Lecture 6-a
Analysis of Quicksort

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Analysis of Quicksort

Assume \textit{all elements are distinct} in the following analysis
Question

Q: Remember that H-PARTITION always chooses A[p] (the first element) as the pivot. What is the runtime of QUICKSORT on an already-sorted array?

a) \( \Theta(n) \)       ✔ c) \( \Theta(n^2) \)

b) \( \Theta(n \log n) \)   ✗ d) cannot provide a tight bound

QUICKSORT (A, p, r)

if \( p < r \) then

\[ q \leftarrow \text{H-PARTITION}(A, p, r) \]

QUICKSORT(A, p, q)

QUICKSORT(A, q +1, r)
Example: An Already Sorted Array

Partitioning always leads to 2 parts of size 1 and \( n-1 \)
Worst Case Analysis of Quicksort

- **Worst case** is when the **PARTITION** algorithm always returns imbalanced partitions (of size 1 and n-1) in every recursive call.
  - This happens when the pivot is selected to be either the min or max element.
  - This happens for **H-PARTITION** when the input array is already sorted or reverse sorted.

\[
T(n) = T(1) + T(n-1) + \Theta(n) \\
= T(n-1) + \Theta(n) \\
= \Theta(n^2) \quad (\text{arithmetic series})
\]
Worst Case Recursion Tree

\[ T(n) = T(1) + T(n-1) + cn \]
Worst Case Recursion Tree

\[ T(n) = T(1) + T(n-1) + cn \]

\[ \sum_{k=1}^{n} ck = \Theta(n^2) \]

\[ T(n) = \Theta(n^2) + \Theta(n) \]

\[ T(n) = \Theta(n^2) \]
Best Case Analysis (for intuition only)

- If we’re *extremely lucky*, H-PARTITION splits the array *evenly* at *every* recursive call

\[ T(n) = 2 \cdot T(n/2) + \Theta(n) = \Theta(n \log n) \]

- Instead of splitting 0.5:0.5, what if every split is 0.1:0.9?

\[ T(n) = T(n/10) + T(9n/10) + \Theta(n) \]

⇒ solve this recurrence
“Almost-Best” Case Analysis

\[ \sum_{i=0}^{\infty} \frac{n^i}{100^i} = \Theta(1) \]
“Almost-Best” Case Analysis

\[ \Theta(1) \rightarrow \frac{n}{100} \]

\[ \frac{n}{10} \rightarrow \frac{9n}{100} \]

\[ \frac{n}{10} \rightarrow \frac{9n}{100} \]

\[ \frac{9n}{10} \rightarrow \frac{81n}{100} \]

\[ \leq cn \rightarrow \leq cn \]

\[ \Theta(1) \rightarrow \leq cn \]
“Almost-Best” Case Analysis

\[ T(n) = \Theta(n \log_{10} n) \]

\[ h_{\min} = \log_{10} n \]
\[ h_{\max} = \log_{10/9} n \]

\[ cn h_{\min} \leq T(n) \leq cn h_{\max} \]
\[ cn \log_{10} n \leq T(n) \leq cn \log_{10/9} n \]

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Balanced Partitioning

- We have seen that if H-PARTITION always splits the array with 0.1-to-0.9 ratio, the runtime will be $\Theta(n \log n)$.
- Same is true with a split ratio of 0.01-to-0.99, etc.

- Possible to show that if the split has always constant ($\Theta(1)$) proportionality, then the runtime will be $\Theta(n \log n)$.

- In other words, for a constant $\alpha$ ($0 < \alpha \leq 0.5$):
  
  $\alpha$–to–$(1-\alpha)$ proportional split yields $\Theta(n \log n)$ total runtime
Balanced Partitioning

- In the rest of the analysis, assume that all input permutations are equally likely.
  - This is only to gain some intuition
  - We cannot make this assumption for average case analysis
  - We will revisit this assumption later
- Also, assume that all input elements are distinct.

- What is the probability that H-PARTITION returns a split that is more balanced than 0.1-to-0.9?
Balanced Partitioning

**Reminder**: *H-PARTITION* will place the pivot in the right partition unless the pivot is the smallest element in the arrays.

**Question**: If the pivot selected is the $m^{\text{th}}$ smallest value ($1 < m \leq n$) in the input array, what is the size of the left region after partitioning?

$1 \quad q \quad n$

- there are $m-1$ elements less than the pivot
- pivot is placed in the right region

$q = m - 1$
Balanced Partitioning

**Question**: What is the probability that the pivot selected is the \(m^{th}\) smallest value in the array of size \(n\)?

\[
\frac{1}{n} \text{ (since all input permutations are equally likely)}
\]

**Question**: What is the probability that the left partition returned by \(H\)-PARTITION has size \(m\), where \(1 < m < n\)?

\[
\frac{1}{n} \text{ (due to the answers to the previous 2 questions)}
\]
Balanced Partitioning

**Question**: What is the probability that H-PARTITION returns a split that is more balanced than 0.1-to-0.9?

The partition boundary will be in this region for a more balanced split than 0.1-to-0.9.

\[
\text{Probability} = \sum_{q=0.1n+1}^{0.9n-1} \frac{1}{n} = \frac{1}{n} \left( 0.9n - 1 - 0.1n - 1 + 1 \right) = 0.8 \frac{1}{n}
\]

\[\approx 0.8 \text{ for large } n\]
Balanced Partitioning

- The probability that $H$-PARTITION yields a split that is more balanced than 0.1-to-0.9 is 80% on a random array.

- Let $P_\alpha >$ be the probability that $H$-PARTITION yields a split more balanced than $\alpha$-to-$(1-\alpha)$, where $0 < \alpha \leq 0.5$

- Repeat the analysis to generalize the previous result
Balanced Partitioning

**Question**: What is the probability that H-PARTITION returns a split that is more balanced than $\alpha$-to-$\alpha$?

The partition boundary will be in this region for a more balanced split than $\alpha n$-to-$(1-\alpha)n$.

![Diagram showing partition boundaries]

The probability can be calculated as:

$$\text{Probability} = \sum_{q=\alpha n+1}^{(1-\alpha)n-1} \frac{1}{n} = \frac{1}{n} ((1-\alpha)n - 1 - \alpha n - 1 + 1) = (1 - 2\alpha) - \frac{1}{n}$$

$$\approx (1 - 2\alpha) \text{ for large } n$$
Balanced Partitioning

- We found $P_{\alpha} = 1 - 2\alpha$

  Examples: $P_{0.1} = 0.8$ \hspace{1cm} $P_{0.01} = 0.98$

- Hence, *H-PARTITION* produces a split
  - more balanced than a
    - 0.1-to-0.9 split 80% of the time
    - 0.01-to-0.99 split 98% of the time
  - less balanced than a
    - 0.1-to-0.9 split 20% of the time
    - 0.01-to-0.99 split 2% of the time
Intuition for the Average Case

- **Assumption**: All permutations are equally likely
  - Only for intuition; we’ll revisit this assumption later

- **Unlikely**: Splits always the same way at every level

- **Expectation**:
  - Some splits will be *reasonably balanced*
  - Some splits will be *fairly unbalanced*

- **Average case**: A mix of good and bad splits
  - *Good* and *bad* splits distributed randomly thru the tree
Intuition for the Average Case

- Assume for intuition: Good and bad splits occur in the alternate levels of the tree
  
  *Good split*: Best case split
  
  *Bad split*: Worst case split
Intuition for the Average Case

Compare 2-successive levels of avg case vs. 1 level of best case
In terms of the remaining subproblems, two levels of avg case is slightly better than the single level of the best case.

The avg case has extra divide cost of $\Theta(n)$ at alternate levels.
Intuition for the Average Case

- The extra divide cost $\Theta(n)$ of bad splits absorbed into the $\Theta(n)$ of good splits.
- Running time is still $\Theta(n \log n)$
Intuition for the Average Case

- Running time is still $\Theta(n \log n)$
  - But, slightly larger hidden constants, because the height of the recursion tree is about twice of that of best case.
Intuition for the Average Case

- Another way of looking at it:
  - Suppose we alternate lucky, unlucky, lucky, unlucky, …
  - We can write the recurrence as:
    \[ L(n) = 2 U(n/2) + \Theta(n) \quad \text{lucky split (best)} \]
    \[ U(n) = L(n-1) + \Theta(n) \quad \text{unlucky split (worst)} \]
  - Solving:
    \[ L(n) = 2 (L(n/2-1) + \Theta(n/2)) + \Theta(n) \]
    \[ = 2L(n/2-1) + \Theta(n) \]
    \[ = \Theta(n \log n) \]

How can we make sure we are usually lucky for all inputs?
Summary: Quicksort Runtime Analysis

Worst case: Unbalanced split at every recursive call

\[ T(n) = T(1) + T(n-1) + \Theta(n) \]
\[ \Rightarrow T(n) = \Theta(n^2) \]

Best case: Balanced split at every recursive call (extremely lucky)

\[ T(n) = 2T(n/2) + \Theta(n) \]
\[ \Rightarrow T(n) = \Theta(n \log n) \]
**Summary: Quicksort Runtime Analysis**

**Almost-best case:** Almost-balanced split at every recursive call

\[ T(n) = T(n/10) + T(9n/10) + \Theta(n) \]
\[ \text{or} \quad T(n) = T(n/100) + T(99n/100) + \Theta(n) \]
\[ \text{or} \quad T(n) = T(\alpha n) + T((1-\alpha)n) + \Theta(n) \]

for any constant \( \alpha, 0 < \alpha \leq 0.5 \)

\[ T(n) = \Theta(n \log n) \]
Summary: Quicksort Runtime Analysis

For a random input array, the probability of having a split

more balanced than $0.1 - 0.9 : 80\%$
more balanced than $0.01 - 0.99 : 98\%$
more balanced than $\alpha - (1-\alpha) : 1 - 2\alpha$

for any constant $\alpha, 0 < \alpha \leq 0.5$
Summary: Quicksort Runtime Analysis

**Avg case intuition**: Different splits expected at different levels

- some balanced (good), some unbalanced (bad)

**Avg case intuition**: Assume the good and bad splits alternate

- i.e. good split → bad split → good split → ...

- \( T(n) = \Theta(n\log n) \)

*(informal analysis for intuition)*