Graph Algorithms

CS 202 – Fundamental Structures of Computer Science II

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Computer Engineering Department

Motivation

- We will now see how several problems in Graph Theory can be modeled and solved using Graph algorithms
- Many real life problems can be modeled with graphs
- We will give algorithms that solve some common graph problems
- We will see how choice of data structures is important in increasing the performance of algorithms

Definitions

A *graph* $G = (V, E)$ consists of a set of *vertices*, V, and a set of *edges*, E.

Each edge is a *pair* (u, w), where $u, w \in E$.

Edges are also called as *arcs*

If the pair is ordered, then the graph is *directed (digraph)*, otherwise it is *unidirected* graph.

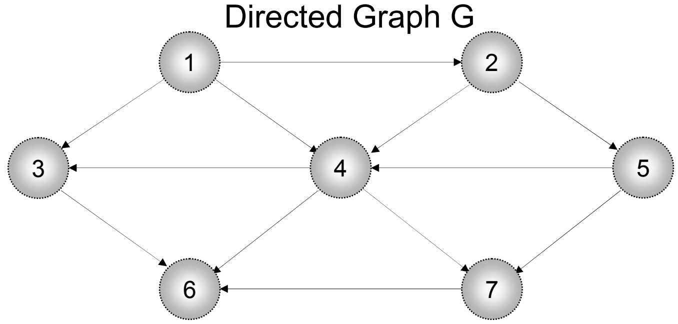
Vertex w is *adjascent* to v if and only if $(v, w) \in E$.

In an undirected graph with edge (v, w), and hence (w, v), w is adjascent to v, and v is adjascent to w.

An edge may have a third component which is called *weight* or *cost*.

Example

Directed Graph G



G = (V,E)
V = {1,2,3,4,5,6,7}
E = {(1,2),(1,4),(1,3),(2,4),(2,5),(3,6),(4,3),(4,6),(4,7),(5,4),(5,7),(7,6)
3 is *adjacent* to 1, but 1 is not adjacent to 3.

Definitions

A *path* in a graph is a sequence of vertices $w_1, w_2, ..., w_N$, such that $(w_i, w_{i+1}) \in E$ for $1 \leq i < N$.

The *length* of such a path is the number of edges on the path, which is equal to $N - 1$.

There can be a path from a vertes to itself. If this path contains no edges, than the path length is 0.

If graph contains a path from a vertex v to itself, then we say that the graph contains a *loop*.

A *simple path* is a path that all vertices are distinct, except that the first and last vertex could be the same.

A *cycle* in a directed graph is a path at least 1 such that $w_1 = w_N$. For undirected graphs, we requires that the edges are distinct.

A directed graph is *acylic* if it has no cycles. Such a graps is also referred as *DAG*.

An undirected graph is *connected* if there is a path from every vertext to every other vertex.

If such a graph is directed, then it is said that it is *strongly connected*.

If a directed graph is not strongly connected, but the underlying graph (without directions) is connected, then the graph is said to be *weakly connected*.

A *complete graph* is a graph in which there is an edge between every pair of vertices.

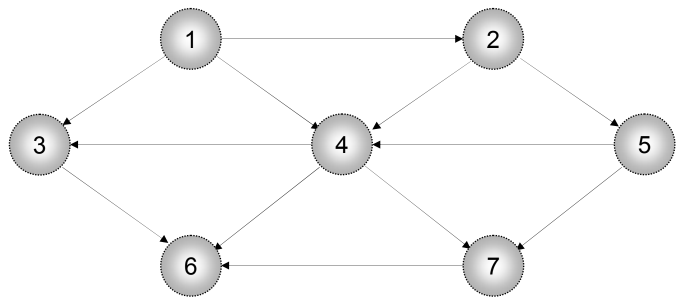
Representation of Graphs

- We will consider representation of directed graphs. Undirected graphs are similarly represented.
- Support we number the vertices, starting from one.
- There are two methods
  1. *Adjacency matrix* representation
  2. *Adjacency list* representation

Some Common Graph Problems and Algorithms

- Topological Sort
- Shortest-Path Algorithms
- Network Flow Problems
- Minimum Spanning Tree
- Depth First Search and Applications

Representation of Graphs



- We will represent the graph above as an example.

Adjacency Matrix Representation

- Use a two-dimensional array A.
- For each edge (u,v), set A[u][v] to true, otherwise to false.
- If the edge has a weight (cost) associated with it than we set the A[u][v] equal to the weight.
  - Use a very large or very small weight as a *sentinel* to indicate nonexistent edges.
- Space requirement $O(|V|^2)$
- Good if the graph is *dense*.

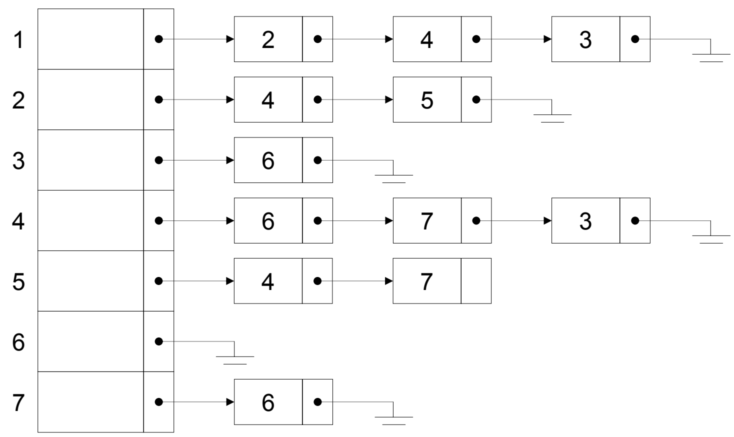
Adjacency Matrix Representation

	1	2	3	4	5	6	7
1	0	1	1	1	0	0	0
2	0	0	0	1	1	0	0
3	0	0	0	0	0	1	0
4	0	0	1	0	0	1	1
5	0	0	0	1	0	0	1
6	0	0	0	0	0	0	0
7	0	0	0	0	0	1	0

Adjacency List Representation

- For each vertex, keep a list of all adjacent vertices.
- Space requirement is O(|E| + |V|)
  - Linear in the size of the graph.
- Standard way to represent graphs
- Undirected graphs can be similarly represented; each edge (u,v) appears in two lists.
  - Space usage doubles.

Adjacency List Representation



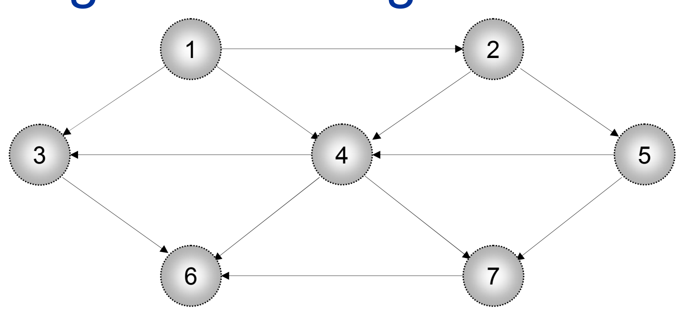
Some Common Graph Problems and Algorithms

- Topological Sort
- Shortest-Path Algorithms
- Network Flow Problems
- Minimum Spanning Tree
- Depth First Search and Applications

Topological Sort

- A *topological sort* is an ordering of vertices in a directed acyclic graph, such that *if there is a path from v_i to v_j, then v_j appears after v_i in ordering.*
  - A topological ordering is not possible if the graph contains cycles, since for two vertices v and w on a cycle, *v precedes w* and *w precedes v*.
- Ordering is not necessarily unique.
  - $v_1, v_2, v_5, v_4, v_3, v_7, v_6$ (one ordering)
  - $v_1, v_2, v_5, v_4, v_7, v_3, v_6$ (an other ordering)

Topological Ordering



A topological ordering 1 2 5 4 3 7 6

An other topological ordering 1 2 5 4 7 3 6

Topological Sort Algorithm Sketch

- 1. Find a vertex, v, with no incoming edges.
- 2. Print this vertex v; and remove it, along with all its edges, from the graph.
- 3. Repeat steps 1 and 2 until graph is empty.

Topological Sort Algorithm - Formally

- **Definition**:
  - *Indegree* of a vertex v is the number of edges in the form (u,v).
- **Algorithm**:
  - Compute the indegrees of all vertices in the graph.
  - Read the graph into an adjacency list.
  - Apply the algorithm in the previous slide.

Topological Sort Algorithm - Formally

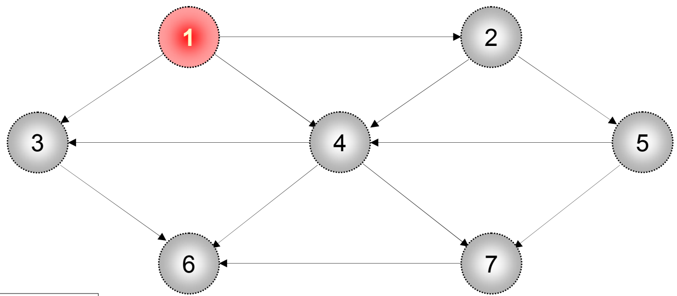
```
void
Graph::topsort()
{
        Vertex v;     // vertex, v, that has indegree equal to 0
        Vertex w;     // vertex adjacent to v
        int counter; // keeps the topological order number: 0,1,2,….

        for (counter = 0; counter < NUM_VERTICES; counter++)
        {
                v = findNewVertexOfDegreeZero();     // O(N)
                if (v == NOT_A_VERTEX)
                        throw CycleFound();
                v.topNum = counter; // index in topological order
                for each w adhacent to v
                        w.indegree--;
        }
}
```

- Running time = $O(|V|^2)$

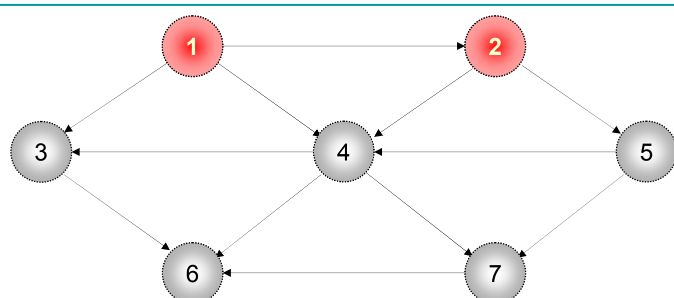
More efficient algorithm

- Keep the vertices which have indegree equal to zero in a separate *box* (stack or queue).
  - 1. Start with an empty box.
  - 2. Scan all the vertices in the graph
  - 3. Put vertices that have indegree equal to zero into the box.
  - 4. While the box (queue) is not empty
    - 4.1. Remove head of queue: vertex v.
    - 4.2. Print v
    - 4.3. Decrease the indegrees of <u>all the vertices adjacent to v</u> by one.
    - 4.4. Go to step 4.



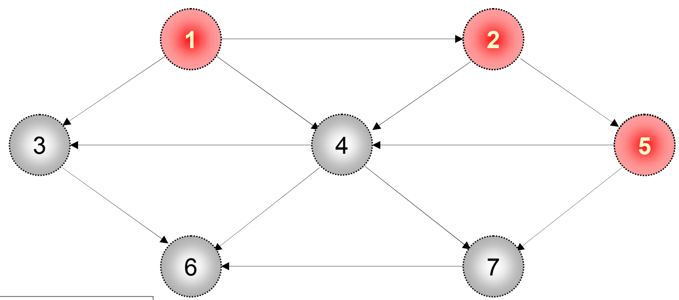
Indegrees	
v_1	0
v_2	1
v_3	2
v_4	3
v_5	1
v_6	3
v_7	2

After Enqueue	*After Dequeue*	*Print*
v_1		v_1
….	….	



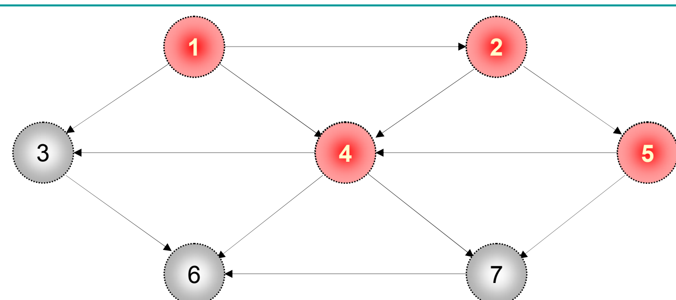
Updated Indegrees	
v_1	0
v_2	0
v_3	1
v_4	2
v_5	1
v_6	3
v_7	2

After Enqueue	*After Dequeue*	*Print*
v_2		v_2
….	….	



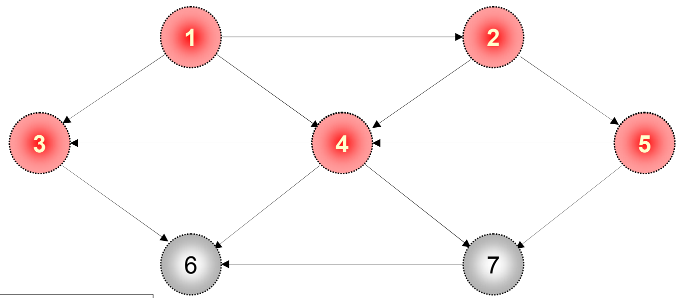
Updated Indegrees	
v_1	0
v_2	0
v_3	1
v_4	1
v_5	0
v_6	3
v_7	2

After Enqueue	*After Dequeue*	*Print*
v_5		v_5
….	….	



Updated Indegrees	
v_1	0
v_2	0
v_3	1
v_4	0
v_5	0
v_6	3
v_7	1

After Enqueue	*After Dequeue*	*Print*
v_4		v_4
….	….	

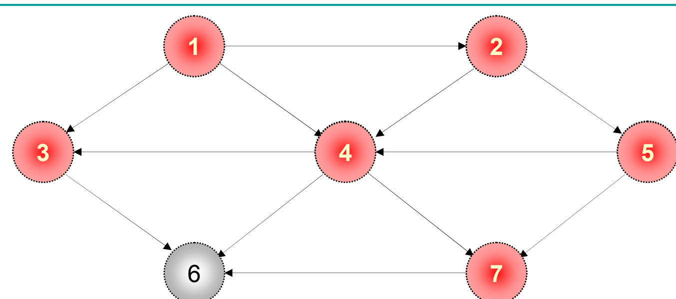


Updated Indegrees	
v_1	0
v_2	0
v_3	0
v_4	0
v_5	0
v_6	2
v_7	0

After Enqueue	*After Dequeue*
v_3	v_7
v_7	
….	….

Print

v_3

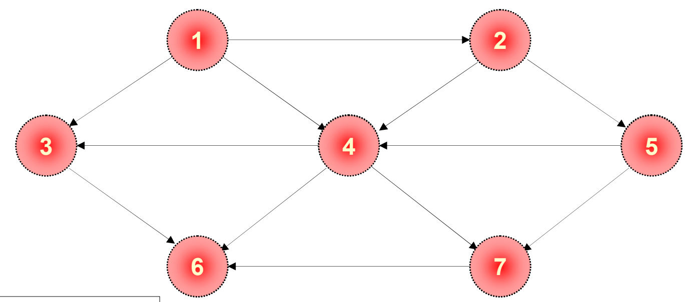


Updated Indegrees	
v_1	0
v_2	0
v_3	0
v_4	0
v_5	0
v_6	1
v_7	0

After Enqueue	*After Dequeue*
v_7	
….	….

Print

v_3



Updated Indegrees

v_1	0
v_2	0
v_3	0
v_4	0
v_5	0
v_6	0
v_7	0

After Enqueue	*After Dequeue*	*Print*
v_6		v_6
….	….	

Finished! Result: 1,2,5,4,3,7,6


```
void Graph::topsort()
{
        Queue q(NUM_VERTICES);
        int counter = 0; //topological order of a vertex:1,2,3,…,NUM_VERTICES
        Vertex v, w;

        q.makeEmpty();
        for each vertex v
                if (v.indegree == 0)
                        q.enqueue(v):
        while (!q.isEmpty())
        {
                v = q.dequeue();
                v.topNum = ++counter;

                for each w adjacent to v
                        if (--w.indegree ==0)
                                q.enqueue(w);
        }
        if (counter != NUM_VERTICES)
                throw CycleFound();
}
```

Pseudocode of Efficient Topological Sort Algorithm