Lecture 12b

Dynamic Tables
Why Dynamic Tables?

In some applications:

• We don't know how many objects will be stored in a table.

• We may allocate space for a table
  – But, later we may find out that it is not enough.
  – Then, the table must be reallocated with a larger size.
    • All the objects stored in the original table
    • Must be copied over into the new table.
Why Dynamic Tables?

• Similarly, if many objects are deleted from the table:
  – it may be worthwhile to reallocate the table with a smaller size.

This problem is called Dynamically Expanding and Contracting a table.
Why Dynamic Tables?

Using amortized analysis we will show that,

The amortized cost of insertion and deletion is $O(1)$.

Even though the actual cost of an operation is large when it triggers an expansion or a contraction.

We will also show how to guarantee that

The unused space in a dynamic table never exceeds a constant fraction of the total space.
Operations

**TABLE-INSERT:**

*Inserts* into the table an item that occupies a single slot.

**TABLE-DELETE:**

*Removes* an item from the table & frees its slot.
Load Factor

Load Factor of a Dynamic Table $T$

$$\alpha(T) = \frac{\text{Number of items stored in the table}}{\text{size}(\text{number of slots}) \text{ of the table}}$$

For an empty table

$$\alpha(T) = \frac{0}{0} = 1$$

by definition
Insertion-Only Dynamic Tables

**Table-Expansion:**
- **Assumption:**
  - Table is allocated as an array of slots
- **A table fills up when**
  - all slots have been used
  - equivalently, when its load factor becomes 1
- **Table-Expansion occurs when**
  - An item is to be inserted into a full table
Insertion-Only Dynamic Tables

• A Common Heuristic
  – Allocate a new table that has twice as many slots as the old one.

• Hence, insertions are performed if only

\[ \frac{1}{2} \leq \alpha(T) \leq 1 \]
Table Insert

TABLE-INSERT (T, x)
    if size[T] = 0 then
        allocate table[T] with 1 slot
        size[T] ← 1
    if num[T] = size[T] then
        allocate new-table with 2.size[T] slots
        copy all items in table[T] into new-table
        free table[T]
        table[T] ← new-table[T]
        size[T] ← 2.size[T]
    insert x into table[T]
    num[T] ← num[T] + 1
end
Table Expansion

• Running time of TABLE-INSERT is proportional to the number of elementary insert operations.

• Assign a cost of 1 to each elementary insertion

• Analyze a sequence of $n$ TABLE-INSERT operations on an initially empty table
Cost of Table Expansion

What is cost $c_i$ of the $i$-th operation?

- If there is room in the current table (or this is the first operation)
  \[ c_i = 1 \] (only one elementary insert operation)
- If the current table is full, an expansion occurs, then the cost is $c_i = i$.
  \[ 1 \] for the elementary insertion of the new item
  \[ i-1 \] for the items that must be copied from the old table to the new table.
Cost of Table Expansion

• If $n$ operations are performed,
  
  The worst case cost of an operation is $O(n)$
  
  Therefore the total running time is $O(n^2)$

• This bound is not tight, because

  Expansion does not occur so often in the course of $n$ operations.
Amortized Analysis of Insert

The Aggregate Method

Table is initially empty.

Observe:

\[ c_i = \begin{cases} i & \text{if } i \text{ is an exact power of } 2 \\ 1 & \text{otherwise} \end{cases} \]
The Aggregate Method

Therefore the total cost of \( n \) TABLE-INSERT operations is

\[
\sum_{i=1}^{n} c_i = n + \sum_{j=0}^{\lceil \lg n \rceil} 2^j < n + \sum_{j=0}^{\lg n} 2^j = n + 2n = 3n
\]

| i  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | ... |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|    |
| \( c_i \) | 1  | 2  | 3  | 1  | 5  | 1  | 1  | 1  | 9  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 17 | 1  | 1  | 1  | ... |
| Expansion cost | 1  | 2  | 4  | 8  |    |    |    |    |    |    |    |    |    |    |    |    |    | 16 |    |    |    |    |

The amortized cost of a single operation is \( 3n/n = 3 = O(1) \)
The Accounting Method

Assign the following amortized costs

- Table-Expansion : 0
- Insertion of a new item : 3

**Insertion of a new item**

a) 1 (as an actual cost) for inserting itself into the table

b) 1 (as a credit) for moving itself in the next expansion

c) 1 (as a credit) for moving another item (in the next expansion) that has already moved in the last expansion
Accounting Method Example

Size of the table: $M$

Immediately after an expansion (just before the insertion)

$num[T] = M/2$ and $size[T] = M$ where $M$ is a power of 2.

Table contains no credits

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>
Accounting Method Example

1st insertion

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>X</td>
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<td></td>
</tr>
</tbody>
</table>

(c)

2nd insertion

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>X</td>
<td>X</td>
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<td>X</td>
<td>X</td>
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</tr>
</tbody>
</table>

(a) $1 for insertion
# Accounting Method Example

## M/2th Insertion

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<table>
<thead>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>X</td>
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<td>X</td>
<td>X</td>
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<td>Z</td>
<td>Z</td>
<td>Z</td>
<td>Z</td>
<td>Z</td>
</tr>
</tbody>
</table>

Thus, by the time the table contains $M$ items and is full

- each item in the table has $\$1$ of credit to pay for its move during the next expansion
The Potential Method

Define a potential function $\Phi$ that is

- 0 immediately after an expansion
- builds to the table size by the time table becomes full

Next expansion can be paid for by the potential.
Definition of $\Phi$

One possible $\Phi$ is:

$$\Phi(T) = 2*\text{num}[T] - \text{size}[T]$$

Immediately after an expansion

$$\text{size}[T] = 2*\text{num}[T] \Rightarrow \Phi(T) = 0$$

Immediately before an expansion

$$\text{size}[T] = \text{num}[T] \Rightarrow \Phi(T) = \text{num}[T]$$

The initial value for the potential is 0
Definition of $\Phi$

Since the table is at least half full

(i.e. $\text{num}[T] \geq \frac{\text{size}[T]}{2}$)

$\Phi(T)$ is always nonnegative.

Thus, the sum of the amortized cost of $n$

TABLE-INSERT operations is an upper bound on the sum of the actual costs.
Analysis of $i$-th Table Insert

$n_i : \text{num}[T]$ after the $i$-th operation

$s_i : \text{size}[T]$ after the $i$-th operation

$\Phi_i : \text{Potential}$ after the $i$-th operation

Initially we have $n_i = s_i = \Phi_i = 0$

Note that, $n_i = n_{i-1} + 1$ always hold.
Insert without Expansion

If the $i$-th TABLE-INSERT does not trigger an expansion,

\[ s_i = s_{i-1} ; c_i = 1 \]

\[ \hat{c}_i = c_i + \phi_i - \phi_{i-1} = 1 + (2n_i - s_i) - (2n_{i-1} - s_{i-1}) \]

\[ = 1 + (2(n_{i-1} + 1) - s_{i-1}) - (2n_{i-1} - s_{i-1}) \]

\[ = 1 + 2\eta_{i-1} + 2 - \$_{i-1} - 2\eta_{i-1} + \$_{i-1} = 3 \]
Insert with Expansion

If the $i$-th TABLE-INSERT does trigger an expansion, then

$$n_{i-1} = s_{i-1}; \quad s_i = 2s_{i-1}; \quad c_i = n_i = n_{i-1} + 1$$

$$\hat{c}_i = c_i + \phi_i - \phi_{i-1} = n_i + (2n_i - s_i) - (2n_{i-1} - s_{i-1})$$

$$= (n_{i-1} + 1) + (2(n_{i-1} + 1) + 2s_{i-1}) - (2n_{i-1} - s_{i-1})$$

$$= n_{i-1} + 1 + 2n_{i-1} + 2 - 2n_{i-1} - 2n_{i-1} + n_{i-1} = 3$$
Adding Delete Operation

**TABLE-DELETE**: Remove the specified item from the table. It is often desirable to **contract** the table. In table **contraction**, we would like to preserve two properties

- **Load factor** of dynamic table is bounded below by a constant
- **Amortized cost** of a table operation is bounded above by a constant

We assume that the cost can be measured in terms of elementary **insertions** and **deletions**
Expansion and Contraction

A natural strategy for expansion and contraction

- Double the table size when an item is inserted into a full table
- Halve the size when a deletion would cause $\alpha(T) < 1/2$

This strategy guarantees $\frac{1}{2} \leq \alpha(T) \leq 1$

Unfortunately, it can cause the amortized cost of an operation to be quite large
Worst-Case for \( \alpha(T) \geq \frac{1}{2} \)

Consider the following worst case scenario

– We perform \( n \) operations on an empty table where \( n \) is a power of 2

– First \( n/2 \) operations are all insertions, cost a total of \( \Theta(n) \)
  
  at the end: we have \( \text{num}[T] = \text{size}[T] = n/2 \)

– Second \( n/2 \) operations repeat the sequence

  \[ \text{I D D I I D I I D I ...} \]
Worst-Case for $\alpha(T) \geq \frac{1}{2}$

Example: $n=16$

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
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</thead>
<tbody>
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<td>...</td>
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<td>I</td>
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<td>$n_i$</td>
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<td>9</td>
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<td>9</td>
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<td>7</td>
<td>8</td>
</tr>
<tr>
<td>$s_i$</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>8</td>
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<td>16</td>
<td>16</td>
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</tbody>
</table>

In the second $n/2$ operations

- The first INSERT cause an expansion
- Two further DELETEs cause contraction
- Two further INSERTs cause expansion ... and so on

Hence there are $n/8$ expansions and $n/8$ contractions

The cost of each expansion and contraction is $\approx n/2$
Worst-Case for $\alpha(T) \geq \frac{1}{2}$

Thus the total cost of $n$ operations is $\Theta(n^2)$ since

- First $n/2$ operations: $3n$
- Second $n/2$ operations: $(n/4)(n/2) = n^2/8$

The amortized cost of an operation is $\Theta(n)$

The difficulty with this strategy is

- After an expansion, we do not perform enough deletions to pay for a contraction
- After a contraction, we do not perform enough insertions to pay for an expansion
Improving Expansion – Contraction

We can improve upon this strategy by allowing $\alpha(T)$ to drop below $\frac{1}{2}$

We continue to double the table size when an item is inserted into a full table

But, we halve the table size (perform contraction)

when a deletion causes $\alpha(T) < \frac{1}{4}$ rather than $\alpha(T) < \frac{1}{2}$,

Therefore, $\frac{1}{4} \leq \alpha(T) \leq 1$
Improving Expansion – Contraction

Hence after an expansion, $\alpha(T) = \frac{1}{2}$, thus at least half of the items in the table must be deleted before a contraction can occur.

Similarly, after a contraction $\alpha(T) = \frac{1}{2}$, thus the number of items in the table must be doubled by insertions before an expansion can occur.
Potential Method

Define the potential function as follows

- $\Phi(T) = 0$ immediately after an expansion or contraction
- Recall that, $\alpha(T) = \frac{1}{2}$ immediately after and expansion or contraction, therefore the potential should build up to $\text{num}[T]$ as $\alpha(T)$ increases to 1 or decreases to $\frac{1}{4}$
- So that the next expansion or contraction can be paid by the potential.
\( \Phi(\alpha) \) w.r.t. \( \alpha(T) \)

\( M = \text{num}[T] \) when an expansion or contraction occurs
Description of New $\Phi$

One such $\Phi$ is

$$\Phi(T) = \begin{cases} 
2\text{num}[T] - \text{size}[T] & \text{if } \alpha(T) \geq \frac{1}{2} \\
\frac{\text{size}[T]}{2} - \text{num}[T] & \text{if } \alpha(T) < \frac{1}{2}
\end{cases}$$

or

$$\Phi(T) = \begin{cases} 
\text{num}[T](2 - 1/\alpha) & \text{if } \alpha(T) \geq \frac{1}{2} \\
\text{num}[T](1/2\alpha - 1) & \text{if } \alpha(T) < \frac{1}{2}
\end{cases}$$
Description of New $\Phi$

- $\Phi = 0$ when $\alpha = \frac{1}{2}$
- $\Phi = \text{num}[T]$ when $\alpha = \frac{1}{4}$
- $\Phi = 0$ for an empty table
  $(\text{num}[T] = \text{size}[T]=0, \alpha[T] = 0)$
- $\Phi$ is always nonnegative
Amortized Analysis

Operations are:

- TABLE-INSERT
- TABLE-DELETE

\[ c_i : \text{Actual Cost} \quad \hat{c}_i : \text{Amortized Cost} \quad \Phi_i : \Phi(T) \]
\[ n_i : \text{num}[T] \quad s_i : \text{size}[T] \quad \alpha_i : \alpha(T) \]

after the \( i \)-th operation
Table Insert

\[ n_i = n_{i-1} + 1 \iff n_{i-1} = n_i - 1 \]

Table contraction may not occur.

- \( \alpha_{i-1} \geq \frac{1}{2} \)
  Analysis is identical to that for table expansion
  Therefore, \( \hat{c}_i = 3 \) whether the table expands or not.

- \( \alpha_{i-1} < \frac{1}{2} \) and \( \alpha_i < \frac{1}{2} \)
  Expansion may not occur (\( \hat{c}_i = 1 \), \( s_i = s_{i-1} \))
  \[ \hat{c}_i = c_i + \Phi_i - \Phi_{i-1} = 1 + \left( \frac{3i}{2} - n_i \right) - \left( \frac{s_{i-1}}{2} - n_{i-1} \right) \]
  \[ = 1 + \frac{s_i}{2} - n_i - \frac{s_i}{2} + (n_i - 1) = 0 \]
Table Insert

- $\alpha_{i-1} < \frac{1}{2}$ and $\alpha_i \geq \frac{1}{2}$

  $\Rightarrow \alpha_i = \frac{1}{2}$

Expansion may not occur ($c_i = 1$ ; $s_i = s_{i-1}$ ; $n_i = s_i / 2$)

$$\hat{c}_i = c_i + \Phi_i - \Phi_{i-1} = 1 + (s_i / 2 - n_i) - (2n_{i-1} - s_{i-1})$$

$$= 1 + s_i / 2 - n_i - 2(n_i - 1) + s_i = 3 - 3s_i / 2 - 3n_i$$

$$= 3 + 3s_i / 2 - 3n_i = 3 + 3s_i / 2 - 3s_i / 2 = 3$$

Thus, amortized cost of a TABLE-INSERT operation is at most 3.
Table Delete

\[ n_i = n_{i-1} - 1 \Rightarrow n_{i-1} = n_i + 1 \]

Table expansion may not occur.

- \( \alpha_{i-1} \leq \frac{1}{2} \) and \( \frac{1}{4} \leq \alpha_i < \frac{1}{2} \) (It does not trigger a contraction)

\[ s_i = s_{i-1} \text{ and } c_i = 1 \text{ and } \alpha_i < \frac{1}{2} \]

\[ \hat{c}_i = c_i + \Phi_i - \Phi_{i-1} = 1 + (s_i / 2 - n_i) - (s_{i-1} / 2 - n_{i-1}) \]

\[ = 1 + s_i / 2 - n_i - s_i / 2 + (n_i + 1) = 2 \]
Table Delete

- $\alpha_{i-1} = \frac{1}{4}$ (It does trigger a contraction)
  $s_i = s_{i-1}/2$; $n_i = s_{i-1}/2$; and $c_i = n_i + 1$
  $\hat{c}_i = c_i + \Phi_i - \Phi_{i-1} = (n_i + 1) + (s_i/2 - n_i) - (s_{i-1}/2 - n_{i-1})$
  $= n_i + 1 + s_i/2 - n_i - s_i + s_i/2 = 1$

- $\alpha_{i-1} > \frac{1}{2}$ ($\alpha_i \geq \frac{1}{2}$)
  Contraction may not occur ($c_i = 1$; $s_i = s_{i-1}$)
  $\hat{c}_i = c_i + \Phi_i - \Phi_{i-1} = 1 + (2n_i - s_i) - (2n_{i-1} - s_{i-1})$
  $= 1 + 2n_i - s_i - 2(n_i + 1) + s_i = -1$
Table Delete

• $\alpha_{i-1} = \frac{1}{2}$ ($\alpha_i < \frac{1}{2}$)

Contraction may not occur

\[
c_i = 1; \quad s_i = s_{i-1}; \quad n_i = s_{i-1}/2; \quad \text{and } \Phi_{i-1} = 0
\]

\[
\hat{c}_i = c_i + \Phi_i - \Phi_{i-1} = 1 + (s_i / 2 - n_i) - 0
\]

\[
= 1 + s_i / 2 - n_i \quad \text{but} \quad n_{i+1} = s_i / 2
\]

\[
= 1 + (n_i + 1) - n_i = 2
\]
Table Delete

Thus, the amortized cost of a TABLE-DELETE operation is at most 2

Since the amortized cost of each operation is bounded above by a constant

The actual time for any sequence of $n$ operations on a Dynamic Table is $O(n)$