \\ & & \ddots & \ddots \end{pmatrix}. \tag{5}$$

That $C_l \geq 0$ for $l \geq 0$ follows from $-D_l^{-1} \geq 0$ and $A_2 \geq 0$.

**4.1. Checking for the LG distribution.** The form of $Z$ in equation (5) together with Remark 5 suggests the next lemma.

LEMMA 1. *If $D_{L+1} = D_L$ for some finite nonnegative integer $L$, then $D_l = D_L$ for $l > L+1$, and $\pi_L = \pi_{L+k} C_L^k$ for $k \geq 0$.*

*Proof.* From Remark 5 we have $D_{L+1} = A_1 - A_2 D_L^{-1} A_0$ and $D_{L+2} = A_1 - A_2 D_{L+1}^{-1} A_0$. If $D_{L+1} = D_L$, then $D_{L+2} = A_1 - A_2 D_L^{-1} A_0 = D_{L+1} = D_L$. The same argument may be used to show that $D_l = D_L$ for $l > L+2$. The second part of the lemma follows from its first part and the second part of Remark 5. □

The next theorem states a condition under which one has an LG distribution.

THEOREM 1. *Let $L$ be the smallest finite nonnegative integer for which $D_{L+1} = D_L$. Then the stationary distribution of $Q$ is LG with parameter less than or equal to $L$.*

*Proof.* From Lemma 1 and equation (5), when $D_{L+1} = D_L$, we have

$$Z = \begin{pmatrix} D_0 & & & & \\ A_2 & D_1 & & & \\ & \ddots & \ddots & & \\ & & A_2 & D_{L-1} & \\ & & & Y_L & Z_L \end{pmatrix}, \tag{6}$$

where

$$Y_L = \begin{pmatrix} A_2 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad \text{and} \quad Z_L = \begin{pmatrix} D_L & & & \\ A_2 & D_L & & \\ & A_2 & D_L & \\ & & \ddots & \ddots \end{pmatrix}.$$

Since $Z$ is a singular matrix and $\pi_l$ of length $m$ is positive for finite $l$ and unique up to a multiplicative constant with $\lim_{l\to\infty} \pi_l = 0$, the identities $(\pi_L, \pi_{L+1}, \ldots) Z_L = 0$ and $(\pi_{L+1}, \pi_{L+2}, \ldots) Z_L = 0$ obtained from equations $\pi Z = 0$ and (6) suggest that $\pi_{l+1} = \alpha \pi_l$ for $l \geq L$, where $\alpha \in (0,1)$. □

COROLLARY 2. *When $B_0 = A_1 - A_2 B_0^{-1} A_0$, the stationary distribution of $Q$ is LG with parameter $L = 0$*

The next remark helps us to state two lemmas which will be used in checking for an LG distribution.

REMARK 6. *The inverse of an irreducible nonsingular matrix having a nonzero diagonal is full unless numerical coincidence occurs [4, p. 72].*

LEMMA 2. *If $A_1$ is irreducible and $A_2 e > 0$, then $D_l$ is irreducible and $C_l > 0$ for $l \geq 1$.*

*Proof.* From Remark 5 we have $D_{l+1} = A_1 + C_l A_0$, where $C_l = -A_2 D_l^{-1} \geq 0$ and $l \geq 0$. Since $A_0 \geq 0$ by definition, we obtain $C_l A_0 \geq 0$. Besides, $A_1$ has nonnegative off-diagonal elements and is assumed to be irreducible. Hence, its sum with the nonnegative $C_l A_0$ will not change the irreducibility thereby implying irreducible $D_{l+1}$ for $l \geq 0$. Alternatively, $D_l$, $l \geq 1$, is irreducible. On the other hand, $-D_l$ is a nonsingular M-matrix from Remark 5, and therefore has a nonzero diagonal and a nonnegative inverse. When $D_l$ is irreducible, its inverse is full due to Remark 6 implying $-D_l^{-1} > 0$. Since $A_2 \geq 0$ and is assumed to have a nonzero in each row, its product with $-D_l^{-1}$ is positive. Hence, $C_l > 0$ for $l \geq 1$. □

LEMMA 3. *If $e^T A_0 > 0$, $A_2 e > 0$, and $D_L$ is irreducible for some finite nonnegative integer $L$, then $D_l$ is irreducible and $C_l > 0$ for $l \geq L$.*

*Proof.* When $D_L$ is irreducible and $A_2$ has a nonzero in each row, we have $C_L > 0$ as in the proof of Lemma 1. Since $A_0 \geq 0$ and is assumed to have a nonzero in each column, we have $C_L A_0 > 0$ thereby implying an irreducible $D_{L+1}$. The same circle of arguments may be used to show that $C_l > 0$ and $D_{l+1}$ is irreducible for $l > L$. □

The next theorem states another condition under which one has an LG distribution.

THEOREM 2. *Let $L$ be the smallest finite nonnegative integer for which $C_l$ is irreducible and $\rho(C_l) = \rho(C_{l+1})$, where $l \geq L$. Then the stationary distribution of $Q$ is LG with parameter $L$.*

*Proof.* From Remark 5 we have $C_l \geq 0$ for $l \geq 0$. If $C_l$, $l \geq L$, is irreducible, then by the Perron-Frobenius theorem $C_l$ has $\rho(C_l) > 0$ as a simple eigenvalue and a corresponding positive left-hand eigenvector. There are no other linearly independent positive left-hand eigenvectors of $C_l$ [10, p. 673]. From Remark 5 we also have $\pi_l = \pi_{l+1} C_l$ and $\pi_l > 0$ with $\lim_{l \to \infty} \pi_l = 0$. Multiplying both sides of $\pi_l = \pi_{l+1} C_l$ by $\rho(C_l)$, we obtain $\rho(C_l)\pi_l = (\rho(C_l)\pi_{l+1})C_l$. Since $\rho(C_l)$ is a simple eigenvalue of $C_l$ for $l \geq L$, we must have $\pi_l$ as its corresponding positive left-hand eigenvector. Therefore, it must also be that $\pi_l = \rho(C_l)\pi_{l+1}$ for $l \geq L$. Since $\rho(C_l) = \rho(C_{l+1})$ for $l \geq L$, we have $\pi_l = \rho(C_L)\pi_{l+1}$, or $\pi_{l+1} = (1/\rho(C_L))\pi_l$ for $l \geq L$. Consequently $Q$ has an LG distribution with parameter $L$. □

**4.2. Computing the LG distribution.** The next theorem gives the value of $\alpha$ in equation (3) and indicates how $\pi_L$ can be computed up to a multiplicative constant when one has an LG distribution with parameter $L$.

THEOREM 3. *If the stationary distribution of $Q$ is LG with parameter $L$, then $\rho(C_L)\pi_L = \pi_L C_L$, where $\alpha = 1/\rho(C_L)$ and $\pi_L > 0$.*

*Proof.* Since $Q$ has an LG distribution with parameter $L$, from equation (3) we have $\pi_{L+1} = \alpha \pi_L$, where $\alpha \in (0,1)$, $\pi_L > 0$ and $\pi_{L+1} > 0$ with $\lim_{l \to \infty} \pi_l = 0$. That is, for finite $L$, $\pi_{L+1}$ is a positive multiple of $\pi_L$. Furthermore, from Remark 5 we have $\pi_L = \pi_{L+1} C_L$, where $C_L \geq 0$. Since $\pi_{L+1}$ is a positive multiple of $\pi_L$, $\pi_L$ is clearly a positive left-hand eigenvector of $C_L$, and therefore corresponds to the eigenvalue $\rho(C_L)$ [2, p. 28]. Combining the two statements, we obtain $\rho(C_L)\pi_L = \pi_L C_L$, where $\alpha = 1/\rho(C_L)$ and $\pi_L > 0$. □

COROLLARY 3. *When the stationary distribution of $Q$ is LG with parameter less than or equal to $L$, where $L > 0$, if $\rho(C_L) \neq \rho(C_{L-1})$, then the parameter is $L$, else the parameter is less than or equal to $L - 1$.*

**5. Examples revisited.** In this section, we demonstrate the results of the previous section using the three examples introduced in section 3.

**5.1. Example 1.** For the first example in section 2, $D_l^{-1}$, $l \geq 0$, is a full matrix, and we have experimentally shown that $D_{l+1} = D_l$ as $l$ approaches infinity. For the particular case of $m = 2$, we have

$$B_0^{-1} = \frac{-1}{\lambda_1(d - \mu_1)} \begin{pmatrix} \lambda_1 + \mu_2 & \lambda_2 \\ \mu_2 & \lambda_1 + \lambda_2 \end{pmatrix} \quad \text{and} \quad C_0 = -A_2 B_0^{-1} = -\mu_1 B_0^{-1},$$

where $d = \lambda_1 + \lambda_2 + \mu_1 + \mu_2$. The correction to $A_1$ is given by $C_0 A_0 = -\lambda_1 \mu_1 B_0^{-1}$, and therefore

$$D_1 = A_1 + C_0 A_0 = \begin{pmatrix} -(d-\mu_2) + \frac{\mu_1(\lambda_1+\mu_2)}{d-\mu_1} & \lambda_2 + \frac{\lambda_2\mu_1}{d-\mu_1} \\ \mu_2 + \frac{\mu_1\mu_2}{d-\mu_1} & -(d-\lambda_2) + \frac{\mu_1(\lambda_1+\lambda_2)}{d-\mu_1} \end{pmatrix} \neq B_0.$$

In a similar manner one can show that $D_{l+1} \neq D_l$ for finite values of $l$. Hence, Theorem 1 does not apply. However, Lemma 3 applies since $A_0$ and $A_2$ are of full-rank and $D_0$ is irreducible, implying irreducible $C_l$ for $l \geq 0$. Consequently, there is reason to guess that the QBD MC has an LG distribution with parameter $L = 0$ from Theorem 2 and to compute the eigenvalue-eigenvector pair $(\rho(C_0),\pi_0)$ using Theorem 3. Then the guessed solution can be verified in $\pi Q = 0$. Although, this approach will sometimes fail, it works in Example 1, and can be recommended for small values of $L$.

For $m = 2$, it is not difficult to find using Theorem 3 that $\rho(C_0) = \mu_1/\lambda_1 > 1$ implying $\alpha = \lambda_1/\mu_1$, and

$$\pi_0 = (1-\alpha)\begin{pmatrix} \frac{1-\nu}{1-\nu^2} & \frac{\nu(1-\nu)}{1-\nu^2} \end{pmatrix},$$

where $\nu = \lambda_2/\mu_2$.

**5.2. Example 2.** Consider the second example in section 2 for which we have

$$B_0^{-1} = \frac{-1}{1-2p}\begin{pmatrix} 1 & 1 \\ 2p & 1 \end{pmatrix} \quad \text{and} \quad C_0 = -A_2 B_0^{-1} = \frac{1-p}{1-2p}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that $C_0$ is reducible. The correction to $A_1$ is given by $C_0 A_0 = (1-p)e_1 e_2^T$, and therefore

$$D_1 = A_1 + C_0 A_0 = \begin{pmatrix} -1 & 1 \\ 2p & -1 \end{pmatrix} = B_0.$$

Hence, in this example, $D_l = D_0$ for $l \geq 1$ from Lemma 1 due to $D_1 = D_0$. From Corollary 2 we conclude Example 1 has an LG distribution with parameter $L = 0$.

Finally, from Theorem 3 we obtain $\rho(C_0) = (1-p)/(1-2p) > 1$ implying $\alpha = (1-2p)/(1-p)$, and $\pi_0 = (1-\alpha)(1/2 \ 1/2)$.

**5.3. Example 3.** Now, consider the third example in section 3 for which we have

$$B_0^{-1} = \frac{-1}{m\lambda}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & 1 \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \quad \text{and} \quad C_0 = -A_2 B_0^{-1} = \frac{\mu}{m\lambda}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & 1 \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}.$$

Note that $C_0$ is reducible and $\rho(C_0) = \mu/(m\lambda)$, which is not necessarily greater than 1. The correction to $A_1$ is given by $C_0 A_0 = \mu e e_1^T$, and therefore

$$D_1 = \begin{pmatrix} -m\lambda & m\lambda & & & \\ \mu & -(m\lambda+\mu) & m\lambda & & \\ \vdots & & \ddots & \ddots & \\ \mu & & & -(m\lambda+\mu) & m\lambda \\ \mu & & & & -(m\lambda+\mu) \end{pmatrix} \neq B_0.$$

Noticing that $D_1 = A_1 + \mu e e_1^T$ in which the correction $\mu e e_1^T$ is of rank-1, the Sherman-Morrison formula [10, p. 124] yields

$$D_1^{-1} = A_1^{-1} - \mu \frac{A_1^{-1} e e_1^T A_1^{-1}}{1 + \mu e_1^T A_1^{-1} e}.$$

Letting $\gamma = m\lambda/(m\lambda + \mu)$, we obtain

$$A_1^{-1} = \frac{-1}{m\lambda + \mu} \begin{pmatrix} 1 & \gamma & \gamma^2 & \cdots & \gamma^{m-1} \\ & 1 & \gamma & \cdots & \gamma^{m-2} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & \gamma \\ & & & & 1 \end{pmatrix}, \quad (1 + \mu e_1^T A_1^{-1} e) = \gamma^m,$$

$$\mu (A_1^{-1} e)(e_1^T A_1^{-1}) = \frac{1}{m\lambda + \mu} \begin{pmatrix} 1-\gamma^m & \gamma(1-\gamma^m) & \cdots & \gamma^{m-1}(1-\gamma^m) \\ 1-\gamma^{m-1} & \gamma(1-\gamma^{m-1}) & \cdots & \gamma^{m-1}(1-\gamma^{m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ 1-\gamma & \gamma(1-\gamma) & \cdots & \gamma^{m-1}(1-\gamma) \end{pmatrix}$$

and after some algebra, $C_1 A_0 = \mu e e_1^T$. Hence, $D_2 = A_1 + C_1 A_0 = D_1$ implying $D_l = D_1$ for $l \geq 2$ from Lemma 1. From Theorem 1 we have an LG distribution with parameter $L \leq 1$. We also remark that the two matrices $C_0$ and $C_1$ introduced in Remark 4 for QBD processes with rank-1 $A_0$ matrices are given in this example as $C_0 = -\mu D_0^{-1}$ and $C_1 = -\mu D_1^{-1}$. Since $\rho(C_0)$ may be less than 1 and therefore different than $\rho(C_1)$, from Corollary 3 we conclude Example 2 has an LG distribution with parameter $L = 1$.

Regarding the computation of $\alpha$, for instance, when $m = 2$

$$C_0 = \eta \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad C_1 = \eta \begin{pmatrix} 1+\eta & 1 \\ \eta & 1 \end{pmatrix},$$

where $\eta = \mu/(2\lambda)$. Hence, we have

$$\rho(C_1) = \eta \left( 1 + \frac{1}{2}\eta + \sqrt{\eta(1 + \frac{1}{4}\eta)} \right).$$

Note that $\rho(C_0) \neq \rho(C_1)$. Now, using $\rho(C_1)\pi_1 = \pi_1 C_1$, $\pi_0 = \pi_1 C_0$, and $\pi_1 e/(1-\alpha) + \pi_0 e = 1$, where $\alpha = 1/\rho(C_1)$, we obtain

$$\pi_1 = \left( \frac{(\rho(C_1) - \eta)(\rho(C_1) - 1)}{\rho^2(C_1) + \eta(\rho(C_1) - 1)(2\rho(C_1) - \eta)} \;\; \frac{\eta(\rho(C_1) - 1)}{\rho^2(C_1) + \eta(\rho(C_1) - 1)(2\rho(C_1) - \eta)} \right)$$

and

$$\pi_0 = \left( \frac{\eta(\rho(C_1) - \eta)(\rho(C_1) - 1)}{\rho^2(C_1) + \eta(\rho(C_1) - 1)(2\rho(C_1) - \eta)} \;\; \frac{\eta\rho(C_1)(\rho(C_1) - 1)}{\rho^2(C_1) + \eta(\rho(C_1) - 1)(2\rho(C_1) - \eta)} \right).$$

Normally the computation would be performed numerically for the given parameters of the problem. For $m \geq 3$, we would first compute $C_0$ and $C_1$. Then we would obtain the eigenvalue-eigenvector pair $(\rho(C_1), \pi_1)$ from $\rho(C_1)\pi_1 = \pi_1 C_1$ (see Theorem 3). Next we would compute $\pi_0 = \pi_1 C_0$. Finally we would normalize $\pi_0$ and $\pi_1$ with $\pi_1 e/(1-\alpha) + \pi_0 e$.

**6. Conclusion.** This paper introduces necessary and sufficient conditions for a homogeneous continuous-time quasi-birth-and-death (QBD) Markov Chain (MC) to possess level-geometric (LG) stationary distribution. Furthermore, it discusses how an LG distribution can be computed when it exists. Results that utilize the matrices $A_0$, $A_1$, $A_2$, and $B_0$ are given showing how one can easily check for and compute an LG distribution with parameter $L \leq 1$. The results are elaborated on three examples. Examples 2 and 3, which have been used in the literature as test cases, are shown to possess LG distributions with parameter $L = 0$. Since the matrices $A_0$, $A_1$, $A_2$, and $B_0$ that arise in applications are usually sparse, the results developed in this paper may be used before resorting to quadratically convergent algorithms to compute the rate matrix, $R$.

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