Comparing Systems via Simulation
(Part 5)

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Comparing Systems via Simulation
(Outline)

• Introduction
• Comparison Problems
  – Comparing Two Systems
  – Screening Problems
  – Selecting the Best
  – Comparison with a Fixed Performance
Introduction

• Simulation experiments are usually performed to compare some alternative solutions/designs.
• Method that is appropriate depends on the type of comparison (problem) and output data.

Introduction
(Statistical Methods)

• Statistical methods are applicable to computer simulations
• Since the important assumptions of statistics can usually be satisfied approximately.
Introduction (Assumptions)

• Normally distributed data can be secured by batching large number of outputs.
• Independence can be obtained by controlling random-number assignments.
• Multiple-stage sampling is feasible because a subsequent stage can be initialized simply;
  – By retaining the final random number seed from the preceding stage or
  – By regenerating the entire sample.

Comparison Problems

• In this lesson, we will examine 4 classes of problems.
• Objective is to select the correct answer to the problem with a high probability.
Comparison Problems

- Comparing Two Systems:
  - To compute the difference in expected performance of two systems.

- Screening problems:
  - To compare substantial number of designs in order to eliminate clearly inferior (not successful) performers.

- Selecting the best:
  - To find the system with the largest or smallest performance measure.

- Comparison with a fixed performance:
  - To find the best system, provided that its performance exceeds a known, fixed performance standard.

Comparison Problems (Background)

- Compare $k$ different system (design points) via simulation.
- Let $Y$ be a random variable that represents the output.
- Let $Y_{ij}$ represents the $j^{th}$ simulation output (replication or batch) of $i^{th}$ system.
- Let $n_i$ be the number of replications/batches from $i^{th}$ system.
- Simulations of design points are either independent or using common random numbers.
Comparison Problems
(Background)

• Let $\overline{Y}_i$ be the sample mean of $i^{th}$ system.
  $$\overline{Y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}$$

• Let $S^2_i$ be the sample variance of $i^{th}$ system.
  $$S^2_i = \frac{1}{n_i-1} \sum_{j=1}^{n_i} (Y_{ij} - \overline{Y}_i)^2$$

• Let $S^2$ be the pooled sample variance of all systems.
  $$S^2 = \frac{1}{k} \sum_{i=1}^{k} S^2_i$$

Comparing Two Systems

• The goal is to compute the difference in expected performance of two systems
  – In order to determine whether one is better or they are practically equivalent.
Comparing Two Systems

- If systems 1 & 2 are simulated with \( n \) replications/batches using independent random number streams then
  - The difference between system 1 and 2 with \((1-\alpha)\) confidence interval is:

\[
\mu_1 - \mu_2 \text{ range limits } = (\bar{Y}_1 - \bar{Y}_2) \pm t^* \sqrt{\frac{S_1^2 + S_2^2}{n}}
\]

\( t^* = t_{\alpha/2, n-1} = (1-\alpha/2) \) probability value for t-distribution with \( n-1 \) degrees of freedom

Comparing Two Systems

- With \((1-\alpha)\) confidence, if the range limits are;
  - Both positive then ( e.g. \([3,9]\) )
    - Performance metric of system 1 is greater than system 2,
  - Both negative then ( e.g. \([-9,-2]\) )
    - Performance metric of system 1 is smaller than system 2,
  - In different sides of zero then ( e.g. \([-1,3]\) )
    - Systems are equivalent.
Screening Problems

• The goal is to compare substantial number of system designs in order to:
  – Group those with similar performance, and
  – Eliminate clearly inferior performers for examining high performers in more details.
• For instance,
  – 20 potential system designs for a company is produced.
  – Response time is the performance measure.
  – You would like to reduce the number of potential designs using a plot study before a more detailed study is performed.

Screening Problems (Techniques)

• Multiple Comparison Approach
• Subset Selection Approach
Screening Problems (Multiple Comparison Approach)

• Approaches the screening problem by forming simultaneous confidence intervals on parameters $\mu_i - \mu_j$ for all $i \neq j$.
• $k(k-1)/2$ confidence intervals will be formed.
• Indicate magnitude & direction of the difference between each pair of alternatives.

Screening Problems (Multiple Comparison Approach)

• Simulate systems;
  – With independent random number streams,
• Compute;
  – Sample means ($\bar{Y}_i$), and
  – Pooled sample variance ($S^2$).
Screening Problems
(Multiple Comparison Approach)

- Simultaneous confidence intervals of \( \mu_i - \mu_l \) for all \( i \neq l \) (Tukey’s procedure):

\[
(\bar{Y}_i - \bar{Y}_l) \pm \frac{S}{\sqrt{2}} \sqrt{\frac{Q_{\alpha, k, \nu}}{n_i} + \frac{1}{n_l}}
\]

- 1-\( \alpha \) quantile of Studentized range distribution with parameter \( k \) and \( \nu \) (degrees of freedom)

\[ \nu = \sum_{i=1}^{k} (n_i - 1) \]

- Difference between sample means
- Pooled standard deviation

Studentized Range Distribution Table
Screening Problems
(Multiple Comparison Approach)

• Suppose that:
  – k = 4 system architectures.
  – n = 6 replications are obtained for each.

\[
\begin{align*}
\bar{Y}_1 &= 72 \\
\bar{Y}_2 &= 85 \\
\bar{Y}_3 &= 76 \\
\bar{Y}_4 &= 62 \\
S^2 &= 100.9
\end{align*}
\]

\[
\bar{Y}_i = \frac{1}{n} \sum_{j=1}^{n} Y_{ij}
\]

Response time is performance metric (smaller is better)

Screening Problems
(Multiple Comparison Approach)

• Suppose that:
  – Objective is to eliminate system architectures with low performance (high response time) with 0.95 confidence.

  – So compare pairs:
    • 1 and 2
    • 1 and 3
    • 1 and 4
    • 2 and 3
    • 2 and 4
    • 3 and 4
Screening Problems 
(Multiple Comparison Approach)

• Suppose that:
  – We compare system 2 and 4:

\[ \nu = \sum_{i=1}^{4} (n_i-1) = (6-1) + (6-1) + (6-1) + (6-1) = 20 \]

\[ (\bar{Y}_2 - \bar{Y}_4) \pm \frac{Q_{1-0.05}}{\sqrt{2}} \sqrt{\frac{1}{n_1} + \frac{1}{n_1}} \]

\[ (85 - 62) \pm \frac{3.96}{\sqrt{2}} \sqrt{\frac{1}{6} + \frac{1}{6}} = 23 \pm 16 \approx [7, 39] \]

With 95% probability, system 2 is different from system 4 in range 7 to 39. Since range of system 2 is greater than 4, and range does not contain 0, system 2 can be eliminated (because shorter response time is preferable).

Screening Problems 
(Subset Selection Approach)

• Approaches the screening problem by producing a subset of designs that contains the best system with a probability \(1-\alpha\).

• Applicable in cases when data from competing designs are:
  – Independent (different random numbers),
  – Balanced \((n_1=n_2=\ldots=n_k=n)\), and
  – Normally distributed with a common variance.
Screening Problems
(Subset Selection Approach)

• Simulate systems;
  – With independent random number streams,
  – With equal number of replications/batches.

• Compute;
  – Sample means ($\bar{Y}_i$), and
  – Pooled sample variance ($S^2$).

Screening Problems
(Subset Selection Approach)

• Include $i^{th}$ design in the subset if;

$$\bar{Y}_i \leq \min_{1 \leq j \leq k} (Y_j) + g S \sqrt{\frac{2}{n}}$$  
  If smaller performance metric is better

$$\bar{Y}_i \geq \max_{1 \leq j \leq k} (Y_j) - g S \sqrt{\frac{2}{n}}$$  
  If greater performance metric is better

$g = T^{(\alpha)}_{k-1, k(n-1)}$ is a critical value from a multivariate t-distribution.
Screening Problems
(Subset Selection Approach)

• Suppose that:
  – \( k = 4 \) system architectures.
  – \( n = 6 \) replications are obtained for each.

\[
\begin{align*}
\bar{Y}_1 &= 72 \\
\bar{Y}_2 &= 85 \\
\bar{Y}_3 &= 76 \\
\bar{Y}_4 &= 62 \\
S^2 &= 100.9
\end{align*}
\]

\[
\begin{align*}
\bar{Y}_i &= \frac{1}{n} \sum_{j=1}^{n} Y_{ij} \\
S_i^2 &= \frac{1}{n-1} \sum_{j=1}^{n} (Y_{ij} - \bar{Y}_i)^2 \\
S^2 &= \frac{1}{k} \sum_{i=1}^{k} S_i^2
\end{align*}
\]

Response time is performance metric
(smaller is better)
Screening Problems
(Subset Selection Approach)

• Suppose that:
  – Objective is to produce a subset including the best system with 0.95 confidence.
  – Smaller response time is preferred so we use:

\[
Y_i \leq \min_{1 \leq j \leq k} (Y_j) + g \sqrt{\frac{2}{n}}
\]

\[
Y_i \leq \min(72, 85, 76, 62) + T(0.05)_{10}, 5.8 = 62 + 2.19 \times 5.8 = 74.7
\]

– Select system 1 and 4 (72 ≤ 74.7 and 62 ≤ 74.7)

Selecting the Best

• The goal is to find the system with the largest or smallest performance measure.
• For instance,
  – 20 potential system designs for a company is produced.
  – Number of jobs processed per hour is the performance measure.
  – Differences of less than about 5 jobs are considered practically equivalent.
  – You reduced the number of alternatives to 4 with a plot study.
  – Now you would like to determine the best one among 4 with a more detailed study.
Selecting the Best (Techniques)

- Multiple Comparison Approach
  - Procedure Rinott + MCB
  - Procedure NM + MCB
  - Procedure Bonferroni + MCB
- Multinomial Selection Approach
  - Procedure BEM
  - Procedure BG

Selecting the Best (Multiple Comparison Approach)

- In stochastic simulations, correct selection can never be guaranteed with certainty.
- A solution offered by Multiple Comparison integrated with Indifference-zone selection is;
  - To guarantee to select the best system with high probability,
    - Whenever it is at least a user-specified amount better than others.
- This practically significant difference is called Indifference-zone ($\delta$) (e.g. $\delta = 5$ jobs).
Selecting the Best (Multiple Comparison Approach)

• If some system happens to be within δ of the best,
  – Then these are considered to be practically equivalent,
  – And probability of selecting one of the good systems is at least 1-α,
  – So any of them can be chosen considering other important metrics (e.g. cost).

Selecting the Best (Multiple Comparison Approach)

• Uses Indifference-zone.
• Approaches the screening problem by forming simultaneous confidence intervals on parameters;
  – Μ_i - max_l ≠ i Μ_l (greater is better) or
  – Μ_i - min_l ≠ i Μ_l (smaller is better) for all i.
• Bound the difference between expected performance of each system and the best of the others, with probability 1-α.
Selecting the Best
(Multiple Comparison Approach)

- Performs multiple comparisons with the best (MCB).
- Combines indifference-zone selection and MCB.
- Provides information about how close each of the inferior systems is to the best,
  - Which is useful if secondary criteria such as cost, ease of installation are not reflected to the performance measure.
  - e.g. $\delta = 5$ jobs difference is equivalent.

Selecting the Best
(Proc. Rinott+MCB (Independent Sampling))

- Takes observations in two stages:
  - First stage uses $n_0 \geq 2$ (10 recommended) independent observations from each system
    - To estimate marginal variance.
  - Second stage uses marginal variance
    - To compute additional number of observations required to meet the indifference-zone probability.
Selecting the Best
( Proc. Rinott+MCB (Independent Sampling) )

• Specify;
  – $\delta$ (indifference-zone)
  – $\alpha$ (confidence interval probability)
  – $n_0$ (first-stage sample size).

• Take $n_0$ independent replications/batches from each of the $k$ systems.

Selecting the Best
( Proc. Rinott+MCB (Independent Sampling) )

• Compute marginal sample variance for all $i$:

$$S_i^2 = \frac{1}{n_0-1} \sum_{j=1}^{n_0} (Y_{ij} - \bar{Y}_i)^2$$

• Compute final sample size for each system:

$$N_i = \max ( n_0, \left( \frac{h_\alpha n_0 S_i}{\delta} \right)^2 )$$

Find $h_\alpha n_0$ from Procedure Rinott+MCB table.
Selecting the Best
( Proc. Rinott+MCB (Independent Sampling) )

• For each system i,
  – Take $N_i - n_0$ additional (or restart all) observations independently of the first-stage.
  – Compute overall sample mean:

$$ \bar{Y}_i = \frac{1}{N_i} \sum_{j=1}^{N_i} Y_{ij} $$

• Select system with;
  – Largest $\bar{Y}_i$ (greater is better) or
  – Smallest $\bar{Y}_i$ (smaller is better) as the best.
Selecting the Best
(Proc. Rinott+MCB (Independent Sampling))

• Simultaneously form MCB confidence interval for each system $i$:
  – If greater is better:
    $\min(0, Y_i - \max_{i \neq l} Y_l - \delta), \max(0, Y_i - \max_{i \neq l} Y_l + \delta)$
  – If smaller is better:
    $\min(0, Y_i - \min_{i \neq l} Y_l - \delta), \max(0, Y_i - \min_{i \neq l} Y_l + \delta)$

• If range of a system $i$ is not on one side of the zero, it is an equivalent of best (within $\delta$).

Selecting the Best
(Proc. NM+MCB (Common Random Numbers))

• In Rinott + MCB,
  – Systems are simulated independently.

• However, under fairly conditions, assigning common random numbers (CRN) to simulation of each system decreases variances of estimates.

• NM+MCB uses CRN.
Selecting the Best
(Proc. NM+MCB (Common Random Numbers))

• Similarly use $\delta$, $\alpha$ and $n_0$.
• Take $n_0$ replications/batches from each of the $k$ systems using CRN across systems.

• Compute approximate sample variance:

$$S^2 = \frac{2}{(k-1)(n_0-1)} \sum_{i=1}^{k} \sum_{j=1}^{n_0} (\bar{Y}_{i,j} - \bar{\bar{Y}}_i + \bar{Y}_{..})^2$$

$\bar{Y}_{i,j}$ means average of all $i$, $\bar{\bar{Y}}_i$ means average of all $i$ and $j$.

• Compute final sample size for all systems:

$$N = \max \left( n_0, \left[ \frac{g S^2}{\delta} \right]^2 \right)$$

Find $g = \frac{T_{(0.1)}}{k-1,(k-1)(n_0-1)}$ from Multivariate t-distribution table.
Selecting the Best
(Proc. NM+MCB (Common Random Numbers))

• For each system i,
  – Take $N - n_0$ additional (or restart all) observations using CRN across systems.
  – Compute overall sample mean:
    \[
    \bar{Y}_i = \frac{1}{N} \sum_{j=1}^{N} Y_{ij}
    \]

• Select system with;
  – Largest $\bar{Y}_i$ (greater is better) or
  – Smallest $\bar{Y}_i$ (smaller is better) as the best.

Selecting the Best
(Proc. NM+MCB (Common Random Numbers))

• Simultaneously from MCB confidence intervals as in Rinott + MCB.
Selecting the Best
(Proc. Bonferroni+MCB (CRN))

• NM + MCB works under a complex set of conditions.
• Conferroni + MCB works under more general conditions.
• But, tends to require more observations, especially when k is large.

Selecting the Best
(Proc. Bonferroni+MCB (CRN))

• Similarly use $\delta$, $\alpha$ and $n_0$.
• Take $n_0$ replications/batches from each of the $k$ systems using CRN across systems.
Selecting the Best
(Proc. Bonferroni+MCB (CRN))

- Compute sample variances of differences for all $i \neq l$:

$$S^2_i = \frac{1}{n_0 - 1} \sum_{j=1}^{n_0} [(Y_{ij} - \overline{Y}_i) - (\overline{Y}_i - \overline{Y}_l)]^2$$

- Compute final sample size for all systems:

$$N = \max (n_0, \max_{i \neq l} \left( \frac{fS_{il}}{\delta} \right)^2)$$

Find $t = t_{(\alpha/2, k-1)}$ from t-distribution table.

Selecting the Best
(Proc. Bonferroni+MCB (CRN))

- For each system $i$,
  - Take $N - n_0$ additional (or restart all) observations using CRN across systems.
  - Compute overall sample mean:

$$\overline{Y}_i = \frac{1}{N} \sum_{j=1}^{N} Y_{ij}$$

- Select system with
  - Largest $\overline{Y}_i$ (greater is better) or
  - Smallest $\overline{Y}_i$ (smaller is better) as the best.
Selecting the Best
( Proc. Bonferroni+MCB (CRN) )

• Simultaneously from MCB confidence intervals as in Rinott + MCB.

Selecting the Best
( Techniques )

• Multiple Comparison Approach
  – Procedure Rinott + MCB
  – Procedure NM + MCB
  – Procedure Bonferroni + MCB

• Multinomial Selection Approach
  – Procedure BEM
  – Procedure BG
Selecting the Best
(Multinomial Selection Approach)

• Solution offered by Multinomial Selection is;
  – To select the best system with probability 1-\(\alpha\)

• Whenever the ratio of selecting the best to the second best \(p_i\) is greater than a user-specified constant.

• This practically significant smallest ratio worth detecting is called \textit{Indifference-constant} (\(\theta>1\)) (e.g. \(\theta = 1.2\)).

\[
\theta = \min \frac{p_{\text{best}}}{p_{\text{second best}}} \text{ required}
\]

Selecting the Best
(Procedure BEM)

• Specify;
  – \(\theta\) (indifference-constant)
  – \(\alpha\) (confidence interval probability)

• Take a random sample of \(n\) independent multinomial observations from each of the \(k\) systems,
  – Where \(n\) is found from Multinomial procedure table using \(\alpha\), \(\theta\) and \(k\).
Selecting the Best
(Procedure BEM)

- A random sample is taken from each of the $k$ systems for each replication.

So we have a matrix;

\[
\begin{bmatrix}
Y_{11}, Y_{21}, Y_{31}, \ldots, Y_{k1} \\
Y_{12}, Y_{22}, Y_{32}, \ldots, Y_{k2} \\
\vdots \\
Y_{1n}, Y_{2n}, Y_{3n}, \ldots, Y_{kn}
\end{bmatrix}
\]

replications

system designs
Selecting the Best (Procedure BEM)

• From \( Y \), determine multinomial observations \( X \).

\[
\begin{bmatrix}
Y_{11}, Y_{21}, Y_{31}, \ldots, Y_{k1} \\
Y_{11}', Y_{21}', Y_{31}', \ldots, Y_{k1}' \\
Y_{12}, Y_{22}, Y_{32}, \ldots, Y_{k2} \\
\vdots \\
Y_{1n}, Y_{2n}, Y_{3n}, \ldots, Y_{kn}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
X_{11}, X_{21}, X_{31}, \ldots, X_{k1} \\
X_{11}', X_{21}', X_{31}', \ldots, X_{k1}' \\
X_{12}, X_{22}, X_{32}, \ldots, X_{k2} \\
\vdots \\
X_{1n}, X_{2n}, X_{3n}, \ldots, X_{kn}
\end{bmatrix}
\]

replications

\[X_{ij} = \begin{cases} 
1 & \text{if } Y_{ij} > \max_{l \neq i} Y_{lj} \\
0 & \text{otherwise}
\end{cases}\]

On \( j \)th replication, if system \( i \) is best, set \( X_{ij} = 1 \) else \( X_{ij} = 0 \).

Thus there will be only a single 1 on each row.

Selecting the Best (Procedure BEM)

• From \( X \), determine \( W \).

\[
\begin{bmatrix}
X_{11}, X_{21}, \ldots, X_{k1} \\
X_{11}', X_{21}', \ldots, X_{k1}' \\
X_{12}, X_{22}, \ldots, X_{k2} \\
\vdots \\
X_{1n}, X_{2n}, \ldots, X_{kn}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
W_1, W_2, W_3, \ldots, W_k
\end{bmatrix}
\]

\[W_i = \sum_{j=1}^{n} X_{ij}\]

The number of times system \( i \) was the best

• Select the design having largest \( W_i \) as the best (one with highest probability).

• If there are equal \( W_i \)'s, pick any of them
Selecting the Best (Procedure BG)

• Procedure BEM, uses a fixed number of replications to select the best.
• This may sometimes be inefficient.
• Procedure BG;
  – Uses a more efficient but complex procedure, and
  – Stops when one design is sufficiently ahead of the others.

Selecting the Best (Procedure BG)

• Specify;
  – $\theta$ (indifference-constant)
  – $\alpha$ (confidence interval probability)
• To select the best system among $k$ systems.
• Uses incremental number of observations.
• Upper limit of observations, the truncation number ($n_T$), is found from Multinomial procedure table using $\alpha$, $\theta$ and $k$. 
Selecting the Best (Procedure BG)

- The method advances stage by stage.
- A stage contains one observation from each system totally making \( k \) observations.

\[
\begin{bmatrix}
Y_{11}, & Y_{21}, & Y_{31}, & \ldots, & Y_{k1} \\
Y_{12}, & Y_{22}, & Y_{32}, & \ldots, & Y_{k2} \\
\vdots & & & \ddots & \vdots \\
Y_{1r}, & Y_{2r}, & Y_{3r}, & \ldots, & Y_{kr}
\end{bmatrix}
\]

A stage advance stage by stage until stopping criteria is satisfied.

Selecting the Best (Procedure BG)

- At the \( m^{th} \) stage of observations,
  - We determine \( X_m \) from \( Y \):

\[
\begin{bmatrix}
Y_{1m}, & Y_{2m}, & Y_{3m}, & \ldots, & Y_{km}
\end{bmatrix}
\rightarrow
X_m = \begin{bmatrix}
X_{1m}, & X_{2m}, & X_{3m}, & \ldots, & X_{km}
\end{bmatrix}
\]

\[
X_i = \begin{cases} 
1 & \text{if } Y_{ij} = \max_{i \neq j} Y_{ij} \\
0 & \text{otherwise}
\end{cases}
\]
Selecting the Best
(Procedure BG)

• Then compute $W_m$ from $X$:

$$
W_m = \begin{bmatrix}
W_{1m}, & W_{2m}, & \ldots, & W_{nm}
\end{bmatrix}
$$

- $W_m = \sum_{j=1}^{n} X_{ij}$
- $W_m$ is the index of $i$th smallest $W_{jm}$

• And sort $W_m$ ascending:

$W_{[1]m} \leq W_{[2]m} \leq W_{[3]m} \leq \ldots \leq W_{[k]m}$

Selecting the Best
(Procedure BG)

• Compute $Z_m$:

$$
Z_m = \sum_{i=1}^{k-1} (1/\theta) W_m^i - W_{jm}
$$

• Stop sampling when any of the following occur:

$Z_m \leq \alpha/(1-\alpha)$

$m = n_T$

$(W_{[k]m} - W_{[k-1]m}) \geq (n_T - m)$

• Take system $i$ with the largest $W_i$ as the best.
Comparison With a Fixed Performance

• To find the best system, provided that its performance exceeds a known, fixed performance standard ($\mu_0$).
• For instance,
  – There are 4 potential risky investment strategies for a company.
  – But it is also possible to get a known, fixed return if the money is deposited in a bank.
  – Therefore, we would like to choose none of the strategies unless its expected return is larger than the fixed return.

Comparison With a Fixed Performance (Procedure BT)

• Takes observations in two stages:
  – First stage uses $n_0 \geq 2$ (10 recommended) independent observations from each system
    • To estimate marginal variance.
  – Second stage uses marginal variance
    • To compute number of observations required to meet the probability requirement.
Comparison With a Fixed Performance (Procedure BT)

• Specify;
  – $\mu_0$ (fixed performance standard)
  – $\delta$ (indifference-zone)
  – $\alpha$ (confidence interval probability)
  – $n_0$ (first-stage sample size).
• Take $n_0$ independent replications/batches from each of the $k$ systems.

Comparison With a Fixed Performance (Procedure BT)

• Compute estimated sample variance:

$$S^2 = \frac{\sum_{i=1}^{k} \sum_{j=1}^{n_0} (Y_{ij} - \bar{Y}_i)^2}{k(n_0-1)}$$

• Compute final sample size for all systems:

$$N = \max \left( n_0, \left\lceil \left( \frac{gS}{\delta} \right)^2 \right\rceil \right)$$

Find $g$, $h$ from Procedure BT table.
Comparison With a Fixed Performance (Procedure BT)

- For each system i,
  - Take $N - n_0$ additional (or restart all) independent observations.
  - Compute overall sample mean:
    \[
    \bar{Y}_i = \frac{1}{N} \sum_{j=1}^{N} Y_{ij}
    \]
Comparison With a Fixed Performance 
( Procedure BT )

• If greatest performance measure is better:
  
  If \( \max \bar{Y}_i > \mu_0 + h\delta/g \) then
  Select the strategy \( i \) as best,
  Otherwise retain the standard as best.

• If smallest performance measure is better:
  
  If \( \min \bar{Y}_i < \mu_0 - h\delta/g \) then
  Select the strategy \( i \) as best,
  Otherwise retain the standard as best.