CS473-Algorithms I

Lecture 14-A

Graph Searching: Breadth-First Search

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Graph Searching: Breadth-First Search

Graph G =(V, E), directed or undirected with adjacency list repres.

GOAL: Systematically explores edges of G to

- discover every vertex reachable from the source vertex s
- compute the shortest path distance of every vertex from the source vertex s
- produce a breadth-first tree (BFT) G_{Π} with root s
  - BFT contains all vertices reachable from s
  - the unique path from any vertex v to s in G_{Π} constitutes a shortest path from s to v in G

IDEA: Expanding frontier across the breadth -greedy-

- propagate a wave 1 edge-distance at a time
- using a FIFO queue: O(1) time to update pointers to both ends

Breadth-First Search Algorithm

Maintains the following fields for each $u \in V$

- color[u]: color of u
  - WHITE : not discovered yet
  - GRAY : discovered and to be or being processed
  - BLACK: discovered and processed
- $\Pi[u]$: parent of u (NIL of $u = s$ or u is not discovered yet)
- $d[u]$: distance of u from s

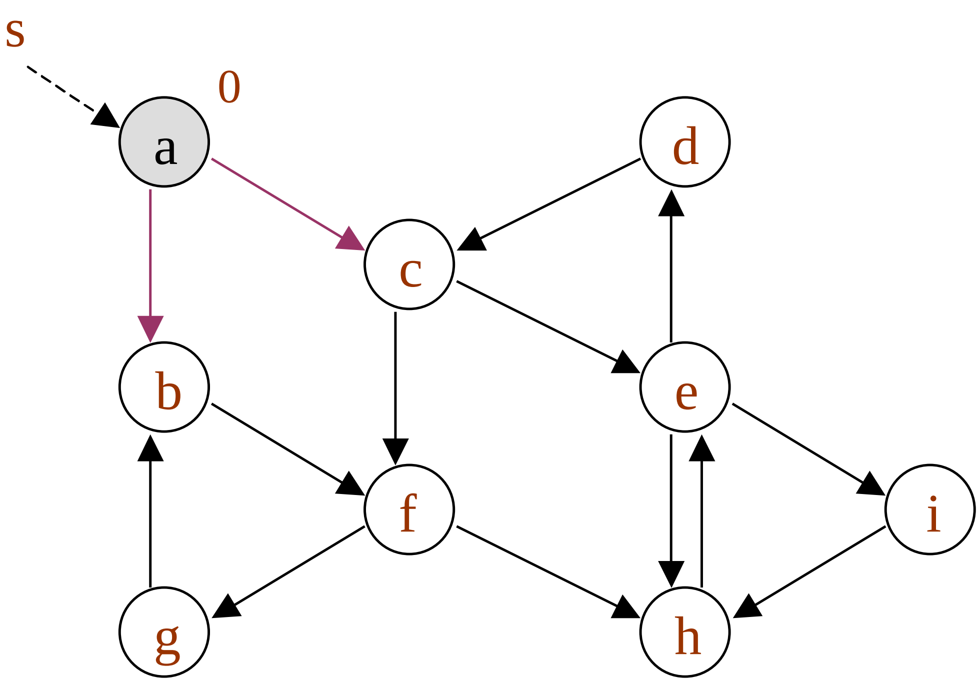
Processing a vertex = scanning its adjacency list

Breadth-First Search Algorithm

BFS(G, s)
    for each $u \in V - \{s\}$ do
        color[u] $\leftarrow$ WHITE
        $\Pi[u] \leftarrow$ NIL; $d\,[u] \leftarrow \infty$
    color[s] $\leftarrow$ GRAY
    $\Pi[s] \leftarrow$ NIL; $d\,[s] \leftarrow 0$
    $Q \leftarrow \{s\}$
    while $Q \neq \varnothing$ do
        $u \leftarrow$ head[Q]
        for each v in Adj[u] do
            if color[v] = WHITE then
                color[v] $\leftarrow$ GRAY
                $\Pi[v] \leftarrow u$
                $d\,[v] \leftarrow d\,[u] + 1$
                ENQUEUE(Q, v)
        DEQUEUE(Q)
        color[u] $\leftarrow$ BLACK

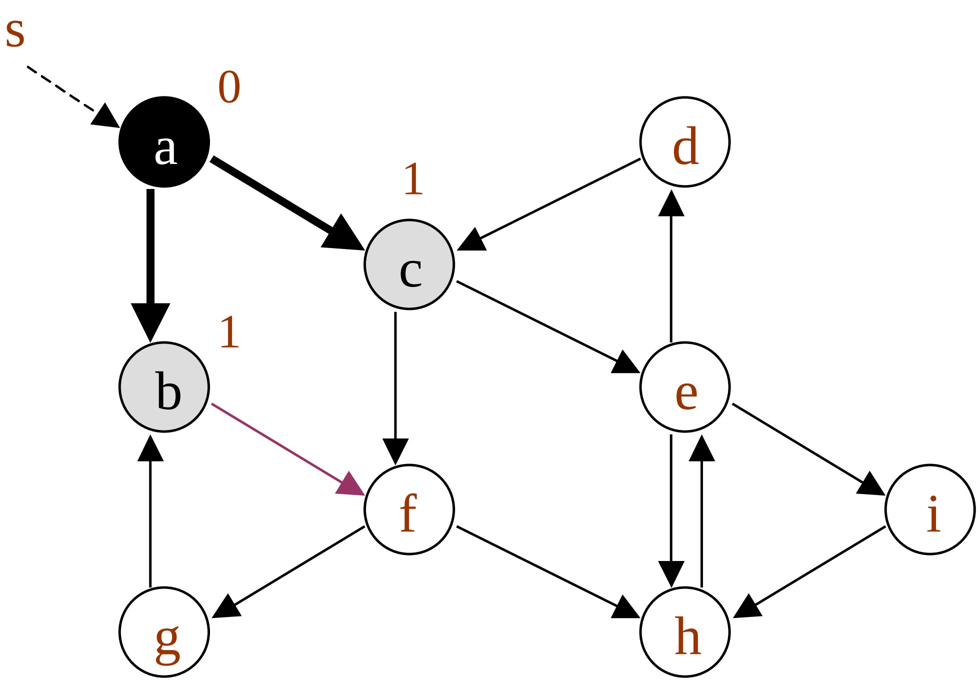
Breadth-First Search

Sample Graph:



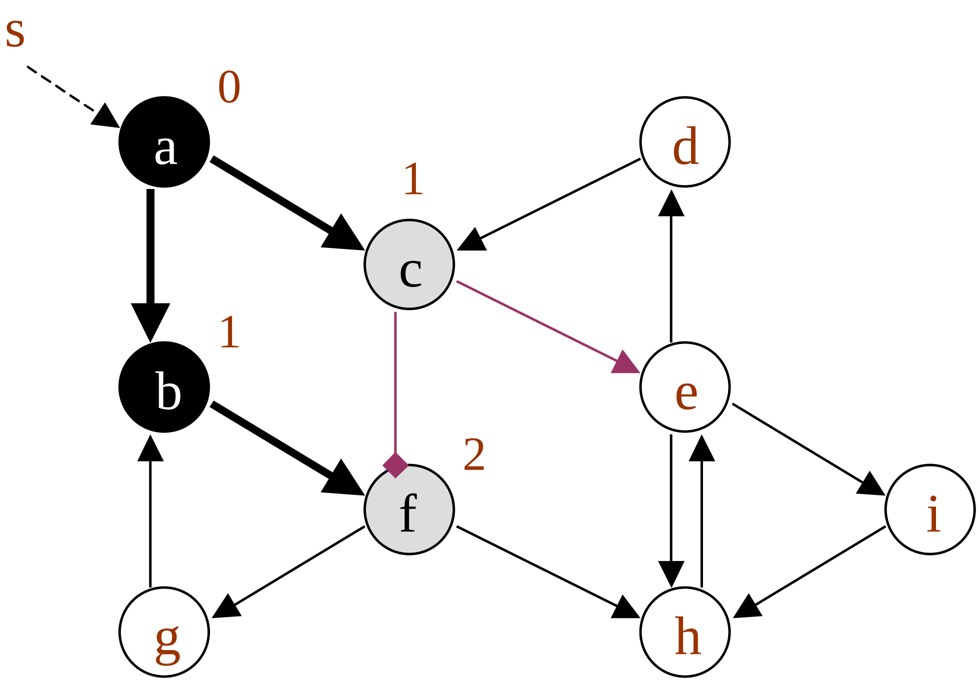
FIFO queue Q	just after processing vertex
⟨a⟩	-

Breadth-First Search



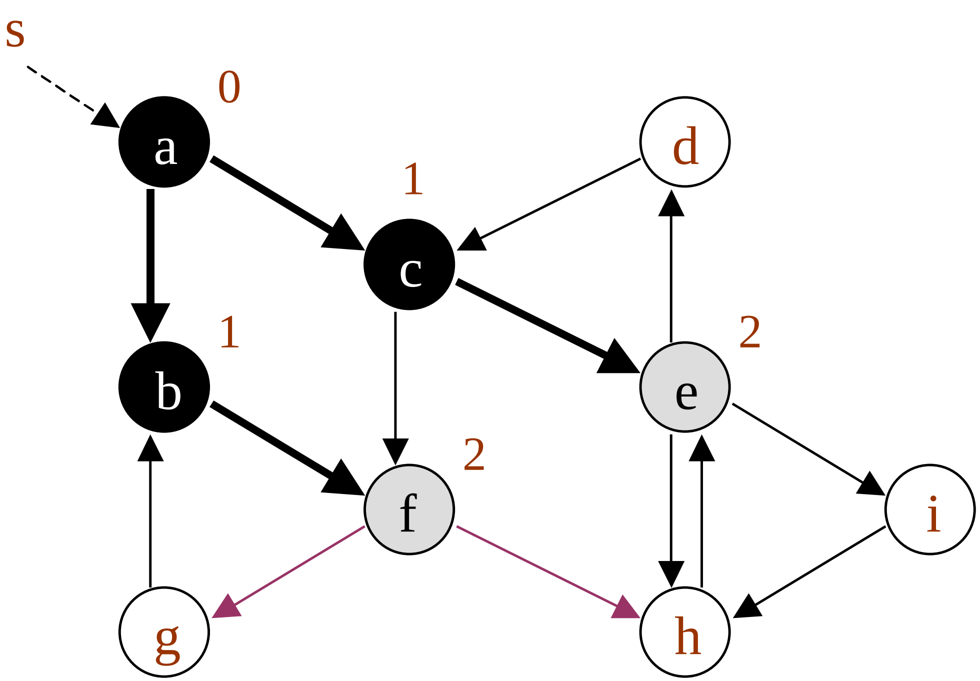
FIFO queue Q	just after processing vertex
⟨a⟩	-
⟨a,b,c⟩	a

Breadth-First Search



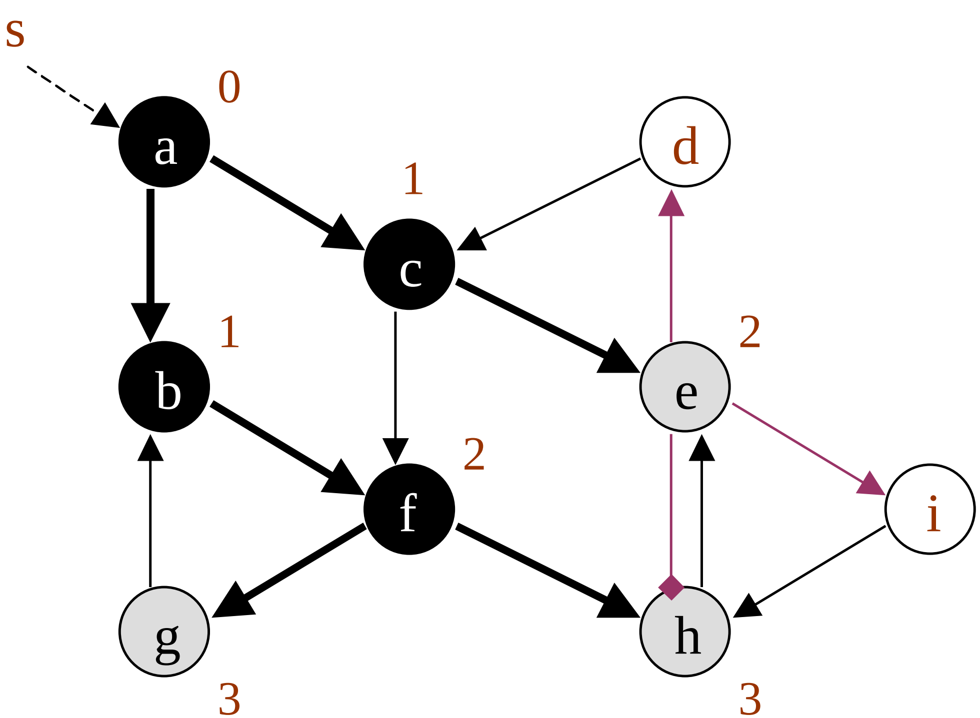
FIFO queue Q	just after processing vertex
⟨a⟩	-
⟨a,b,c⟩	a
⟨a,b,c,f⟩	b

Breadth-First Search



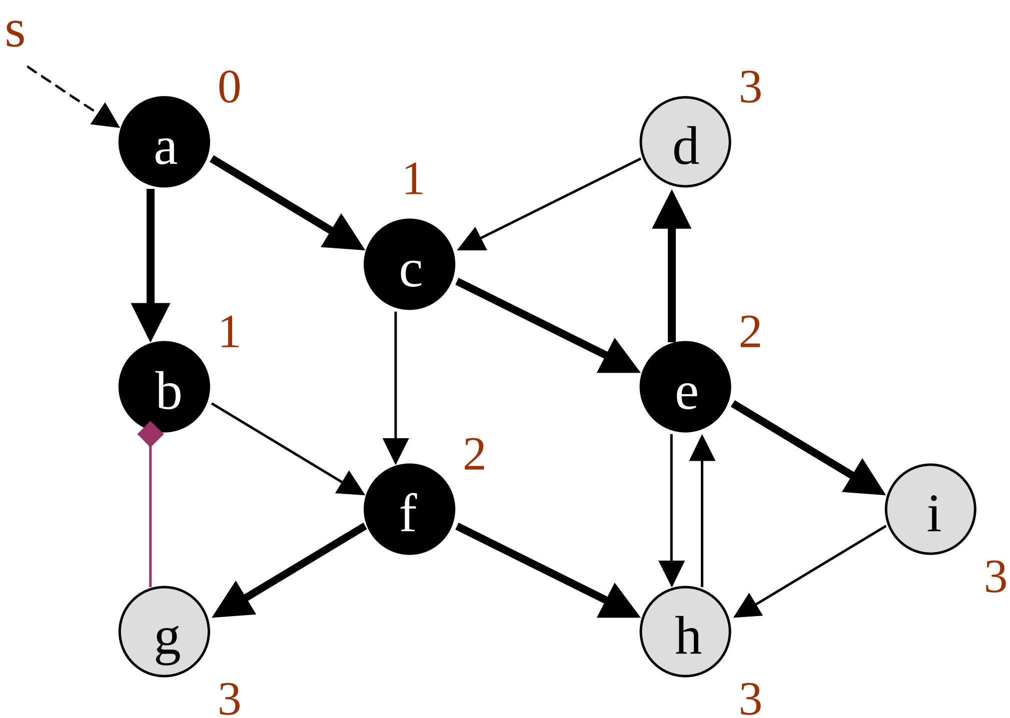
FIFO queue Q	just after processing vertex
$\langle a \rangle$	-
$\langle a,b,c \rangle$	a
$\langle a,b,c,f \rangle$	b
$\langle a,b,c,f,e \rangle$	c

Breadth-First Search



FIFO queue Q	just after processing vertex
⟨a⟩	-
⟨a,b,c⟩	a
⟨a,b,c,f⟩	b
⟨a,b,c,f,e⟩	c
⟨a,b,c,f,e,g,h⟩	f

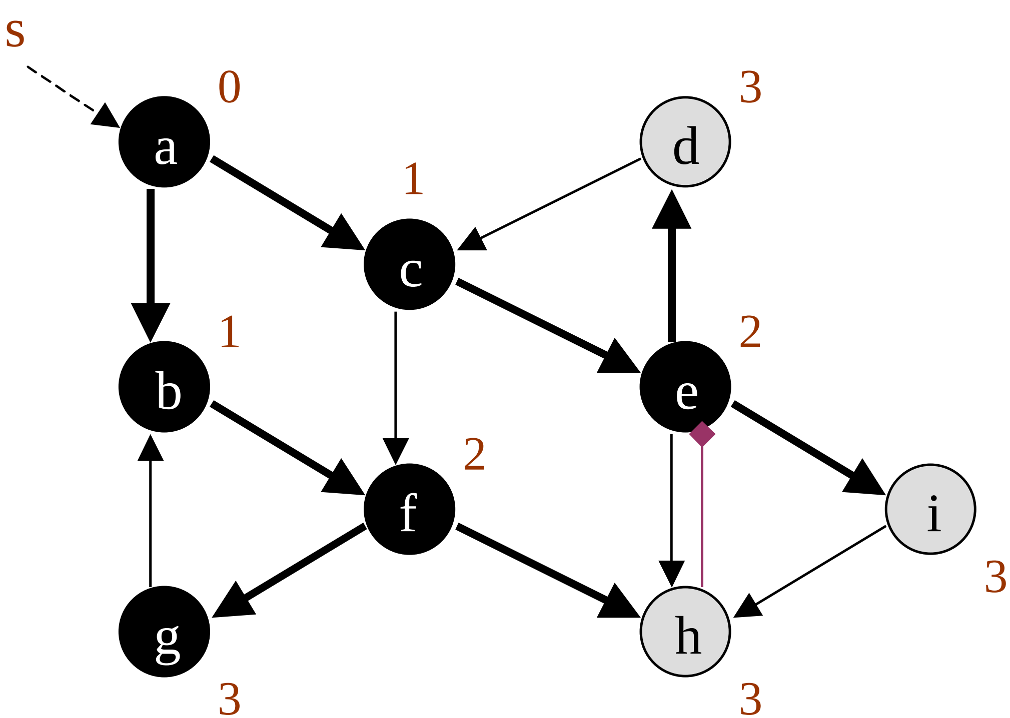
Breadth-First Search



FIFO queue Q	just after processing vertex
⟨a⟩	-
⟨a,b,c⟩	a
⟨a,b,c,f⟩	b
⟨a,b,c,f,e⟩	c
⟨a,b,c,f,e,g,h⟩	f
⟨a,b,c,f,e,g,h,d,i⟩	e

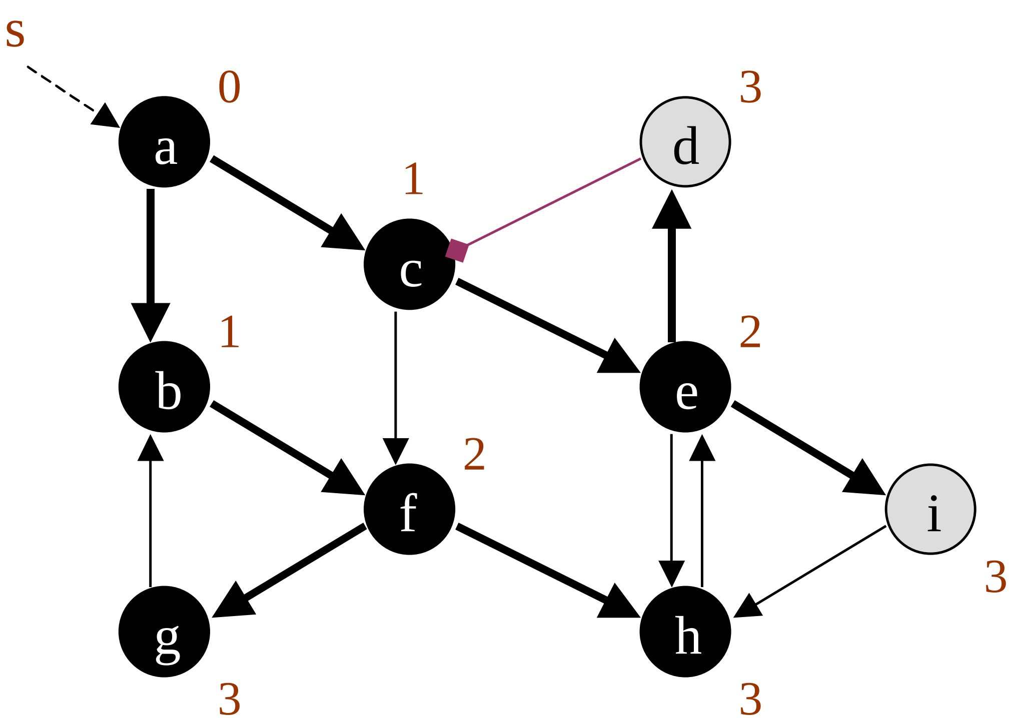
all distances are filled in after processing e

Breadth-First Search



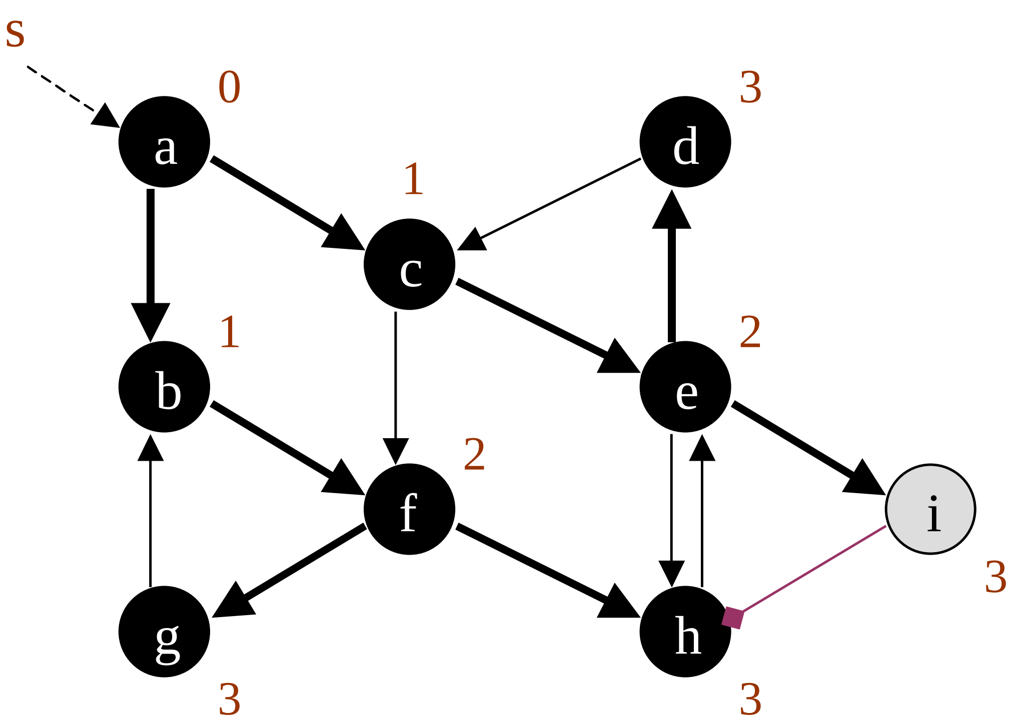
FIFO queue Q	just after processing vertex
⟨a⟩	-
⟨a,b,c⟩	a
⟨a,b,c,f⟩	b
⟨a,b,c,f,e⟩	c
⟨a,b,c,f,e,g,h⟩	f
⟨a,b,c,f,e,g,h,d,i⟩	g

Breadth-First Search



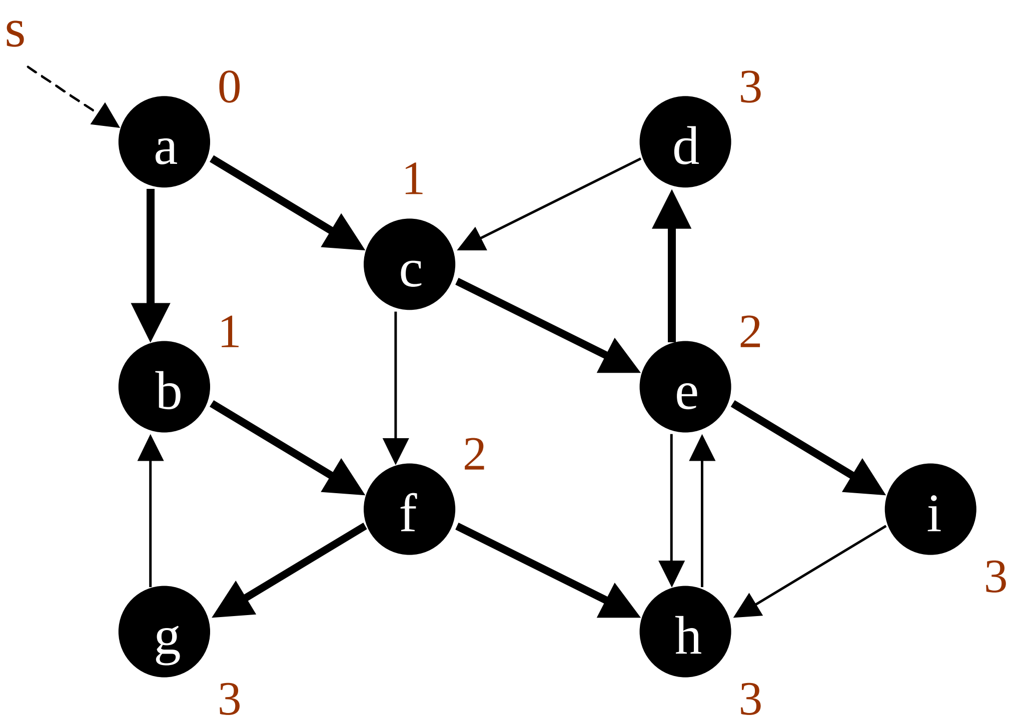
FIFO queue Q	just after processing vertex
⟨a⟩	-
⟨a,b,c⟩	a
⟨a,b,c,f⟩	b
⟨a,b,c,f,e⟩	c
⟨a,b,c,f,e,g,h⟩	f
⟨a,b,c,f,e,g,h,d,i⟩	h

Breadth-First Search



FIFO queue Q	just after processing vertex
⟨a⟩	-
⟨a,b,c⟩	a
⟨a,b,c,f⟩	b
⟨a,b,c,f,e⟩	c
⟨a,b,c,f,e,g,h⟩	f
⟨a,b,c,f,e,g,h,d,i⟩	d

Breadth-First Search



FIFO queue Q	just after processing vertex
⟨a⟩	-
⟨a,b,c⟩	a
⟨a,b,c,f⟩	b
⟨a,b,c,f,e⟩	c
⟨a,b,c,f,e,g,h⟩	f
⟨a,b,c,f,e,g,h,d,i⟩	i

algorithm terminates: all vertices are processed

Breadth-First Search Algorithm

Running time: $O(V+E)$ = considered linear time in graphs

- initialization: $\Theta(V)$
- queue operations: $O(V)$
  - each vertex enqueued and dequeued at most once
  - both enqueue and dequeue operations take $O(1)$ time
- processing gray vertices: $O(E)$
  - each vertex is processed at most once and

    $$\sum_{u \in V} |Adj[u]| = \Theta(E)$$

Theorems Related to BFS

DEF: $\delta(s, v)$ = shortest path distance from s to v

LEMMA 1: for any $s \in V$ & $(u, v) \in E$; $\delta(s, v) \le \delta(s, u) + 1$

For any BFS(G, s) run on $G=(V,E)$

LEMMA 2: $d\,[v] \ge \delta(s, v) \quad \forall v \in V$

LEMMA 3: at any time of BFS, the queue $Q=\langle v_1, v_2, \ldots, v_r \rangle$ satisfies

- $d\,[v_r] \le d\,[v_1] + 1$
- $d\,[v_i] \le d\,[v_{i+1}]$, for $i = 1, 2, \ldots, r-1$

THM1: BFS(G, s) achieves the following

- discovers every $v \in V$ where $s \to v$ (i.e., v is reachable from s)
- upon termination, $d\,[v] = \delta(s, v) \quad \forall v \in V$
- for any $v \ne s$ & $s \to v$; sp$(s, \Pi[v]) \sim (\Pi[v], v)$ is a sp(s, v)

Proofs of BFS Theorems

DEF: shortest path distance $\delta(s, v)$ from s to v

$\delta(s, v)$ = minimum number of edges in any path from s to v

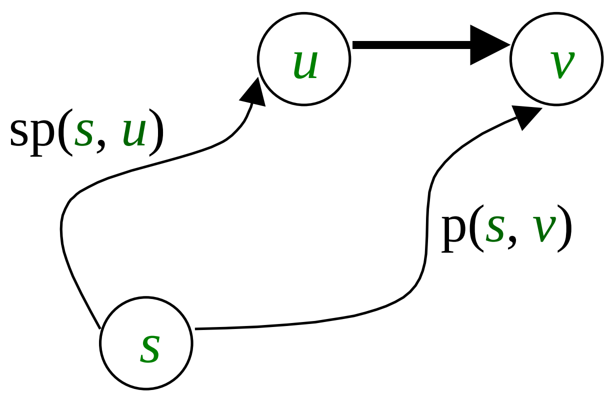
= ∞ if no such path exists (i.e., v is not reachable from s)

L1: for any $s \in V$ & $(u, v) \in E$; $\delta(s, v) \le \delta(s, u) + 1$

PROOF: $s \to u \Rightarrow s \to v$. Then,

consider the path $p(s, v) = sp(s, u) \sim (u, v)$

- $|p(s, v)| = |\, sp(s, u) \,| + 1 = \delta(s, u) + 1$
- therefore, $\delta(s, v) \le |p(s, v)| = \delta(s, u) + 1$



Proofs of BFS Theorems

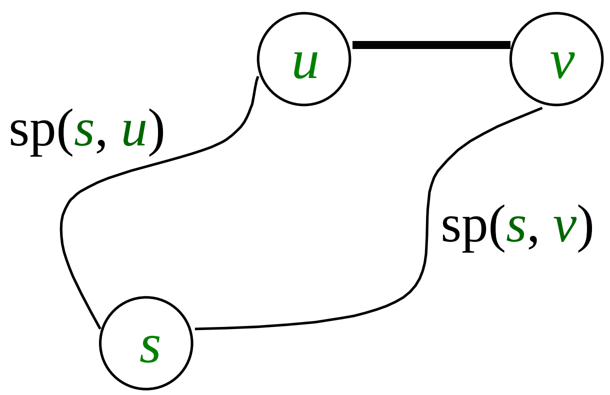
DEF: shortest path distance $\delta(s, v)$ from s to v

$\delta(s, v)$ = minimum number of edges in any path from s to v

L1: for any $s \in V$ & $(u, v) \in E$; $\delta(s, v) \le \delta(s, u) + 1$

C1 of L1: if $G=(V,E)$ is undirected then $(u, v) \in E \Rightarrow (v, u) \in E$

- $\delta(s, v) \le \delta(s, u) + 1$ and $\delta(s, u) \le \delta(s, v) + 1$
- $\Rightarrow \delta(s, u) - 1 \le \delta(s, v) \le \delta(s, u) + 1$ and
  $\delta(s, v) - 1 \le \delta(s, u) \le \delta(s, v) + 1$
- $\Rightarrow \delta(s, u)$ & $\delta(s, v)$ differ by at most 1



Proofs of BFS Theorems

L2: upon termination of BFS(G, s) on G=(V,E);

$d[v] \geq \delta(s, v) \quad \forall v \in V$

PROOF: by induction on the number of ENQUEUE operations

- basis: immediately after 1st enqueue operation ENQ(Q, s): $d[s] = \delta(s, s)$
- hypothesis: $d[v] \geq \delta(s, v)$ for all v inserted into Q
- induction: consider a white vertex v discovered during scanning Adj[u]
- $d[v] = d[u] + 1$ due to the assignment statement

  $\geq \delta(s, u) + 1$ due to the inductive hypothesis since $u \in Q$

  $\geq \delta(s, v)$ due to L1
- vertex v is then enqueued and it is never enqueued again

  $d[v]$ never changes again, maintaining inductive hypothesis

Proofs of BFS Theorems

L3: Let $Q = \langle v_1, v_2, \ldots, v_r \rangle$ during the execution of BFS(G, s), then, $d[v_r] \le d[v_1] + 1$ and $d[v_i] \le d[v_{i+1}]$ for $i = 1, 2, \ldots, r{-}1$

PROOF: by induction on the number of QUEUE operations

- basis: lemma holds when $Q \leftarrow \{s\}$
- hypothesis: lemma holds for a particular Q (i.e., after a certain # of QUEUE operations)
- induction: must prove lemma holds after both DEQUEUE & ENQUEUE operations
- DEQUEUE(Q): $Q = \langle v_1, v_2, \ldots, v_r \rangle \Rightarrow Q' = \langle v_2, v_3, \ldots, v_r \rangle$
  - $d[v_r] \le d[v_1] + 1$ & $d[v_1] \le d[v_2]$ in $Q \Rightarrow$ $d[v_r] \le d[v_2] + 1$ in Q'
  - $d[v_i] \le d[v_{i+1}]$ for $i = 1, 2, \ldots, r{-}1$ in $Q \Rightarrow$ $d[v_i] \le d[v_{i+1}]$ for $i = 2, \ldots, r{-}1$ in Q'

Proofs of BFS Theorems

- ENQUEUE(Q, v): $Q = \langle v_1, v_2, \ldots, v_r \rangle \Rightarrow$
  $Q' = \langle v_1, v_2, \ldots, v_r, v_{r+1} = v \rangle$
  - v was encountered during scanning Adj[u] where $u = v_1$
  - thus, $d[v_{r+1}] = d[v] = d[u] + 1 = d[v_1] + 1 \Rightarrow$
    $d[v_{r+1}] = d[v_1] + 1$ in Q'
  - but $d[v_r] \le d[v_1] + 1 = d[v_{r+1}]$
  - $\Rightarrow d[v_{r+1}] = d[v_1] + 1$ and $d[v_r] \le d[v_{r+1}]$ in Q'

C3 of L3 (monotonicity property):

if: the vertices are enqueued in the order $v_1, v_2, \ldots, v_n$

then: the sequence of distances is monotonically increasing,

i.e., $d[v_1] \le d[v_2] \le \ldots\ldots\ldots\ldots \le d[v_n]$

Proofs of BFS Theorems

THM (correctness of BFS): BFS(G, s) achieves the following on G=(V,E)

- discovers every $v \in V$ where $s \to v$
- upon termination: $d[v] = \delta(s, v) \quad \forall v \in V$
- for any $v \neq s$ & $s \to v$; sp(s, $\Pi[v]$) ~ ($\Pi[v]$, v) = sp(s, v)

PROOF: by induction on k, where $V_k = \{v \in V: \delta(s, v) = k\}$

- hypothesis: for each $v \in V_k$, $\exists$ exactly one point during execution of BFS at which color[v] $\leftarrow$ GRAY, $d[v] \leftarrow k$, $\Pi[v] \leftarrow u \in V_{k-1}$, and then ENQUEUE($Q$, v)
- basis: for $k = 0$ since $V_0 = \{s\}$; color[s] $\leftarrow$ GRAY, $d[s] \leftarrow 0$ and ENQUEUE(Q, s)
- induction: must prove hypothesis holds for each $v \in V_{k+1}$

Proofs of BFS Theorems

Consider an arbitrary vertex $v \in V_{k+1}$, where $k \geq 0$

- monotonicity (L3) + $d\,[v] \geq k + 1$ (L2) + inductive hypothesis

  $\Rightarrow v$ must be discovered after all vertices in V_k were enqueued
- since $\delta(s, v) = k + 1$, $\exists u \in V_k$ such that $(u, v) \in E$
- let $u \in V_k$ be the first such vertex grayed (must happen due to hyp.)
- $u \leftarrow$ head(Q) will be ultimately executed since BFS enqueues every grayed vertex
  - v will be discovered during scanning Adj[u]

    color[v]=WHITE since v isn't adjacent to any vertex in V_j for $j<k$
  - color[v] $\leftarrow$ GRAY, $d\,[v] \leftarrow d\,[u] + 1$, $\Pi[v] \leftarrow u$
  - then, ENQUEUE(Q, v) thus proving the inductive hypothesis

To conclude the proof

- if $v \in V_{k+1}$ then due to above inductive proof $\Pi[v] \in V_k$
  - thus sp(s, $\Pi[v]$) ~ ($\Pi[v]$, v) is a shortest path from s to v

Theorems Related to BFS

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Breadth-First Tree Generated by BFS

LEMMA 4: predecessor subgraph $G_\Pi=(V_\Pi, E_\Pi)$ generated by BFS(G, s) , where $V_\Pi=\{v \in V: \Pi[v] \neq \text{NIL}\}\cup\{s\}$ and

$$E_\Pi=\{(\Pi[v],v) \in E: v \in V_\Pi-\{s\}\}$$

is a breadth-first tree such that

- V_Π consists of all vertices in V that are reachable from s
- $\forall v \in V_\Pi$, unique path p(v, s) in G_Π constitutes a sp(s, v) in G

Prints out vertices on a $s\rightarrow v$ shortest path

PRINT-PATH(G, s, v)

if $v = s$ **then** print s
else if $\Pi[v]$ = NIL **then**
print no "$s\rightarrow v$ path"
else
PRINT-PATH(G, s, $\Pi[v]$)
print v

Breadth-First Tree Generated by BFS

BFS(G,*a*) terminated	BFT generated by BFS(G,*a*)