Lecture 15

Graph Searching:

Depth-First Search and Topological Sort
Depth-First Search

- Graph $G = (V,E)$ directed or undirected
- Adjacency list representation
- **Goal**: Systematically explore every vertex and every edge
- **Idea**: search deeper whenever possible
  - Using a LIFO queue (Stack; FIFO queue used in BFS)
Depth-First Search

• Maintains several fields for each $v \in V$
• Like BFS, colors the vertices to indicate their states. Each vertex is
  – Initially white,
  – grayed when discovered,
  – blackened when finished
• Like BFS, records discovery of a white $v$ during scanning $\text{Adj}[u]$ by $\pi[v] \leftarrow u$
Depth-First Search

• Unlike BFS, predecessor graph $G_\pi$ produced by DFS forms spanning forest

• $G_\pi = (V, E_\pi)$ where

  $$E_\pi = \{ (\pi[v], v) : v \in V \text{ and } \pi[v] \neq \text{NIL} \}$$

• $G_\pi =$ depth-first forest (DFF) is composed of disjoint depth-first trees (DFTs)
Depth-First Search

- DFS also timestamps each vertex with two timestamps:
  - $d[v]$: records when $v$ is first discovered and grayed
  - $f[v]$: records when $v$ is finished and blackened
- Since there is only one discovery event and finishing event for each vertex we have $1 \leq d[v] < f[v] \leq 2|V|$
Depth-First Search

**DFS(G)**

for each \( u \in V \) do

- \( \text{color}[u] \leftarrow \text{white} \)
- \( \pi[u] \leftarrow \text{NIL} \)

\( time \leftarrow 0 \)

for each \( u \in V \) do

- if \( \text{color}[u] = \text{white} \) then
  - \( \text{DFS-Visit}(G, u) \)

**DFS-Visit(G, u)**

- \( \text{color}[u] \leftarrow \text{gray} \)
- \( d[u] \leftarrow \text{time} \leftarrow \text{time} + 1 \)

for each \( v \in \text{Adj}[u] \) do

  - if \( \text{color}[v] = \text{white} \) then
    - \( \pi[v] \leftarrow u \)
    - \( \text{DFS-Visit}(G, v) \)

- \( \text{color}[u] \leftarrow \text{black} \)
- \( f[u] \leftarrow \text{time} \leftarrow \text{time} + 1 \)
Depth-First Search

- Running time: $\Theta(V+E)$
- Initialization loop in $\text{DFS}$: $\Theta(V)$
- Main loop in $\text{DFS}$: $\Theta(V)$ exclusive of time to execute calls to $\text{DFS-Visit}$
- $\text{DFS-Visit}$ is called exactly once for each $v \in V$ since
  - $\text{DFS-Visit}$ is invoked only on white vertices and
  - $\text{DFS-Visit}(G, u)$ immediately colors $u$ as gray
- For loop of $\text{DFS-Visit}(G, u)$ is executed $|\text{Adj}[u]|$ time
- Since $\Sigma |\text{Adj}[u]| = E$, total cost of executing loop of $\text{DFS-Visit}$ is $\Theta(E)$
Depth-First Search: Example
Depth-First Search: Example
Depth-First Search: Example
Depth-First Search: Example
Depth-First Search: Example
Depth-First Search: Example
Depth-First Search: Example
Depth-First Search: Example

\[
\begin{array}{ccc}
\text{s} & \text{x} & \text{z} \\
\text{w} & \text{y} & \text{u} \\
\text{1} & \text{2} & \text{3}  \\
\end{array}
\]
Depth-First Search: Example
Depth-First Search: Example
Depth-First Search: Example

- The graph represents a depth-first search process.
- The nodes are labeled with letters (s, t, u, v, w, x, y, z).
- The edges connect these nodes, showing the search progression.
Depth-First Search: Example

\[\begin{align*}
\text{Depth} & \quad \text{First Search: Example} \\
\begin{array}{cccccc}
\text{s} & \text{w} & \text{x} & \text{y} & \text{z} \\
\text{s} & \text{t} & \text{u} & \text{v} & \text{w} & \text{x} & \text{y} \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\end{align*}\]
Depth-First Search: Example
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Depth-First Search: Example
Depth-First Search: Example

**DFS(G) terminated**

**Depth-first forest (DFF)**