CS473-Algorithms I

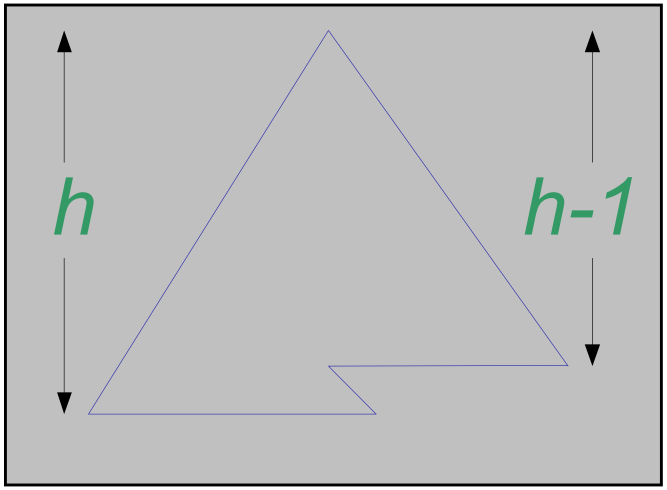
Lecture 8

Heapsort

Introduction

- $O(n \lg n)$ worst case
- Sorts in place
- Another design paradigm
  - Use of a data structure (heap) to manage information during execution of algorithm

Heap Data Structure

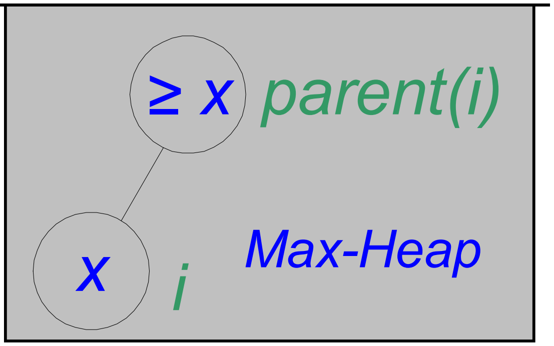


- Nearly complete binary tree
  - Completely filled on all levels, except possibly the lowest level
  - Lowest level is filled from left to right
  - Each node of the tree stores an element
- Height of a node
  - Length of the longest simple downward path from the node to a leaf
  - ▷ Height of the tree: height of the root
- Depth of a node
  - Length of the simple downward path from the root to the node

Heap Property

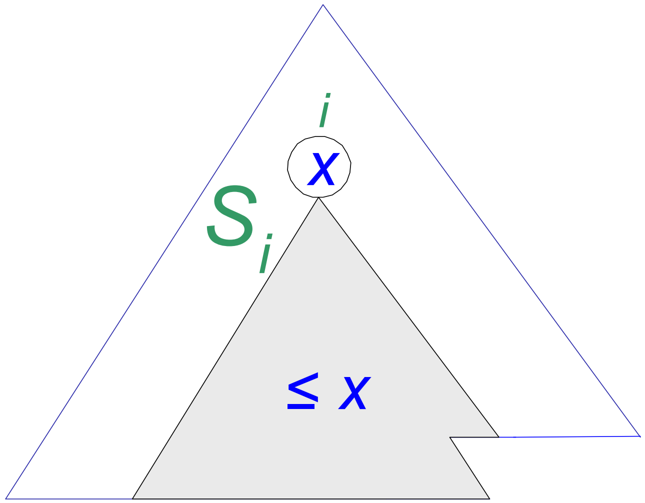
- For every node *i* other than root
  - Max-Heap: A[parent(*i*)] ≥ A[*i*]
  - Min-Heap: A[parent(*i*)] ≤ A[*i*]

  Where A[*i*] denotes the element stored at node *i*

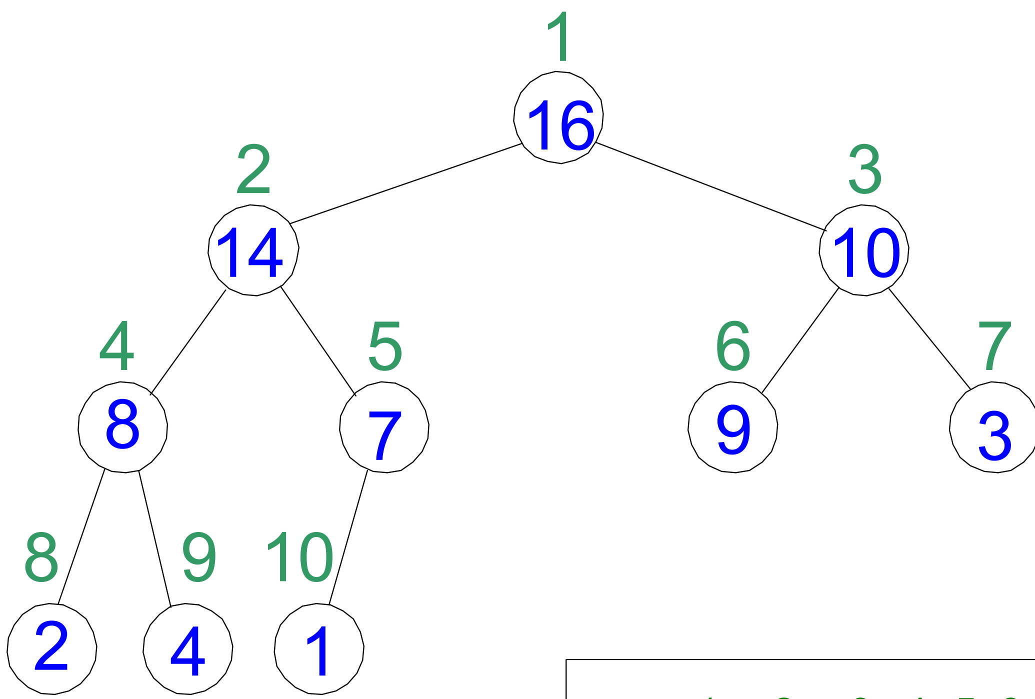


- Will discuss Max-Heap

Fact: Largest element in a subtree of a heap is at the root of the subtree.



Example



	1	2	3	4	5	6	7	8	9	10
A:	16	14	10	8	7	9	3	2	4	1

Heap Data Structure

- Store a heap in an array with implicit links
  - Left child: left(i)=2i
  - Right child: right(i)= 2i+1

  Computing 2i is fast: left shift in binary
  - Parent of i is: parent(i)=$\lfloor i/2 \rfloor$

  Computing $\lfloor i/2 \rfloor$ is fast: right shift in binary
- A[1]: element stored at the root
- Array has two attributes
  - length[A]: number of elements in A
  - heap-size[A]=n: number of elem. in heap stored in A

$$\boxed{n \leq \text{length}[A]}$$

Heap Operations

EXTRACT-MAX(A, n)

max ← A[1]

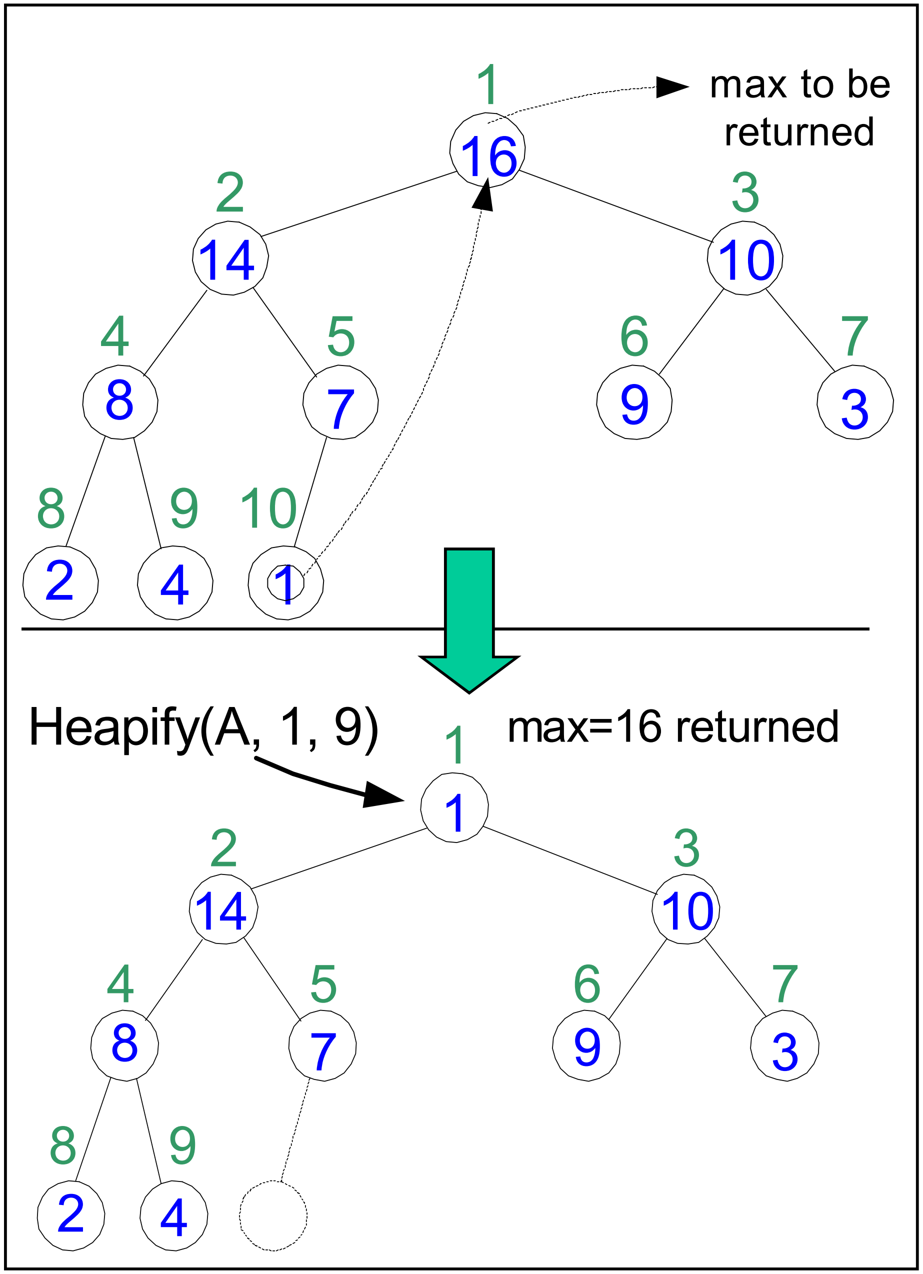
A[1] ← A[n]

$n \leftarrow n - 1$

HEAPIFY(A, 1, n)

return max

O(1) + heapify time



Heap Operations

Maintaining heap property:

Subtrees rooted at left[i] and right[i] are already heaps.

But, A[i] may violate the heap property (i.e., may be smaller than its children)

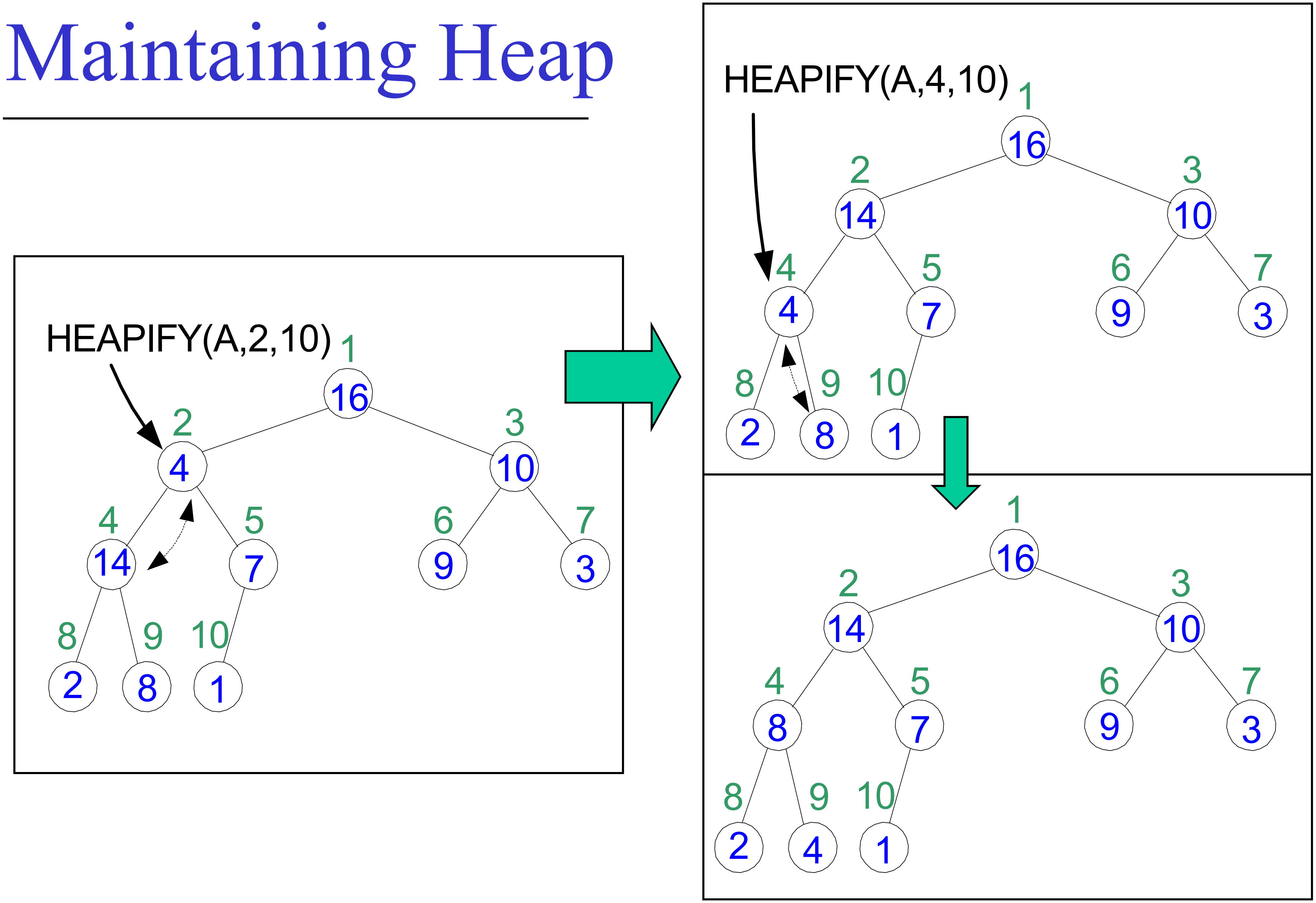
Idea: Float down the value at A[i] in the heap so that subtree rooted at i becomes a heap.

HEAPIFY(A, i, n)

if $2i \le n$ **and** A[$2i$] > A[i]
  then largest $\leftarrow 2i$
else largest $\leftarrow i$

if $2i + 1 \le n$ **and** A[$2i$+1] > A[largest]
  then largest $\leftarrow 2i + 1$
if largest $\ne i$ **then**
  exchange A[i]$\leftrightarrow$ A[largest]
  HEAPIFY(A, largest, n)
else return

Maintaining Heap



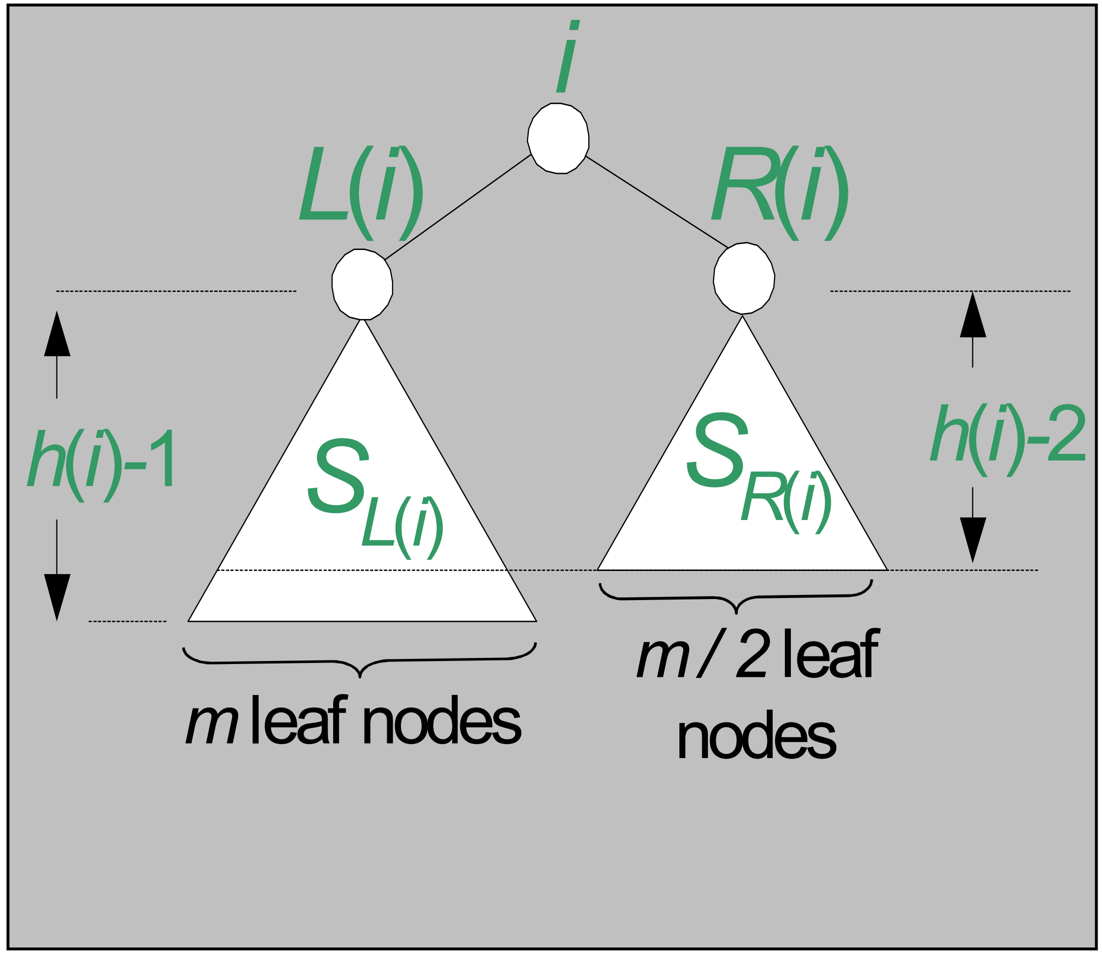
Intuitive Analysis of HEAPIFY

- Consider HEAPIFY(A, i, n)
  - let $\mathrm{h}(i)$ be the height of node i
  - at most $\mathrm{h}(i)$ recursion levels
    - Constant work at each level: $\Theta(1)$
  - Therefore $\mathrm{T}(i) = \mathrm{O}(\mathrm{h}(i))$
- Heap is almost-complete binary tree

  $\triangleright$ $\mathrm{h}(i) = \mathrm{O}(\lg n)$
- Thus $\boxed{\mathrm{T}(n) = \mathrm{O}(\lg n)}$

Formal Analysis of HEAPIFY

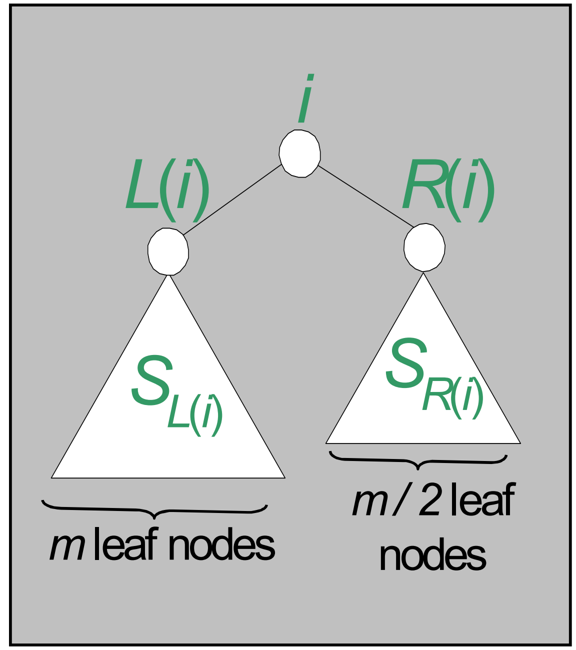
- Worst case occurs when last row of the subtree S_i rooted at node i is half full



- $T(n) \le T(|S_{L(i)}|) + \Theta(1)$
- $S_{L(i)}$ and $S_{R(i)}$ are complete binary trees of heights $h(i) - 1$ and $h(i) - 2$, respectively

Formal Analysis of HEAPIFY

- Let m be the number of leaf nodes in $S_{L(i)}$
- $|S_{L(i)}| = \underbrace{m}_{\text{ext}} + \underbrace{(m-1)}_{\text{int}} = 2m - 1$;
- $|S_{R(i)}| = \overbrace{m/2}^{\text{ext}} + \overbrace{(m/2 - 1)}^{\text{int}} = m - 1$



- $|S_{L(i)}| + |S_{R(i)}| + 1 = n$

  $(2m-1) + (m-1) + 1 = n \Rightarrow m = (n+1)/3$

  $|S_{L(i)}| = 2m - 1 = 2(n+1)/3 - 1 = (2n/3 + 2/3) - 1 = 2n/3 - 1/3 \leq 2n/3$

- $T(n) \leq T(2n/3) + \Theta(1) \Rightarrow \boxed{T(n) = O(\lg n)}$

By case 2 of Master Thm

Maintaining Heap Property: Efficiency Issues

Recursion vs iteration:

•In the absence of tail recursion iterative version is in general more efficient.

Because of the pop/push operations to/from stack at each level of recursion.

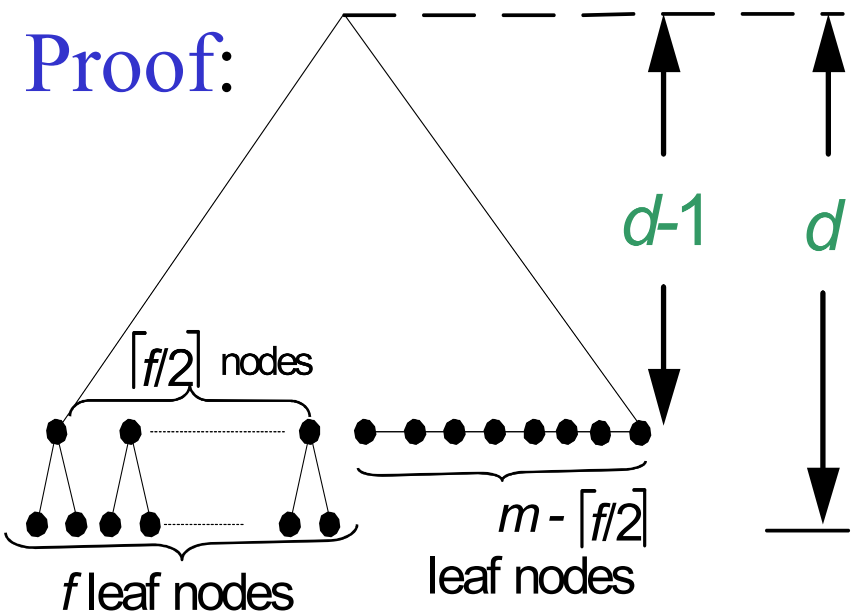
```
HEAPIFY(A, i, n)
  j ← i
  while true do
      if 2j ≤ n and A[2j] > A[j]
          then largest ← 2j
      else largest ← j
      if 2j +1 ≤ n and A[2j+1] > A[largest]
          then largest ← 2j +1
      if largest ≠ j  then
          exchange A[j] ↔ A[largest]
          j ← largest
      else return
```

Building Heap

- Use HEAPIFY in a bottom-up manner
  - This processing order guarantees that $S_{L(i)}$ and $S_{R(i)}$ are already heaps when HEAPIFY is run on node i

Lemma: last $\lceil n/2 \rceil$ nodes of a heap are all leaves

Proof:



$m = 2^{d-1}$: # nodes at level $d-1$

f: # nodes at level d (last level)

Proof of Lemma

- # of leaves$=f + (m - \lceil f/2 \rceil)$

  $= m + \lfloor f/2 \rfloor$

$m + (m - 1) + f = n$

$2m + f = n + 1$

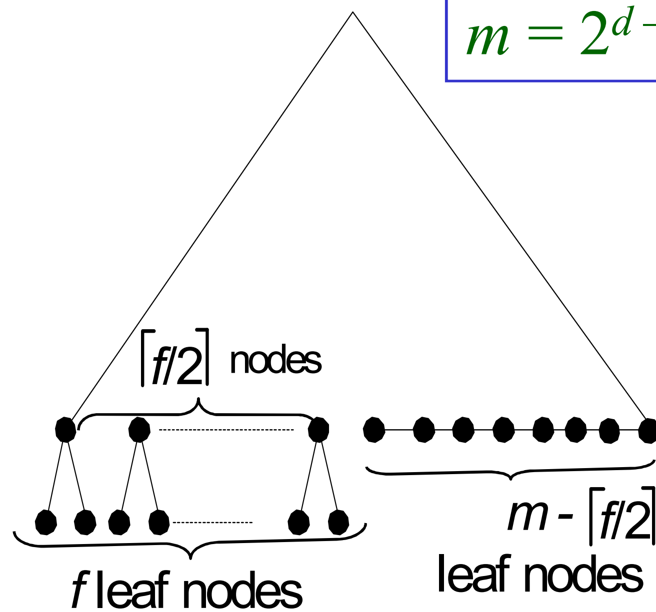
$\lfloor \frac{1}{2}(2m + f) \rfloor = \lfloor \frac{1}{2}(n+1) \rfloor$

$\lfloor m + f/2 \rfloor = \lceil n/2 \rceil$

$m + \lfloor f/2 \rfloor = \lceil n/2 \rceil$

- # of leaves$=\lceil n/2 \rceil$

$m = 2^{d-1}$



Q.E.D

Building Heap

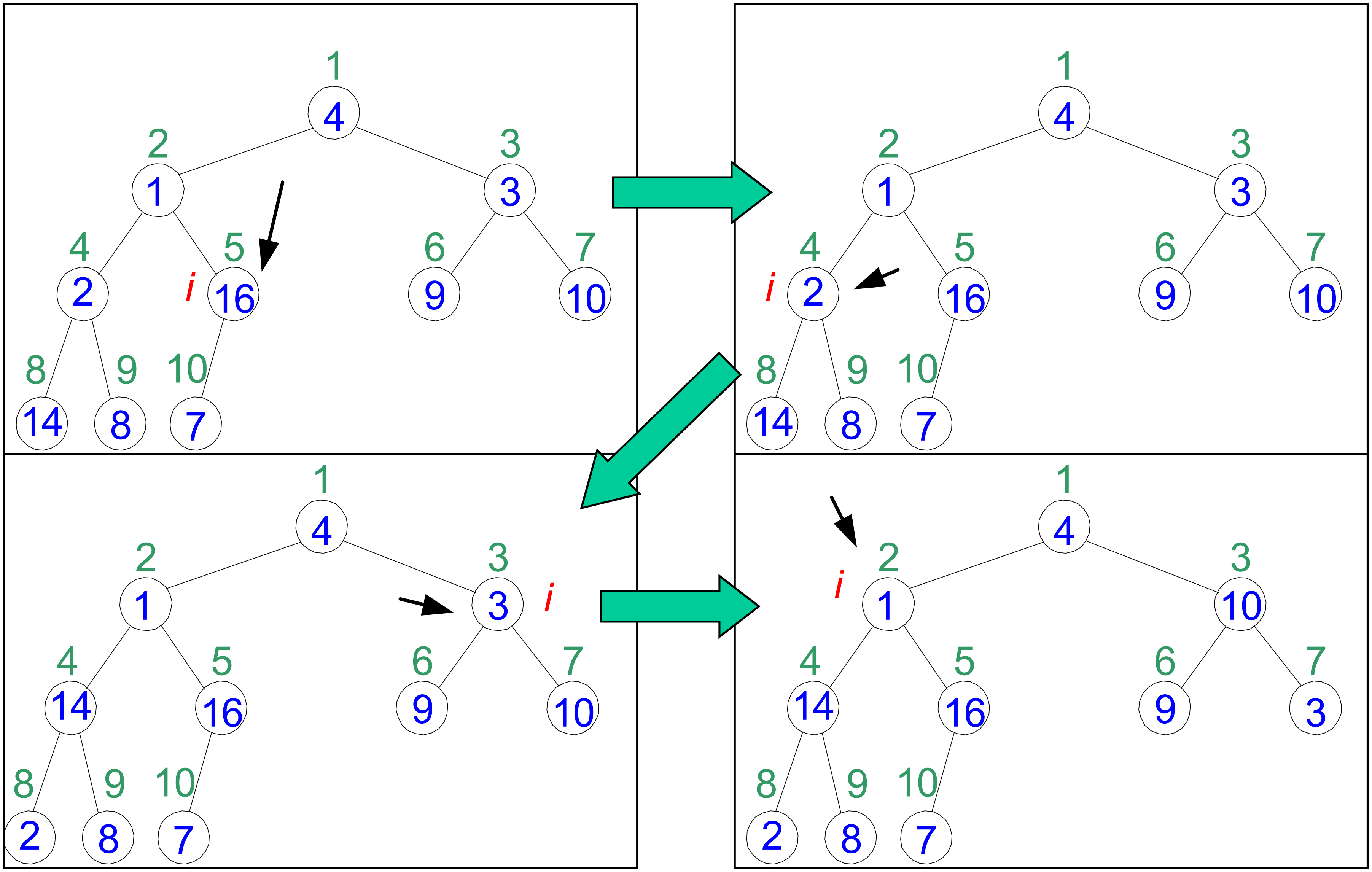
```
BUILD-HEAP(A, n)
    for i ← ⌊n/2⌋ downto 1 do
        HEAPIFY(A, i, n)
```

Running time analysis

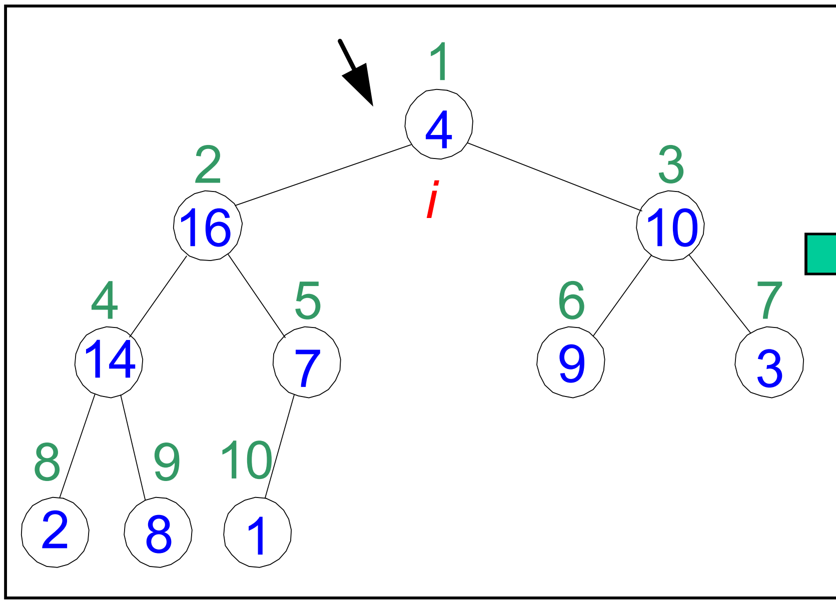
- Get simple O($n$lg$n$) bound
  - n calls to HEAPIFY each of which takes O(lg$n$) time
  - Loose bound
  - A good approach in general
    - Start by proving easy bound
    - Then, try to tighten it

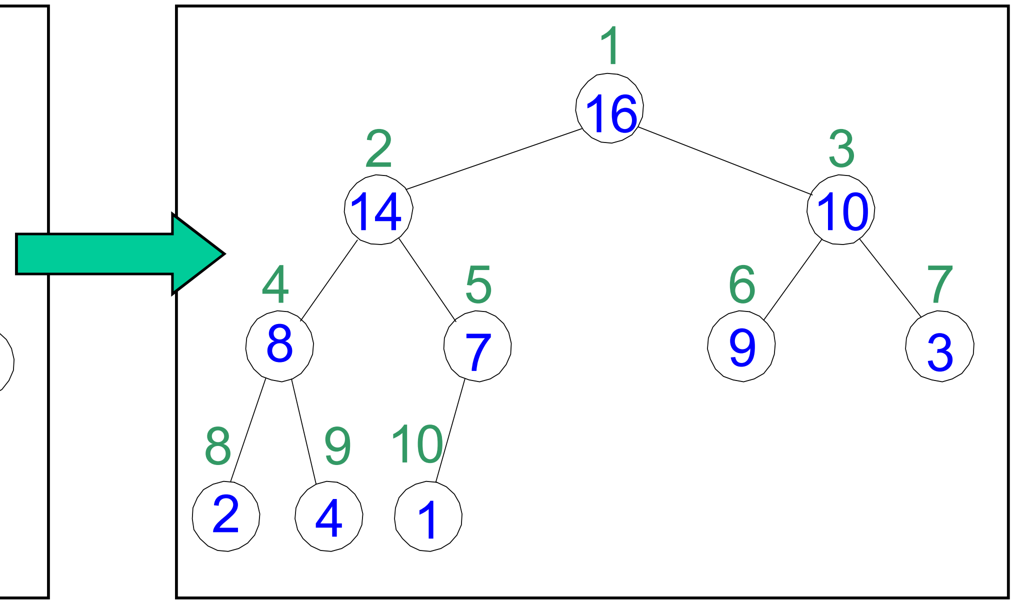
Build-Heap: Example

1	2	3	4	5	6	7	8	9	10
4	1	3	2	16	9	10	14	8	7

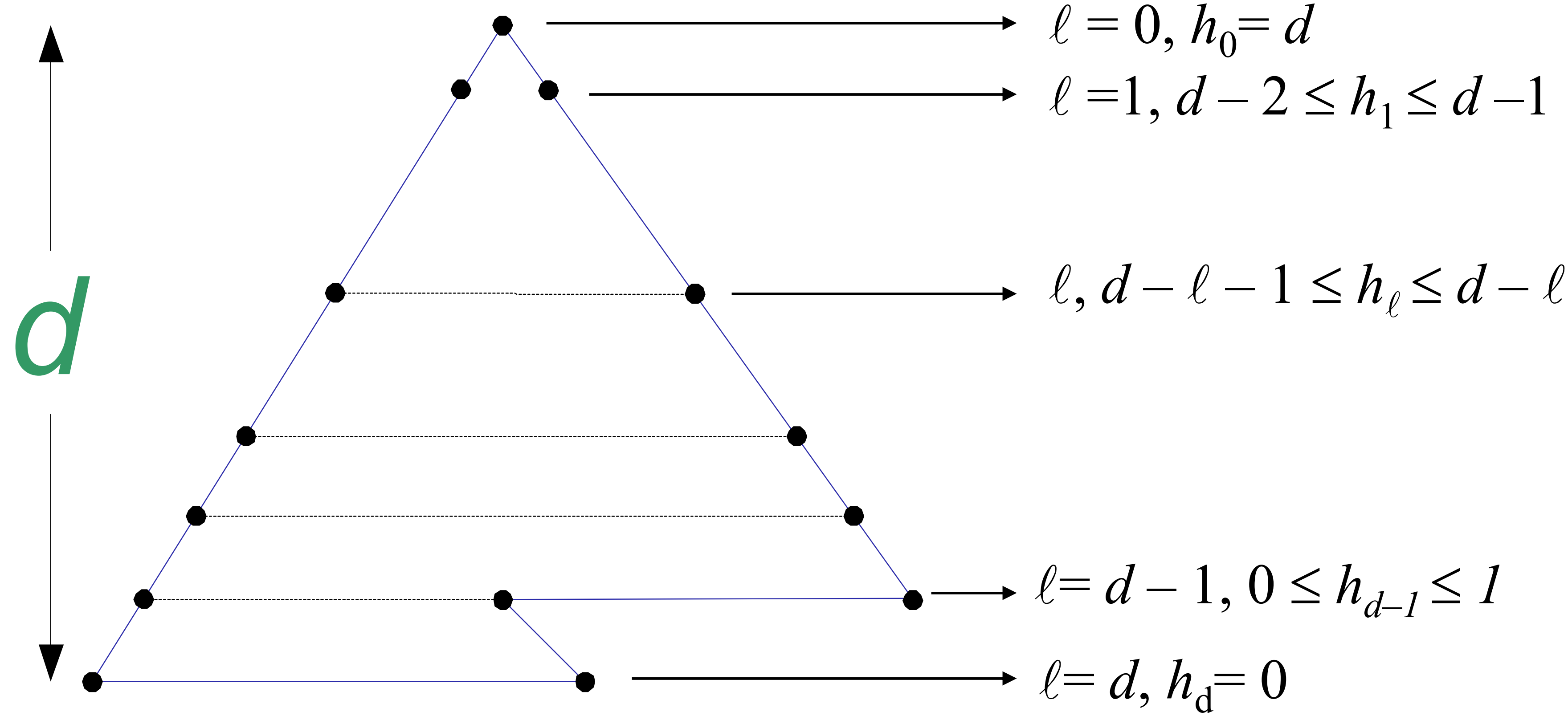


Build-Heap: Example(cont')





Build-Heap: tighter running time analysis



If the heap is complete binary tree then $h_\ell = d - \ell$

Otherwise, nodes at a given level do not all have the same height

But we have $d - \ell - 1 \le h_\ell \le d - \ell$

Build-Heap: **tighter** running time analysis

Assume that all nodes at level $\ell = d - 1$ are processed

$$T(n) = \sum_{\ell=0}^{d-1} n_\ell O(h_\ell) = O\Big(\sum_{\ell=0}^{d-1} n_\ell \, h_\ell\Big) \quad \begin{cases} n_\ell = 2^\ell = \#\text{ of nodes at level } \ell \\ h_\ell = \text{height of nodes at level } \ell \end{cases}$$

$$\therefore T(n) = O\Big(\sum_{\ell=0}^{d-1} 2^\ell \, (d - \ell)\Big)$$

Let $h = d - \ell \Rightarrow \ell = d - h$ (change of variables)

$$T(n) = O\Big(\sum_{h=1}^{d} h \, 2^{d-h}\Big) = O\Big(\sum_{h=1}^{d} h \, 2^d/2^h\Big) = O\Big(2^d \sum_{h=1}^{d} h \, (1/2)^h\Big)$$

but $2^d = \Theta(n) \Rightarrow T(n) = O\Big(n \sum_{h=1}^{d} h \, (1/2)^h\Big)$

Build-Heap: **tighter** running time analysis

$$\sum_{h=1}^{d} h(1/2)^h \le \sum_{h=0}^{d} h(1/2)^h \le \sum_{h=0}^{\infty} h(1/2)^h$$

recall infinite decreasing geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ where } |x| < 1$$

differentiate both sides

$$\sum_{k=0}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

Build-Heap: **tighter** running time analysis

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

then, multiply both sides by x

$$\sum_{k=0}^{\infty} kx^{k} = \frac{x}{(1-x)^2}$$

in our case: $x = 1/2$ and $k = h$

$$\therefore \sum_{h=0}^{\infty} h(1/2)^h = \frac{1/2}{(1-1/2)^2} = 2 = O(1)$$

$$\therefore T(n) = O(n \sum_{h=1}^{d} h(1/2)^h) = O(n)$$

Heapsort Algorithm

The HEAPSORT algorithm

(1) Build a heap on array $A[1 \ldots n]$ by calling BUILD-HEAP(A, n)

(2) The largest element is stored at the root $A[1]$

Put it into its correct final position $A[n]$ by $A[1] \leftrightarrow A[n]$

(3) Discard node n from the heap

(4) Subtrees (S_2 & S_3) rooted at children of root remain as heaps

but the new root element may violate the heap property

Make $A[1 \ldots n-1]$ a heap by calling HEAPIFY(A, 1, $n-1$)

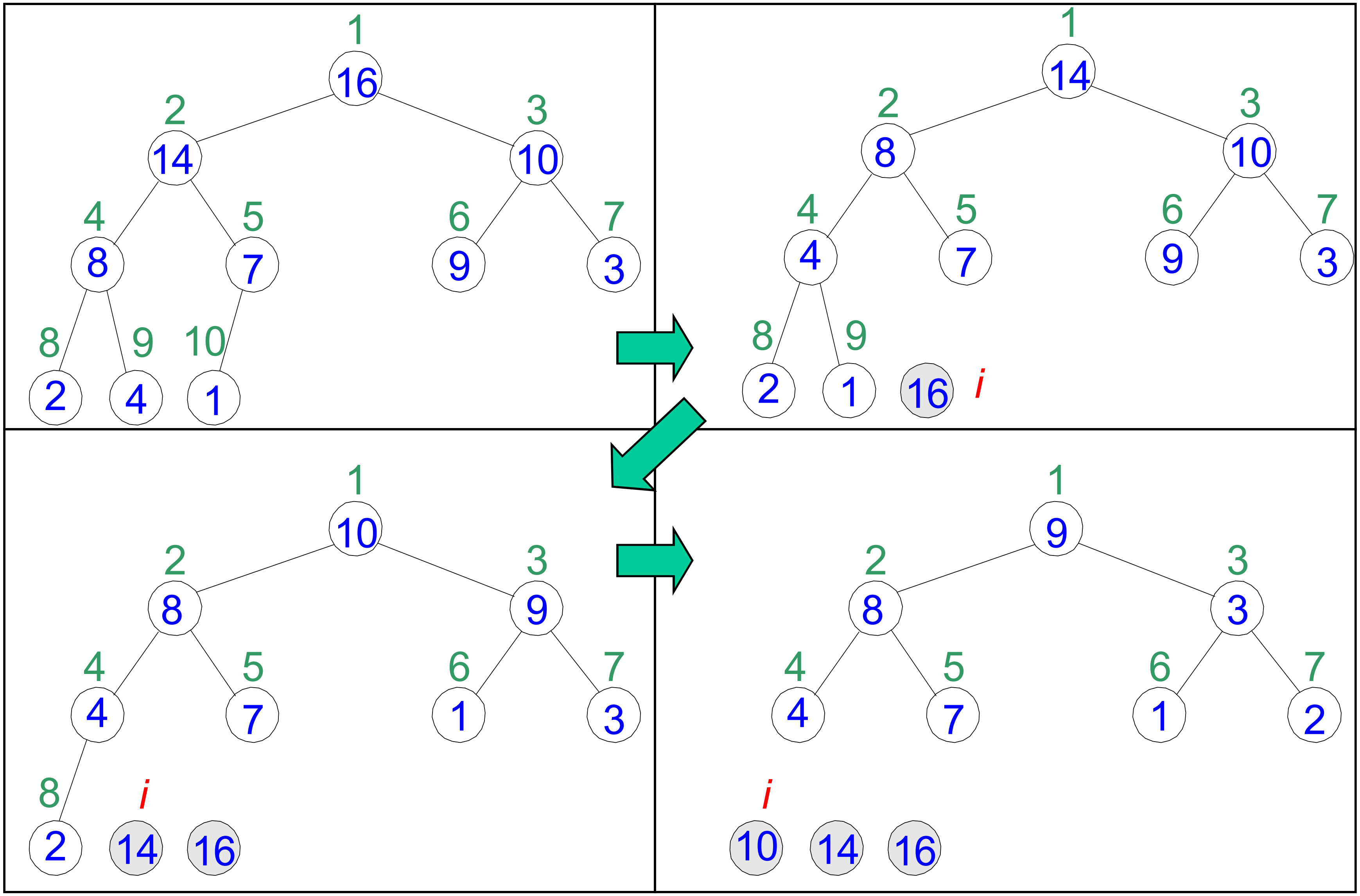
(5) $n \leftarrow n - 1$

(6) Repeat steps 2–4 until $n = 2$

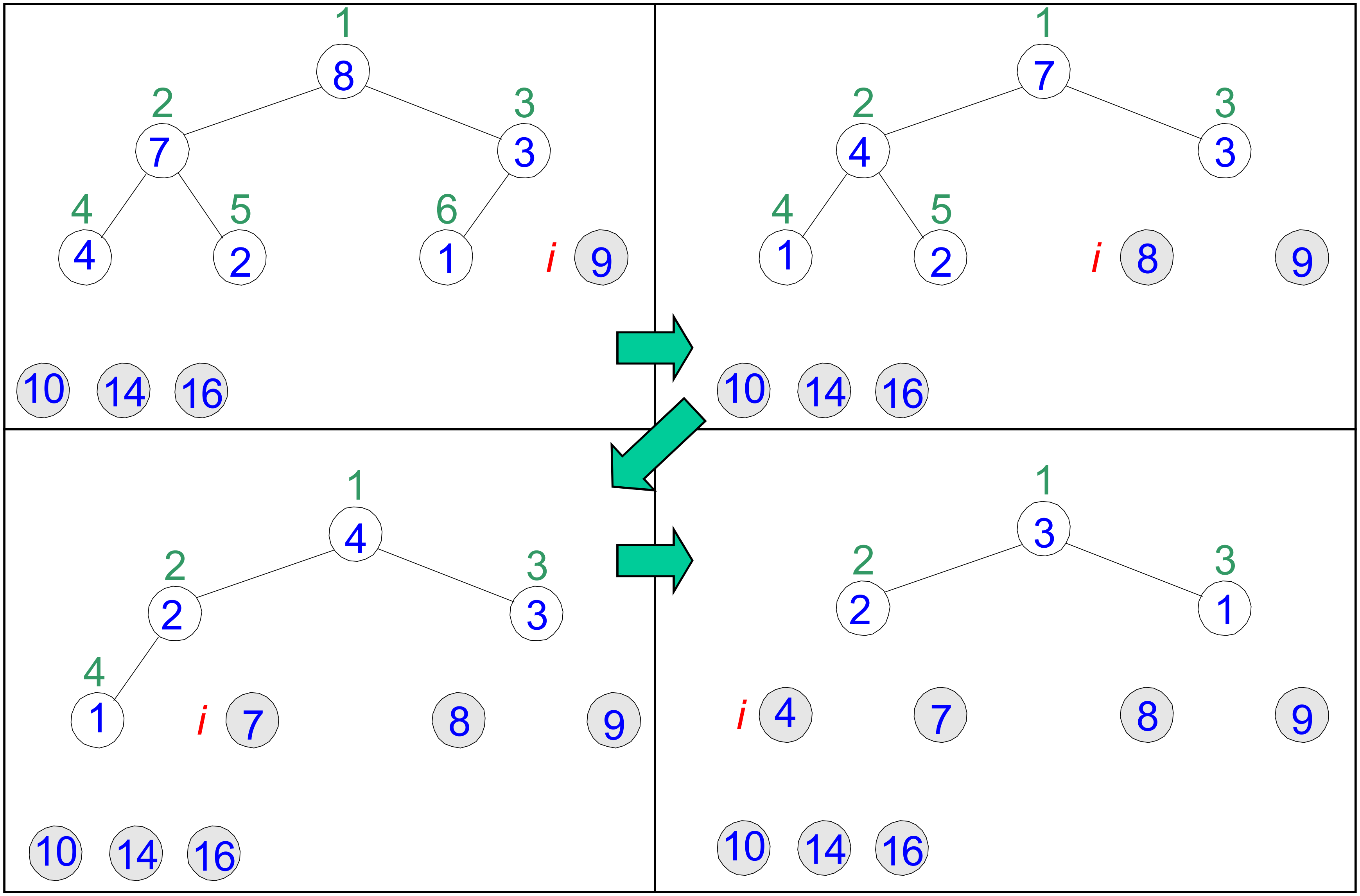
Heapsort Algorithm

HEAPSORT(A, n)
  BUILD-HEAP(A, n)
  for $i \leftarrow n$ downto 2 do
    exchange A[1] $\leftrightarrow$ A[i]
    HEAPIFY(A, 1, $i - 1$)

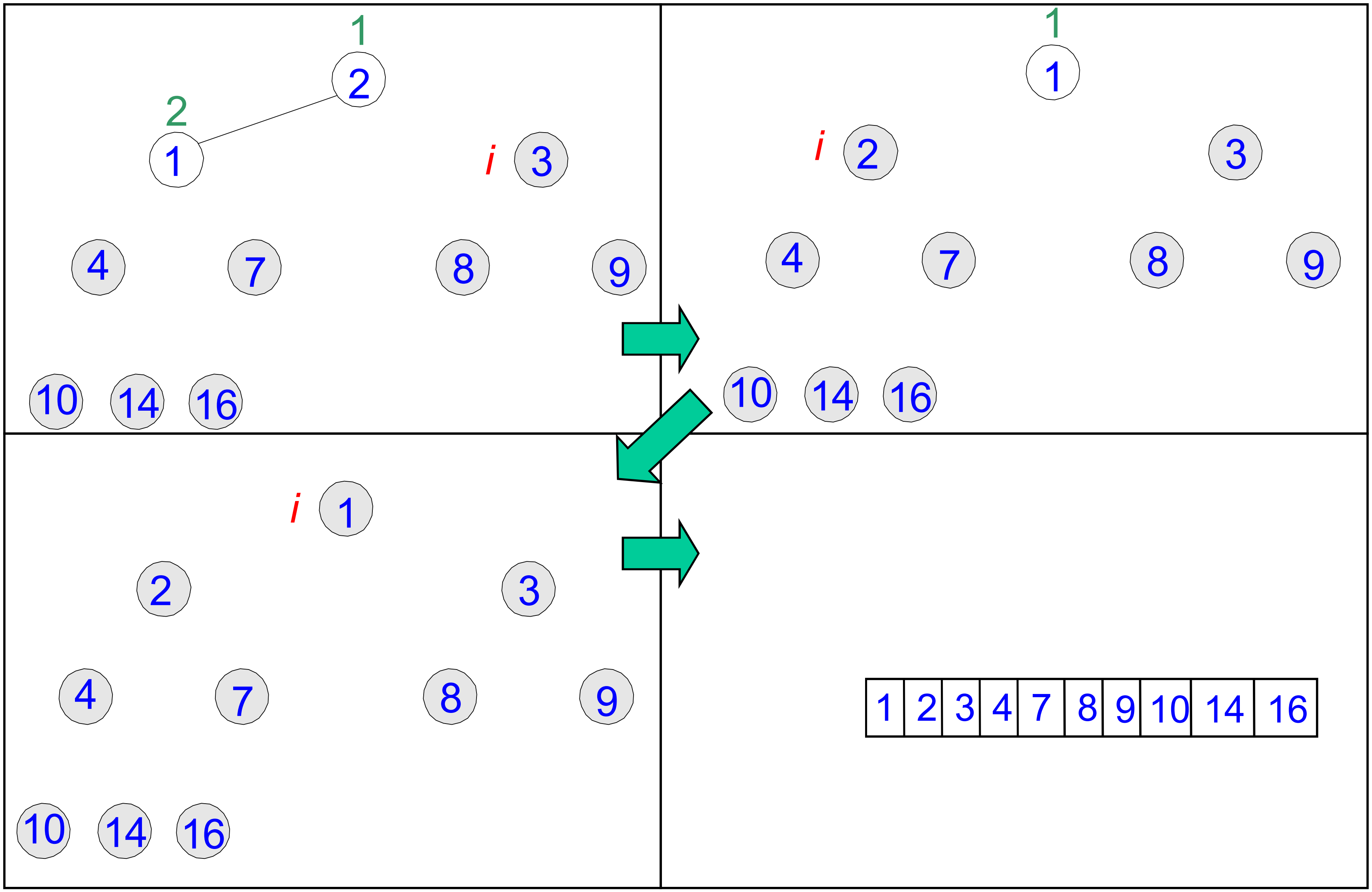
Heapsort: Example



Heapsort: Example



Heapsort: Example



Heapsort Run Time Analysis

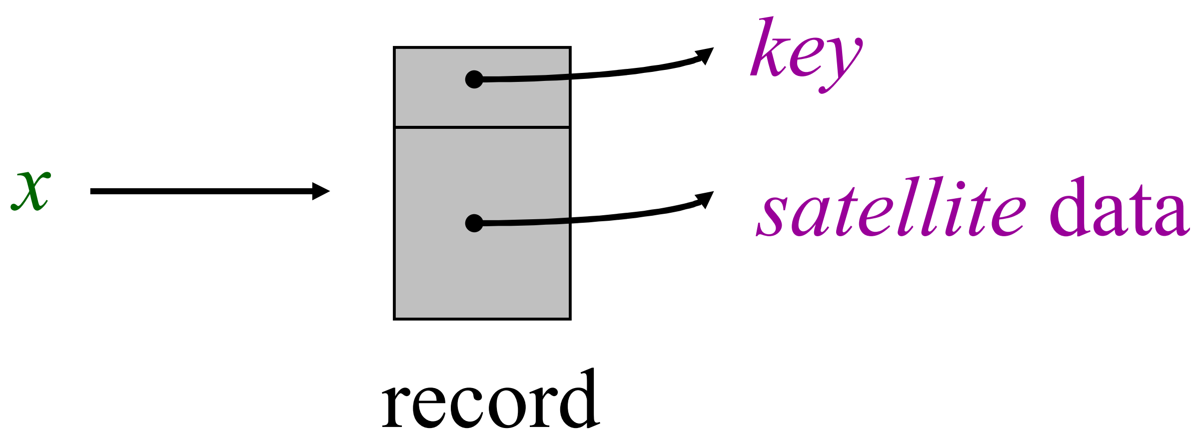
- BUILD-HEAP takes O(n) time
- i-th iteration of for loop takes O(lg(n – i)) time

$$T(n) = \sum_{i=1}^{n-1} O(\lg(n-i)) = \sum_{k=1}^{n-1} O(\lg k) = O\left(\sum_{k=1}^{n-1} \lg k\right) = O(n \lg n)$$

- Heapsort is a very good algorithm but, a good implementation of quicksort always beats heapsort in practice
- However, heap data structure has many popular applications, and it can be efficiently used for implementing priority queues

Data structures for Dynamic Sets

- Consider sets of records having *key* and *satellite* data



Operations on Dynamic Sets

- Queries: Simply return info; Modifying operations: Change the set
  - INSERT(S, x): (Modifying) $S \leftarrow S \cup \{x\}$
  - DELETE(S, x): (Modifying) $S \leftarrow S - \{x\}$
  - MAX(S) / MIN(S): (Query) return $x \in S$ with the largest/smallest *key*
  - EXTRACT-MAX(S) / EXTRACT-MIN(S) : (Modifying) return and delete $x \in S$ with the largest/smallest *key*
  - SEARCH(S, k): (Query) return $x \in S$ with $key[x] = k$
  - SUCCESSOR(S, x) / PREDECESSOR(S, x) : (Query) return $y \in S$ which is the next larger/smaller element after x
- Different data structures support/optimize different operations

Priority Queues (*PQ*)

- Supports
  - INSERT
  - MAX / MIN
  - EXTRACT-MAX / EXTRACT-MIN
- One application: Schedule jobs on a shared resource
  - PQ keeps track of jobs and their relative priorities
  - When a jobs is finished or interrupted

    Highest priority job is selected from those pending using EXTRACT-MAX
  - A new job can be added at any time using INSERT

Cevdet Aykanat - Bilkent University
Computer Engineering Department

Priority Queues

- Another application: Event-driven simulation
  - Events to be simulated are the items in the PQ
  - Each event is associated with a time of occurrence which serves as a *key*
  - Simulation of an event can cause other events to be simulated in the future
  - Use EXTRACT-MIN at each step to choose the next event to simulate
  - As new events are produced insert them into the PQ using INSERT

Implementation of Priority Queue

- Sorted linked list: Simplest implementation
  - INSERT
    - O(n) time
    - Scan the list to find place and splice in the new item
  - EXTRACT-MAX
    - O(1) time
    - Take the first element

▹ Fast extraction but slow insertion.

Implementation of Priority Queue

- Unsorted linked list: Simplest implementation
  - INSERT
    - O(1) time
    - Put the new item at front
  - EXTRACT-MAX
    - O(n) time
    - Scan the whole list

▷ Fast insertion but slow extraction

Sorted linked list is better on the average
- Sorted list: on the average, scans $n/2$ elem. per insertion
- Unsorted list: always scans n elem. at each extraction

Heap Implementation of PQ

- INSERT and EXTRACT-MAX are both $O(\lg n)$
  - good compromise between fast insertion but slow extraction and vice versa
- EXTRACT-MAX: already discussed HEAP-EXTRACT-MAX

INSERT: Insertion is like that of Insertion-Sort.

Traverses $O(\lg n)$ nodes, as HEAPIFY does but makes fewer comparisons and assignments

–HEAPIFY: compares parent with both children

–HEAP-INSERT: with only one

HEAP-INSERT(A, *key*, *n*)

$n \leftarrow n + 1$

$i \leftarrow n$

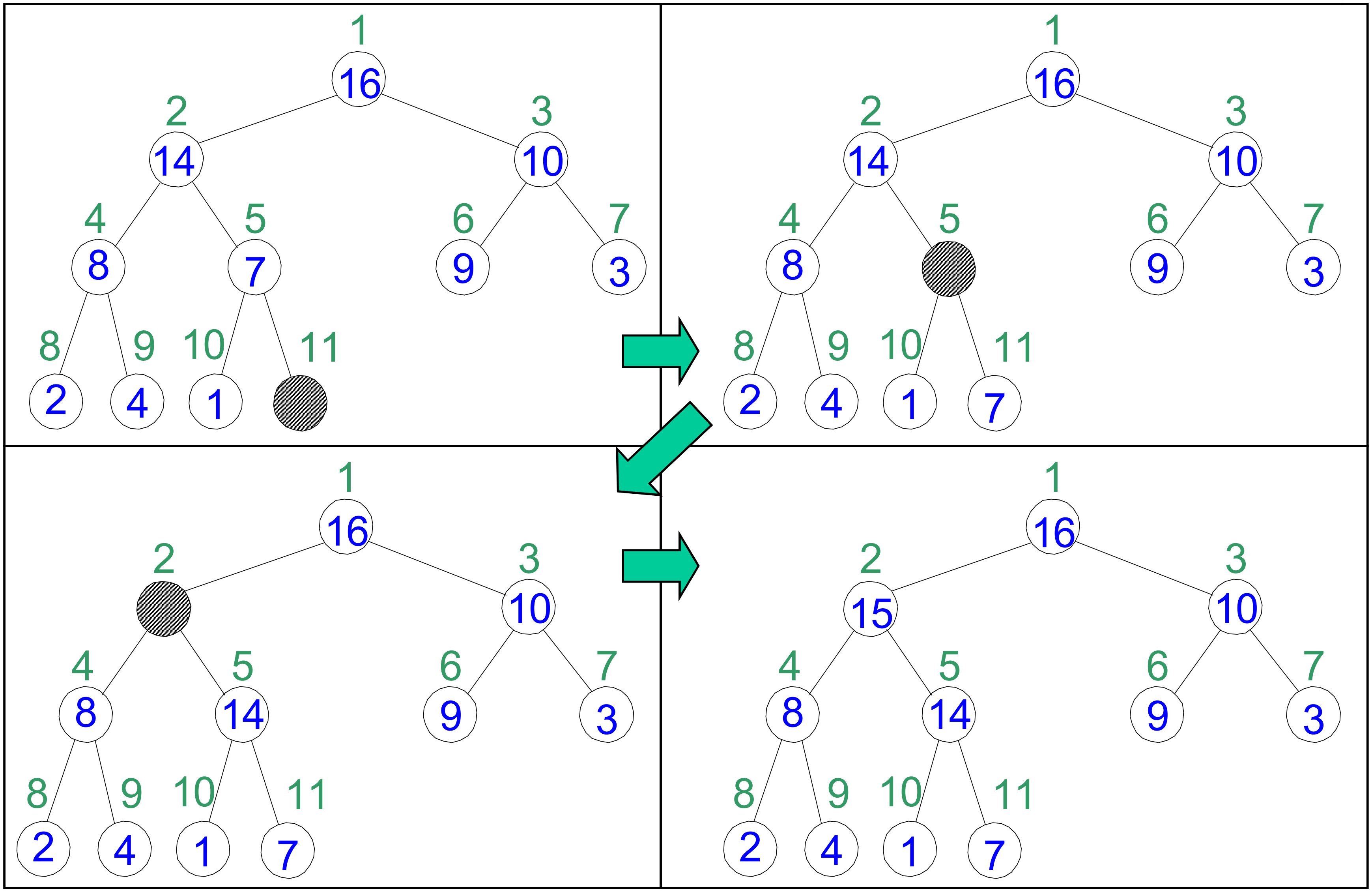
while $i > 1$ **and** $A[\lfloor i/2 \rfloor] < key$ **do**

$A[i] \leftarrow A[\lfloor i/2 \rfloor]$

$i \leftarrow \lfloor i/2 \rfloor$

$A[i] \leftarrow key$

HEAP-INSERT(A, 15)

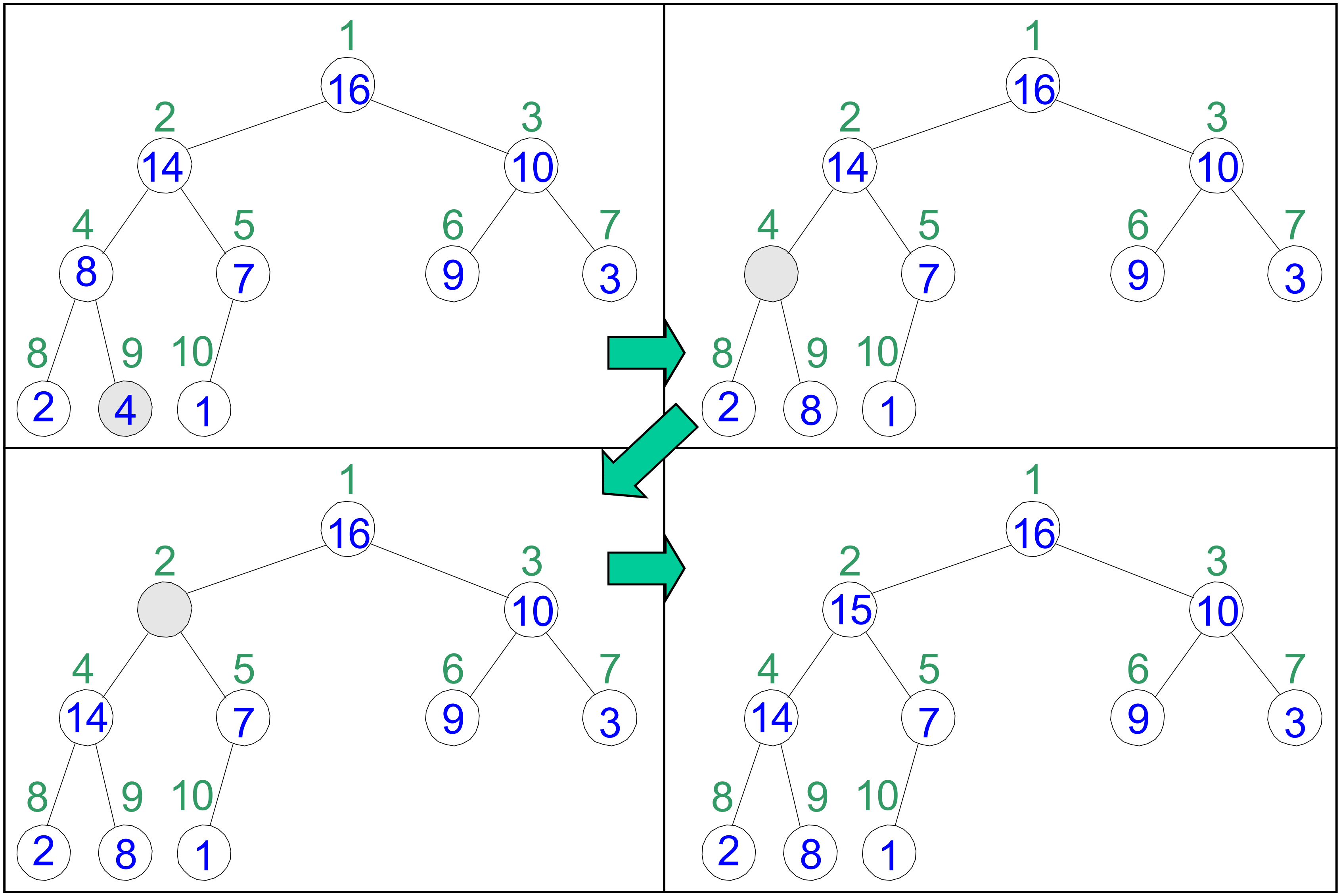


Heap Increase Key

- Key value of i-th element of heap is increased from A[i] to *key*

```
HEAP-INCREASE-KEY(A, i, key)
    if key < A[i] then
        return error
    while i >1 and A[⌊i/2⌋] < key do
        A[i] ← A[⌊i/2⌋]
        i ← ⌊i/2⌋
    A[i] ← key
```

HEAP-INCREASE-KEY(A, 9, 15)



Heap Implementation of PQ

	key	data	H-ptr
a	14		4
b	1		10
c	10		3
d	16		1
e	*		¬
f	9		6
g	2		8
h	15		2
i	*		¬
j	3		7
k	7		5
l	*		¬
m	8		9
n	*		¬
o	*		¬