CS473-Algorithms I

Lecture 2

Asymptotic Notation

O-notation (upper bounds)

- $f(n) = O(g(n))$ if $\exists$ positive constants c, n_0 such that

$$0 \le f(n) \le cg(n), \ \forall n \ge n_0$$

e.g., $2n^2 = O(n^3)$

$$2n^2 \le cn^3 \Rightarrow cn \ge 2 \Rightarrow c = 1 \ \& \ n_0 = 2$$

or

$$c = 2 \ \& \ n_0 = 1$$

Asymptotic running times of algorithms are usually defined by functions whose domain are N={0, 1, 2, …} (natural numbers)

O-notation (upper bounds)

- “=” is funny; “one-way” equality
- O-notation is sloppy, but convenient
- though sloppy, must understand what really means
- think of O(g(n)) as a set of functions:

$$O(g(n)) = \{f(n): \exists \text{ positive constants } c, n_0 \text{ such that } 0 \le f(n) \le cg(n), \forall n \ge n_0\}$$

hence, $2n^2 = O(n^3)$ means that $2n^2 \in O(n^3)$

O-notation

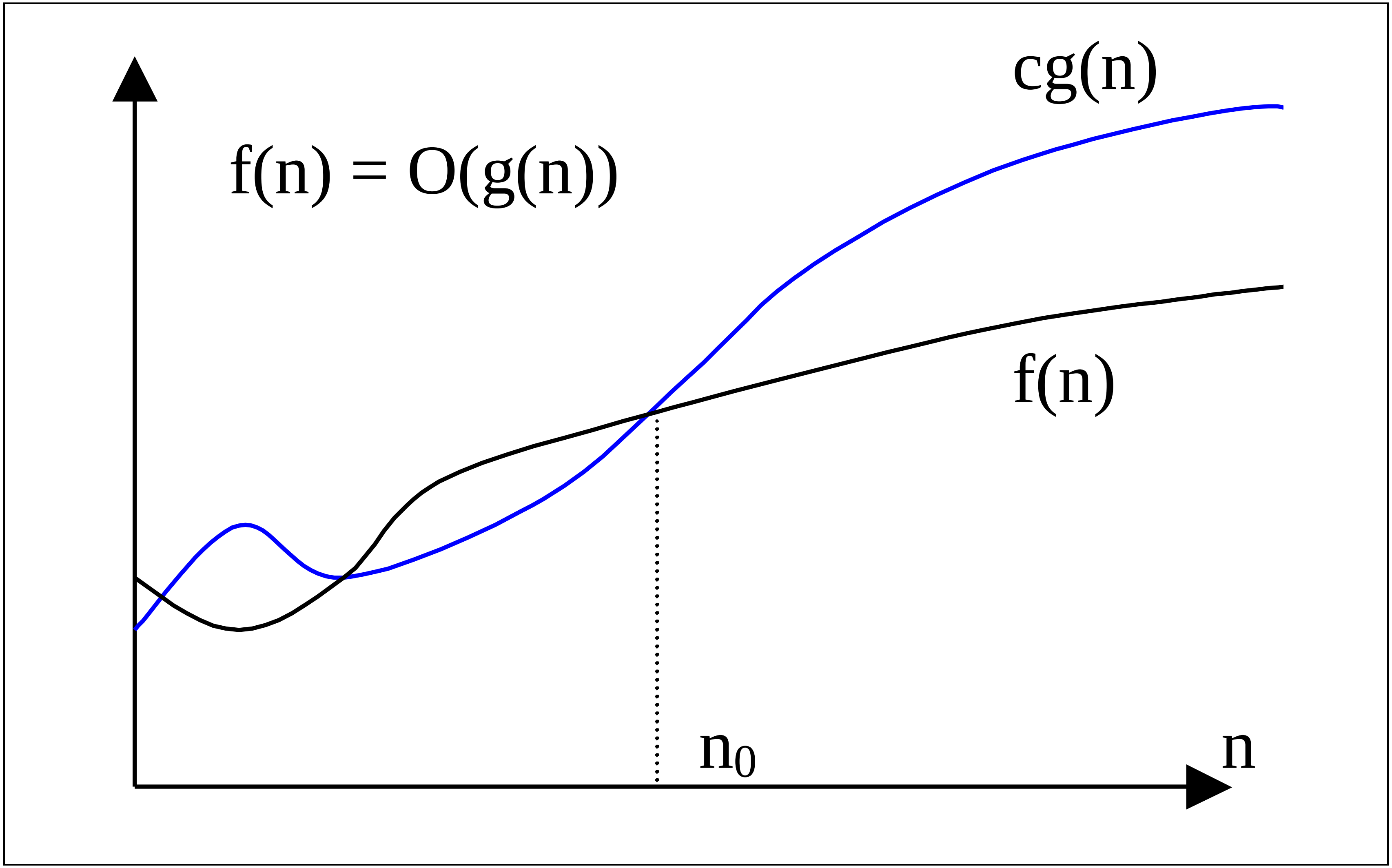
- O-notation is an upper-bound notation
- e.g., makes no sense to say "running time of an algorithm is at least $O(n^2)$". Why?
  - let running time be $T(n)$
  - $T(n) \geq O(n^2)$ means

    $T(n) \geq h(n)$ for some $h(n) \in O(n^2)$
  - however, this is true for any $T(n)$ since

    $h(n) = 0 \in O(n^2)$, & running time > 0,

    so stmt tells nothing about running time

O-notation (upper bounds)

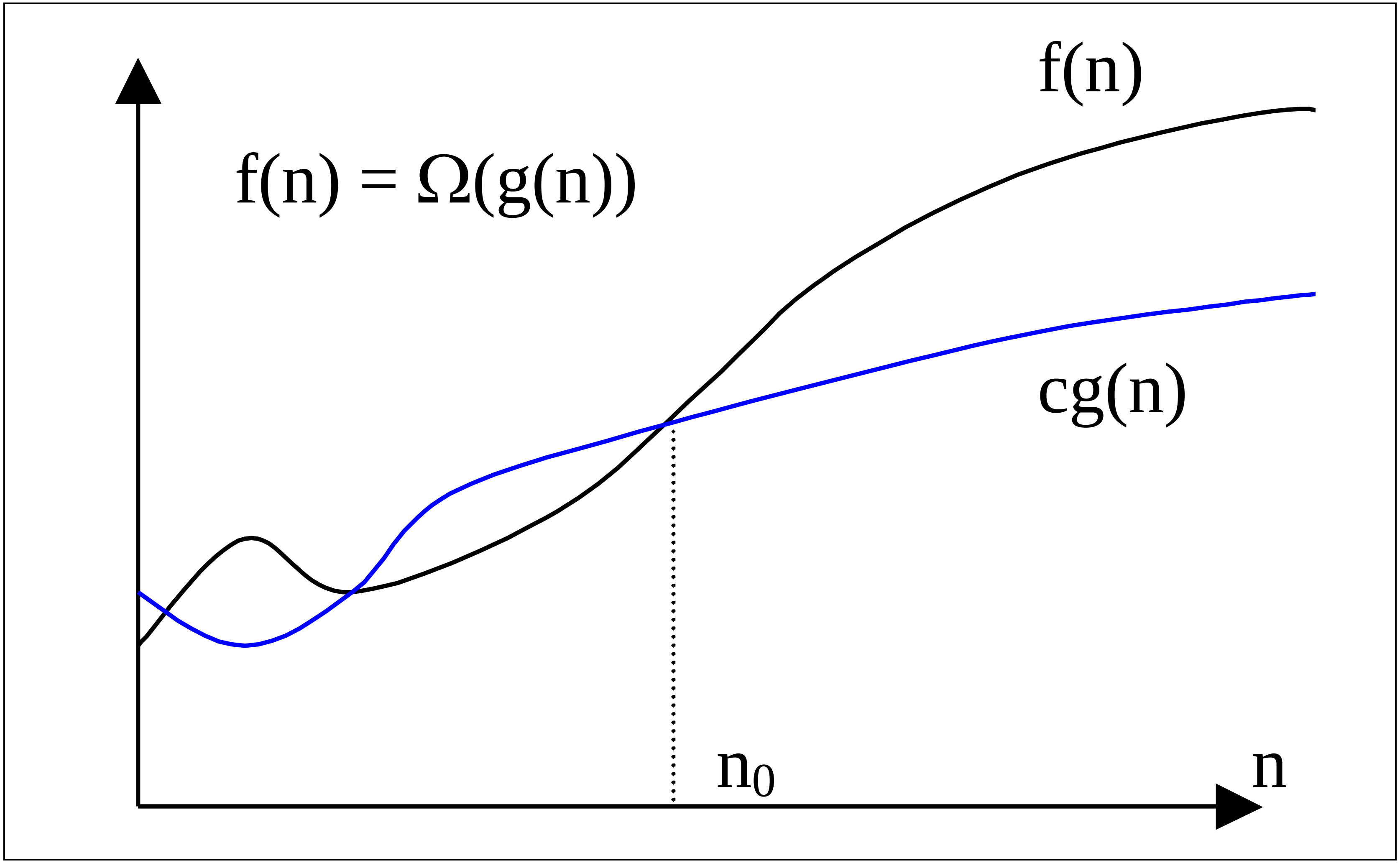


Ω-notation (lower bounds)

- $f(n) = \Omega(g(n))$ if $\exists$ positive constants c, n_0 such that
  $$0 \le cg(n) \le f(n), \forall n \ge n_0$$
  e.g., $\sqrt{n} = \Omega(\lg n)$ $(c = 1, n_0 = 16)$
  i.e., $1 \times \lg n \le \sqrt{n}$ $\forall n \ge 16$

- $\Omega(g(n)) = \{f(n): \exists$ positive constants c, n_0 such that
  $$0 \le cg(n) \le f(n), \forall n \ge n_0\}$$

Ω-notation (lower bounds)



Θ-notation (tight bounds)

- $f(n)=\Theta(g(n))$ if $\exists$ positive constants c_1, c_2, n_0 such that

$$0 \le c_1 g(n) \le f(n) \le c_2 g(n), \ \forall n \ge n_0$$

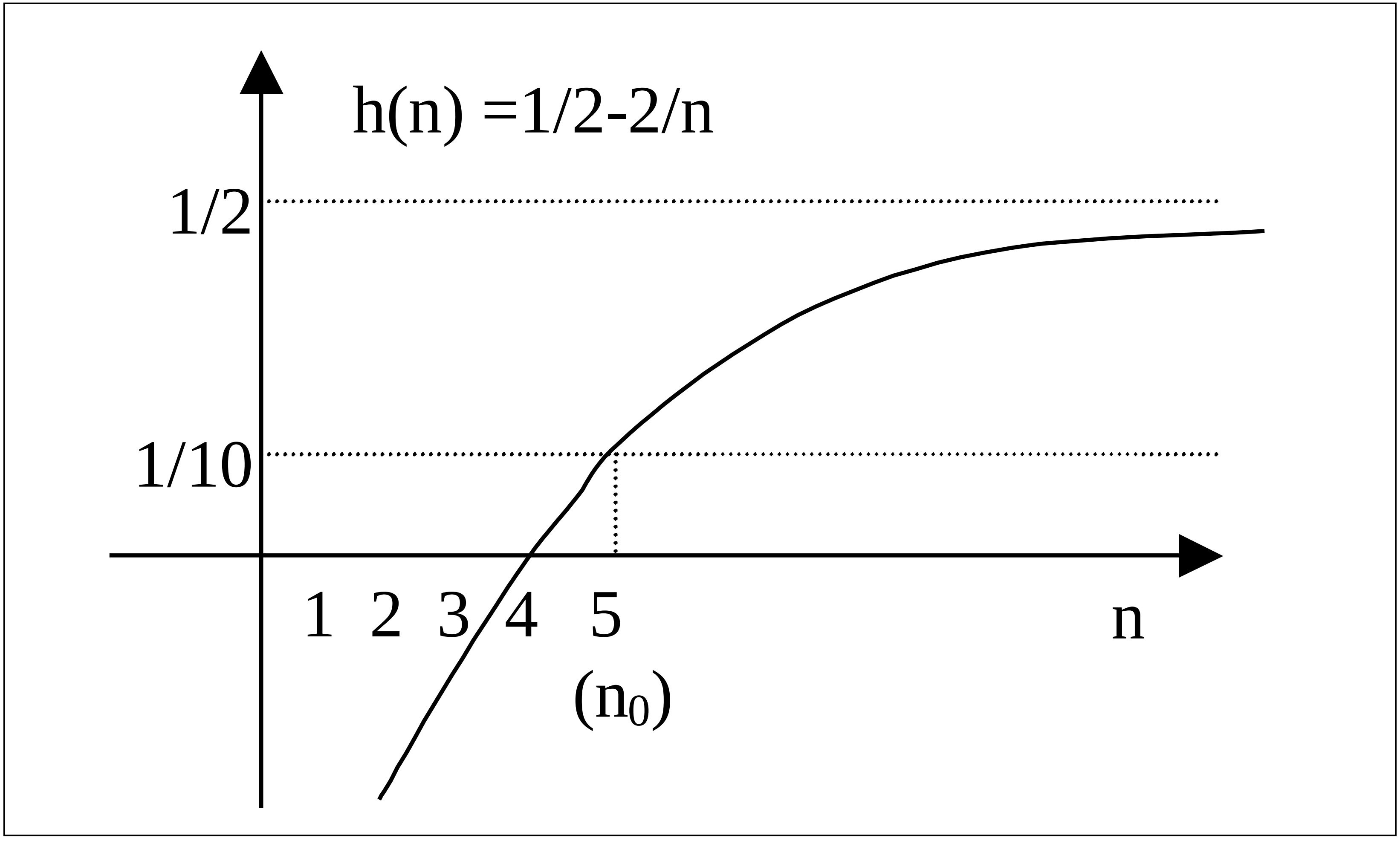
- example:

$$\frac{1}{2}n^2 - 2n = \Theta(n^2)$$

$$0 \le c_1 n^2 \le \frac{1}{2}n^2 - 2n \le c_2 n^2$$

$$c_1 \le \frac{1}{2} - \frac{2}{n} \le c_2$$

Θ-notation: example ($0 < c_1 \le h(n) \le c_2$)



Θ-notation: example ($0 < c_1 \le h(n) \le c_2$)

$$h(n) = \frac{1}{2} - \frac{2}{n} \le \frac{1}{2} = c_2, \forall n \ge 0$$

$$h(n) = \frac{1}{2} - \frac{2}{n} \ge \frac{1}{10} = c_1, \forall n \ge 5$$

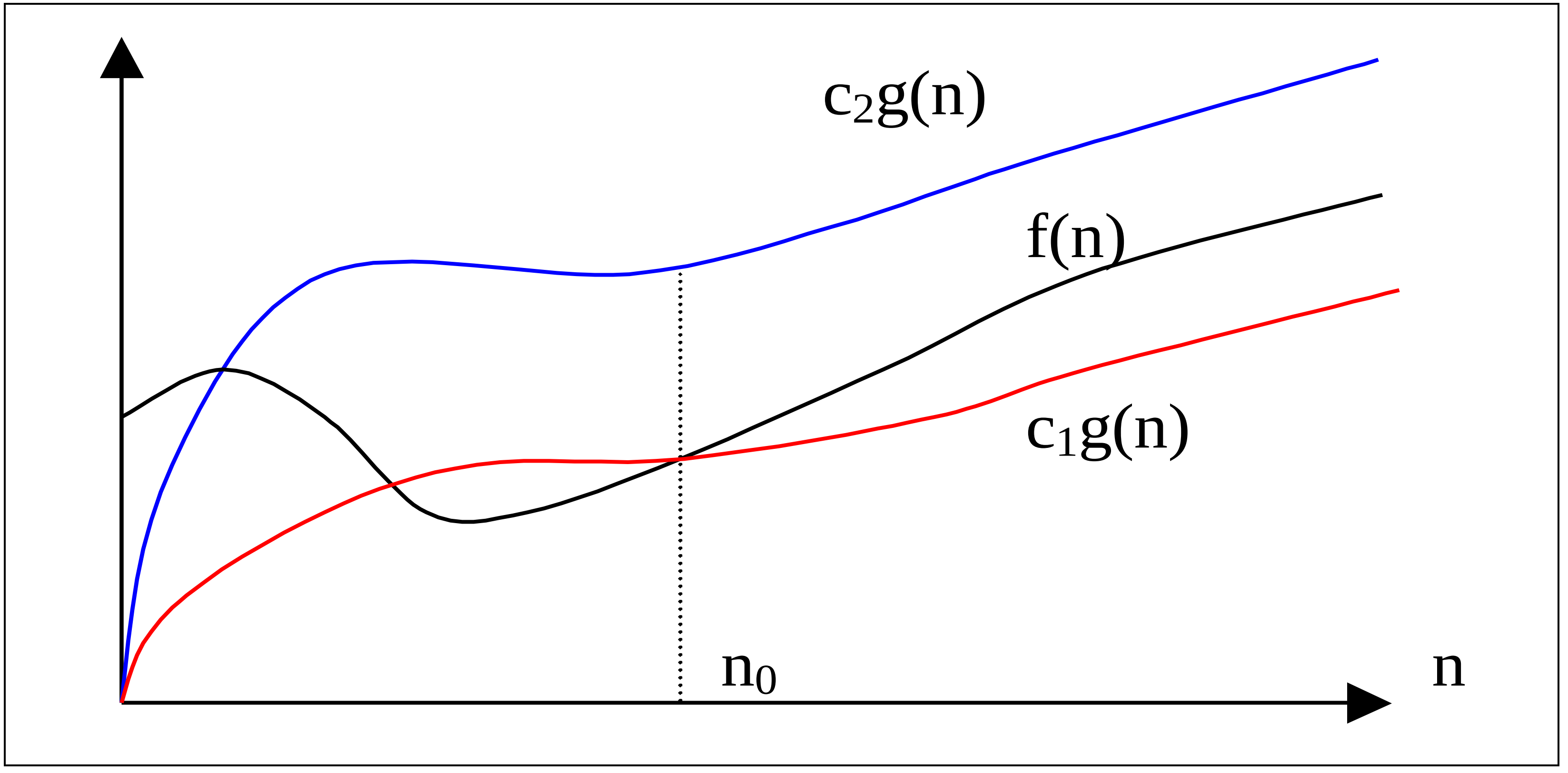
therefore

$$c_1 = \frac{1}{10}, c_2 = \frac{1}{2}, n_0 = 5$$

Θ-notation (tight bounds)

$\Theta(g(n)) = \{f(n)$: $\exists$ positive constants c_1, c_2, n_0 such that

$$0 \le c_1 g(n) \le f(n) \le c_2 g(n),\ \forall n \ge n_0\}$$



Θ-notation (tight bounds)

- Prove that $10^{-8} n^2 \neq \Theta(n)$

  - suppose c_2, n_0 exist such that $10^{-8} n^2 \leq c_2 n$, $\forall n \geq n_0$

  - but then $c_2 \geq 10^{-8} n$

  - contradiction since c_2 is a constant
- Theorem: leading constants & low-order terms don't matter
- Justification: can choose the leading constant large enough to make high-order term dominate other terms

Θ-notation (tight bounds)

- Theorem: (O and Ω) $\Leftrightarrow$ Θ
  - Θ is stronger than both O and Ω
  - i.e., $\Theta(g(n)) \subseteq O(g(n))$ and

    $\Theta(g(n)) \subseteq \Omega(g(n))$

Using asymptotic notation for describing running times

O-notation

- used to bound worst-case running times
  - also bounds running time on arbitrary inputs as well
- e.g., $O(n^2)$ bound on worst-case running time of insertion sort also applies to its running time on every input

Using O-notation for describing running times

- Abuse to say "running time of insertion sort is $O(n^2)$"
  - for a given n, actual running time depends on particular input of size n
  - i.e., running time is not only a function of n
  - however, worst-case running time is only a function of n

Using O-notation for describing running times

- What we really mean by "running time of insertion sort is $O(n^2)$"
  - worst-case running time of insertion sort is $O(n^2)$

    or equivalently

  - no matter what particular input of size n is chosen (for each value of) running time on that set of inputs is $O(n^2)$

Using Ω -notation for describing running times

- used to bound the best-case running times

  $\Rightarrow$ also bounds the running time on arbitrary inputs as well

- e.g., $\Omega(n)$ bound on best-case running time of insertion sort

  $\Rightarrow$ running time of insertion sort is $\Omega(n)$

Using Ω -notation for describing running times

- "running time of an algorithm is $\Omega(g(n))$" means
  - no matter what particular input of size n is chosen (for any n), running time on that set of inputs is at least a constant times $g(n)$, for sufficiently large n
  - however, it is not contradictory to say "worst-case running time of insertion sort is $\Omega(n^2)$" since there exists an input that causes algorithm to take $\Omega(n^2)$ time

Using Θ-notation for describing running times

1) used to bound worst-case & best-case running times of an algorithm if they are not asymptotically equal

2) used to bound running time of an algorithm if its worst & best case running times are asymptotically equal

Using Θ-notation for describing running times

Case (1):

- a Θ-bound on worst-/best-case running time does not apply to its running time on arbitrary inputs

- e.g., $\Theta(n^2)$ bound on worst-case running time of insertion sort does not imply a $\Theta(n^2)$ bound on running time of insertion sort on every input

  since $T(n) = O(n^2)$ & $T(n) = \Omega(n)$ for insertion sort

Using Θ-notation for describing running times

Case (2):

- implies a Θ-bound on every input
  - e.g., merge sort

$$\left.\begin{array}{l} T(n) = O(n\lg n) \\ T(n) = \Omega(n\lg n) \end{array}\right\} \; T(n) = \Theta(n\lg n)$$

Asymptotic notation in equations

- Asymptotic notation appears alone on RHS of an equation
  - means set membership
  - e.g., $n = O(n^2)$ means $n \in O(n^2)$
- Asymptotic notation appears on RHS of an equation
  - stands for some anonymous function in the set
  - e.g., $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$ means that

    $2n^2 + 3n + 1 = 2n^2 + h(n)$, for some $h(n) \in \Theta(n)$

    i.e., $h(n) = 3n + 1$

Asymptotic notation appears on LHS of an equation

- stands for any anonymous function in the set
  - e.g., $2n^2 + \Theta(n) = \Theta(n^2)$ means that

    for any function $g(n) \in \Theta(n)$

    $\exists$ some function $h(n) \in \Theta(n^2)$

    such that $2n^2+g(n) = h(n), \forall n$

- RHS provides coarser level of detail than LHS

Other asymptotic notations

o-notation

- upper bound provided by O-notation
  may or may not be tight
  - e.g., bound $2n^2 = O(n^2)$ is asymptotically tight
    bound $2n = O(n^2)$ is not asymptotically tight
- o-notation denotes an upper bound
  that is not asymptotically tight

o-notation

- $o(g(n)) = \{f(n)$: for any constant $c > 0$,
  $\exists$ a constant $n_0 > 0$
  such that $0 \le f(n) < cg(n), \forall n \ge n_0\}$

- Intuitively, $$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

  - e.g., $2n = o(n^2)$, any positive c satisfies
  - but $2n^2 \ne o(n^2)$, $c = 2$ does not satisfy

ω-notation

- denotes a lower bound that is not asymptotically tight
- $\omega(g(n)) = \{f(n)$: for any constant $c > 0$,

  $\exists$ a constant $n_0 > 0$

  such that $0 \leq cg(n) < f(n), \forall n \geq n_0\}$

- Intuitively $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

  - e.g., $n^2 / 2 = \omega(n)$, any c satisfies
  - but $n^2 / 2 \neq \omega(n^2)$, $c=1/2$ does not satisfy

Asymptotic comparison of functions

- similar to the relational properties of real numbers

- Transitivity: (holds for all)

  e.g., $f(n) = \Theta(g(n))$ & $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$

- Reflexivity: (holds for Θ, O, Ω)

  e.g., $f(n) = O(f(n))$

- Symmetry: (holds only for Θ)

  e.g., $f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$

- Transpose symmetry: (($O \leftrightarrow \Omega$) and ($o \leftrightarrow \omega$))

  e.g., $f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$

Analogy to the comparison of two real numbers

- $f(n) = O(g(n)) \leftrightarrow a \le b$
- $f(n) = \Omega(g(n)) \leftrightarrow a \ge b$
- $f(n) = \Theta(g(n)) \leftrightarrow a = b$
- $f(n) = o(g(n)) \leftrightarrow a < b$
- $f(n) = \omega(g(n)) \leftrightarrow a > b$

Analogy to the comparison of two real numbers

- Trichotomy property of real numbers does not hold for asymptotic notation
  - i.e., for any two real numbers a and b,

    we have either $a < b$, or $a = b$, or $a > b$
  - i.e., for two functions f(n) & g(n), it may be the case that neither $f(n) = O(g(n))$ nor $f(n) = \Omega(g(n))$ holds
  - e.g., n and $n^{1+\sin(n)}$ cannot be compared asymptotically