# CS473-Algorithms I

## Lecture 8

## Heapsort


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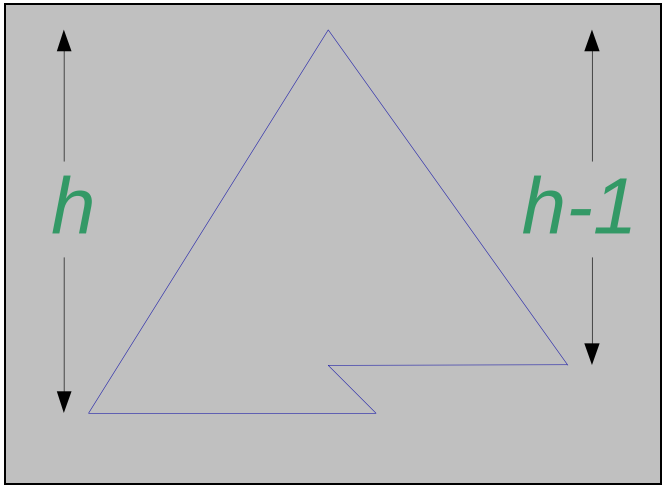
# Introduction

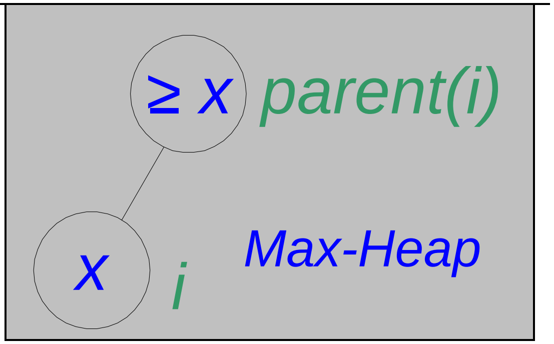
- $O(n \lg n)$ worst case
- Sorts in place
- Another design paradigm
  - Use of a data structure (heap) to manage information during execution of algorithm

# Heap Data Structure




- Nearly complete binary tree
  - Completely filled on all levels, except possibly the lowest level
  - Lowest level is filled from left to right
  - Each node of the tree stores an element
- Height of a node
  - Length of the longest simple downward path from the node to a leaf
  - ▹ Height of the tree: height of the root
- Depth of a node
  - Length of the simple downward path from the root to the node

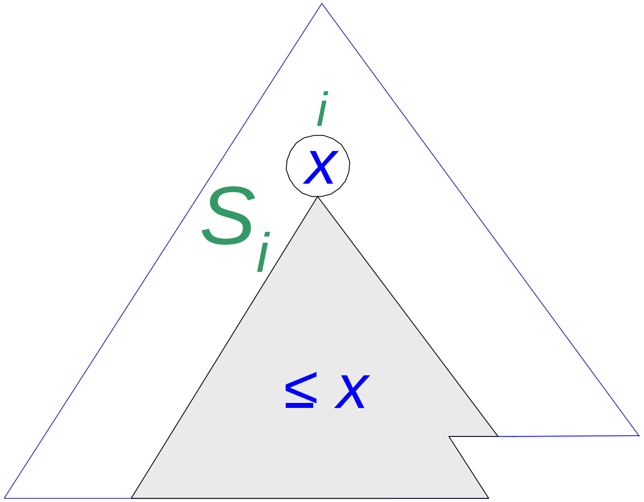
# Heap Property



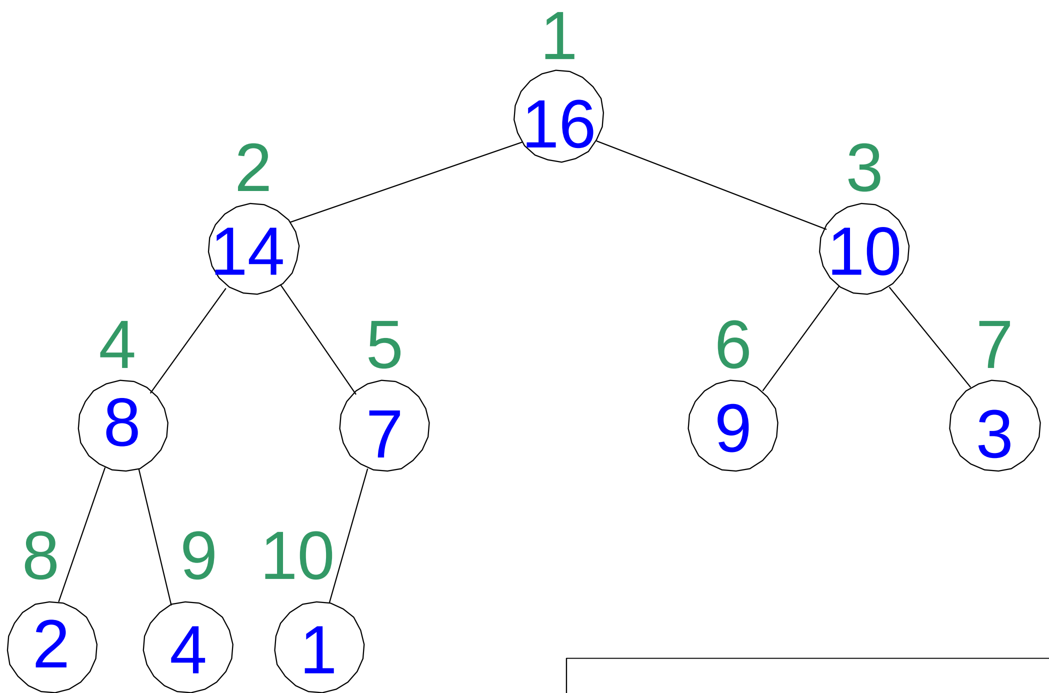

- For every node *i* other than root
  - Max-Heap: A[parent(*i*)] ≥ A[*i*]
  - Min-Heap: A[parent(*i*)] ≤ A[*i*]

  Where A[*i*] denotes the element stored at node *i*
- Will discuss Max-Heap

**Fact**: Largest element in a subtree of a heap is at the root of the subtree.



# Example




A:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 16 | 14 | 10 | 8 | 7 | 9 | 3 | 2 | 4 | 1 |

# Heap Data Structure

- Store a heap in an array with implicit links
  - Left child: left($i$)=2$i$
  - Right child: right($i$)= 2$i$+1

  Computing 2$i$ is fast: left shift in binary
  - Parent of $i$ is: parent($i$)=$\lfloor i/2 \rfloor$

  Computing $\lfloor i/2 \rfloor$ is fast: right shift in binary
- A[1]: element stored at the root
- Array has two attributes
  - length[A]: number of elements in A
  - heap-size[A]=$n$: number of elem. in heap stored in A

$$n \leq \text{length}[A]$$

# Heap Operations

**EXTRACT-MAX**(A, *n*)

max ← A[1]

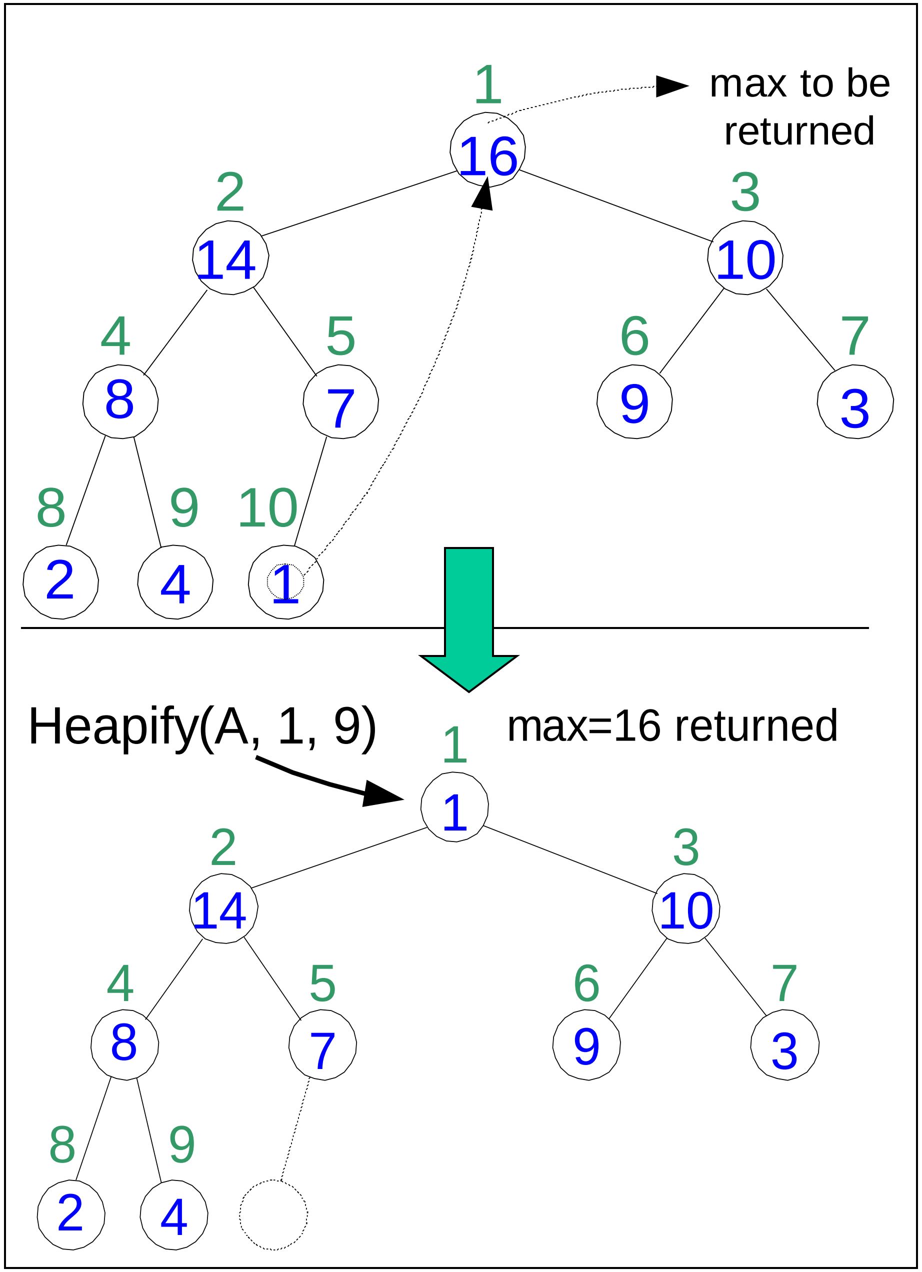
A[1] ← A[*n*]

*n* ← *n* – 1

<u>HEAPIFY</u>(A, 1, *n*)

**return** max

O(1) + heapify time



# Heap Operations

**Maintaining heap property:**

Subtrees rooted at left[$i$] and right[$i$] are already heaps.

But, A[$i$] may violate the heap property (i.e., may be smaller than its children)

Idea: Float down the value at A[$i$] in the heap so that subtree rooted at $i$ becomes a heap.

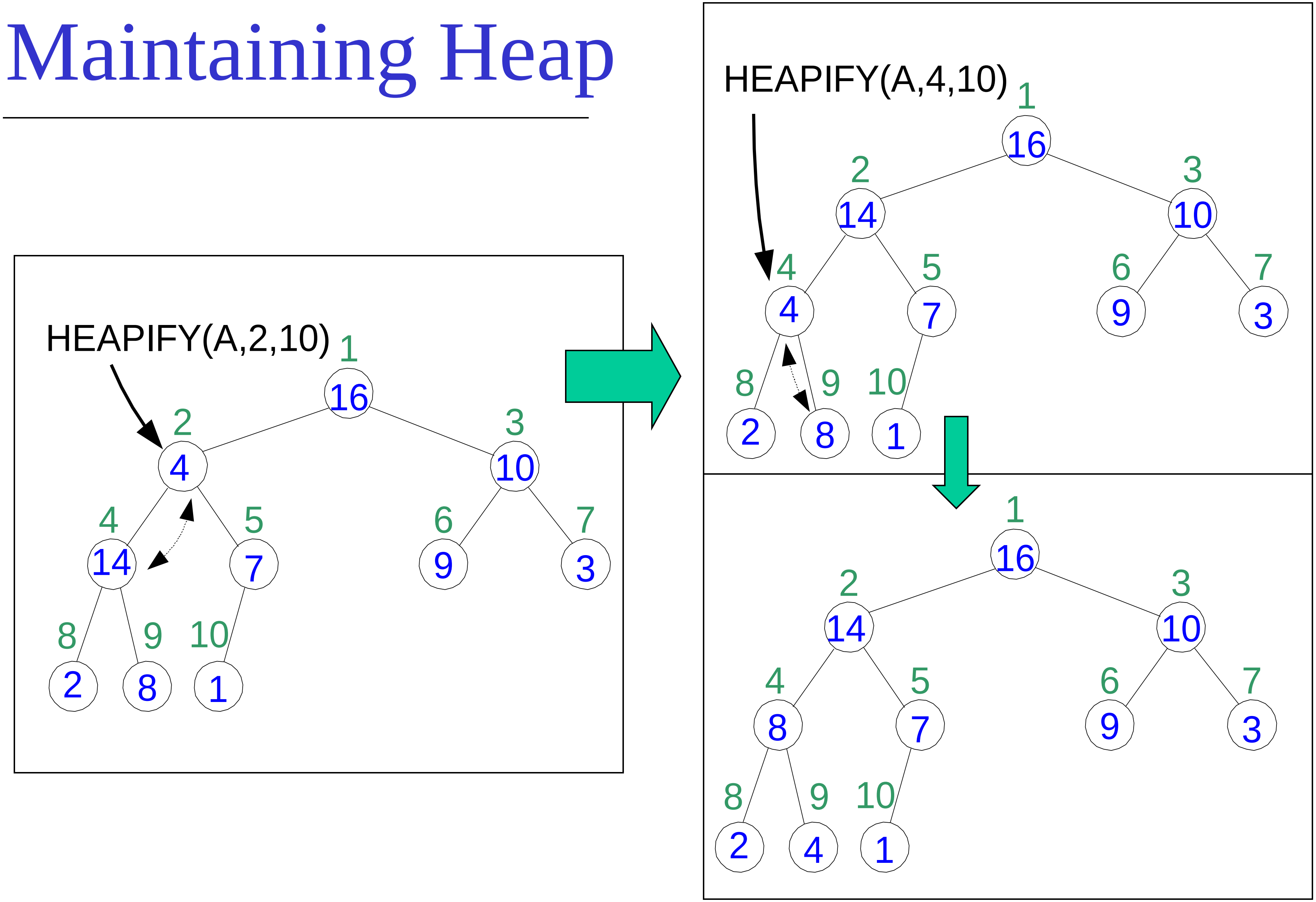
**HEAPIFY**(A, $i$, $n$)

**if** $2i \le n$ **and** A[$2i$] > A[$i$]
  **then** largest ← $2i$
**else** largest ← $i$

**if** $2i + 1 \le n$ **and** A[$2i$+1] > A[largest]
  **then** largest ← $2i + 1$
**if** largest ≠ $i$ **then**
  exchange A[$i$] ↔ A[largest]
  HEAPIFY(A, largest, $n$)

**else return**

# Maintaining Heap



# Intuitive Analysis of HEAPIFY

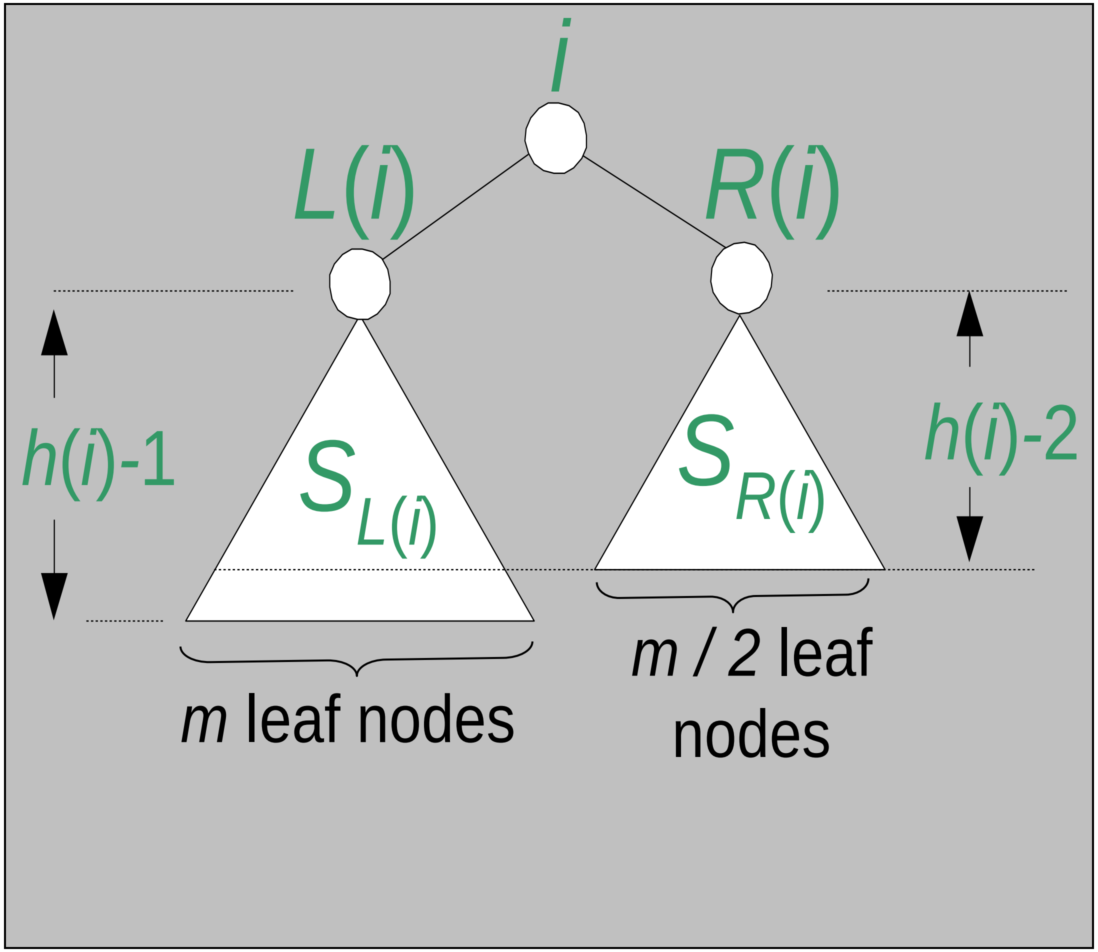
- Consider HEAPIFY(A, $i$, $n$)
  - let h($i$) be the height of node $i$
  - at most h($i$) recursion levels
    - Constant work at each level: $\Theta(1)$
  - Therefore $T(i) = O(h(i))$
- Heap is almost-complete binary tree

  ▷ $h(i) = O(\lg n)$

- Thus $T(n) = O(\lg n)$

# Formal Analysis of HEAPIFY

- Worst case occurs when last row of the subtree $S_i$ rooted at node $i$ is half full



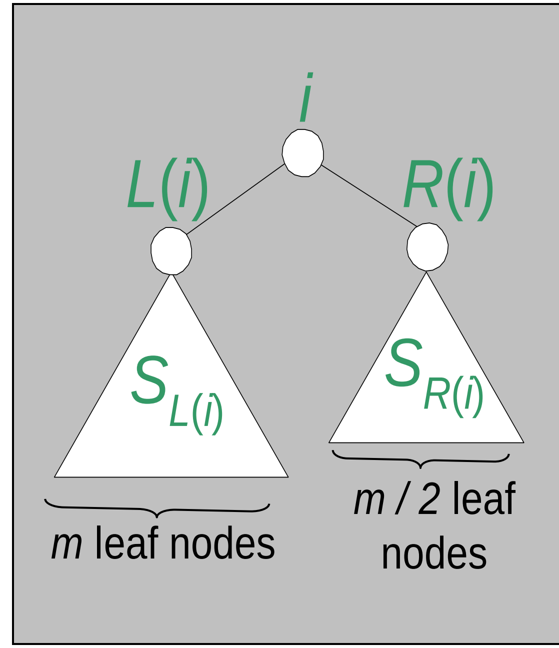

- $T(n) \le T(|S_{L(i)}|) + \Theta(1)$
- $S_{L(i)}$ and $S_{R(i)}$ are complete binary trees of heights $h(i) - 1$ and $h(i) - 2$, respectively

# Formal Analysis of HEAPIFY

- Let m be the number of leaf nodes in $S_{L(i)}$
- $| S_{L(i)} | = \underbrace{m}_{\text{ext}} + \underbrace{(m-1)}_{\text{int}} = 2m - 1$ ;
- $| S_{R(i)} | = \overbrace{m/2} + \overbrace{(m/2 - 1)} = m - 1$
- $| S_{L(i)} | \quad + \quad | S_{R(i)} | + 1 = n$

  $(2m - 1) + (m - 1) + 1 = n \Rightarrow m = (n+1)/3$

  $| S_{L(i)} | = 2m - 1 = 2(n+1)/3 - 1 = (2n/3 + 2/3) - 1 = 2n/3 - 1/3 \leq 2n/3$




- $T(n) \leq T(2n/3) + \Theta(1) \Rightarrow T(n) = O(\lg n)$

By case 2 of Master Thm

# Maintaining Heap Property: Efficiency Issues

**Recursion vs iteration:**

•In the absence of tail recursion iterative version is in general more efficient.

Because of the pop/push operations to/from stack at each level of recursion.

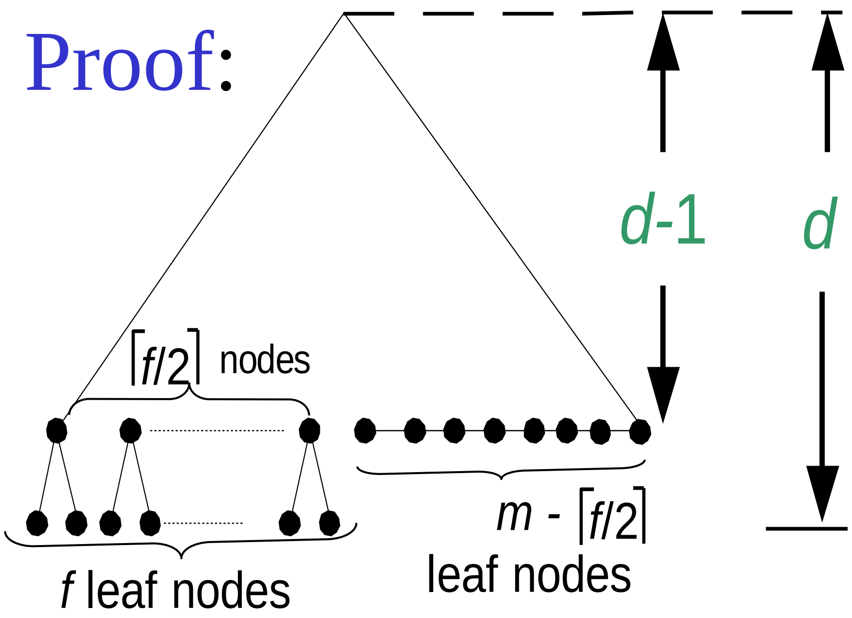
```
HEAPIFY(A, i, n)
  j ← i
  while true do
      if 2j ≤ n and A[2j] > A[j]
           then largest ← 2j
      else largest ← j
      if 2j +1 ≤ n and A[2j+1] > A[largest]
           then largest ← 2j +1
      if largest ≠ j  then
           exchange A[j] ↔ A[largest]
           j ← largest
      else return
```

# Building Heap

- Use HEAPIFY in a bottom-up manner
  - This processing order guarantees that $S_{L(i)}$ and $S_{R(i)}$ are already heaps when HEAPIFY is run on node $i$

Lemma: last $\lceil n/2 \rceil$ nodes of a heap are all leaves

Proof:




$m = 2^{d-1}$: # nodes at level $d - 1$

$f$ : # nodes at level $d$ (last level)

# Proof of Lemma

- # of leaves$=f + (m - \lceil f/2 \rceil)$
  $= m + \lfloor f/2 \rfloor$

$m + (m - 1) + f = n$

$2m + f = n + 1$

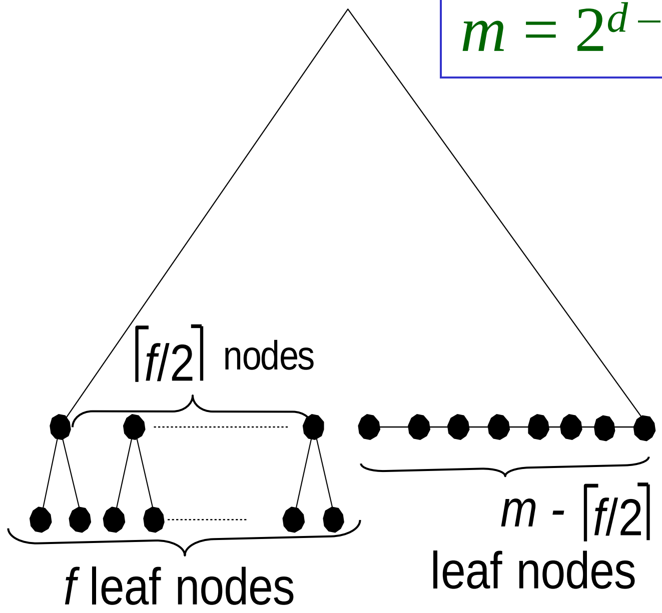
$\lfloor \frac{1}{2}(2m + f) \rfloor = \lfloor \frac{1}{2}(n+1) \rfloor$

$\lfloor m + f/2 \rfloor = \lceil n/2 \rceil$

$m + \lfloor f/2 \rfloor = \lceil n/2 \rceil$

- # of leaves$= \lceil n/2 \rceil$

$m = 2^{d-1}$




Q.E.D

# Building Heap

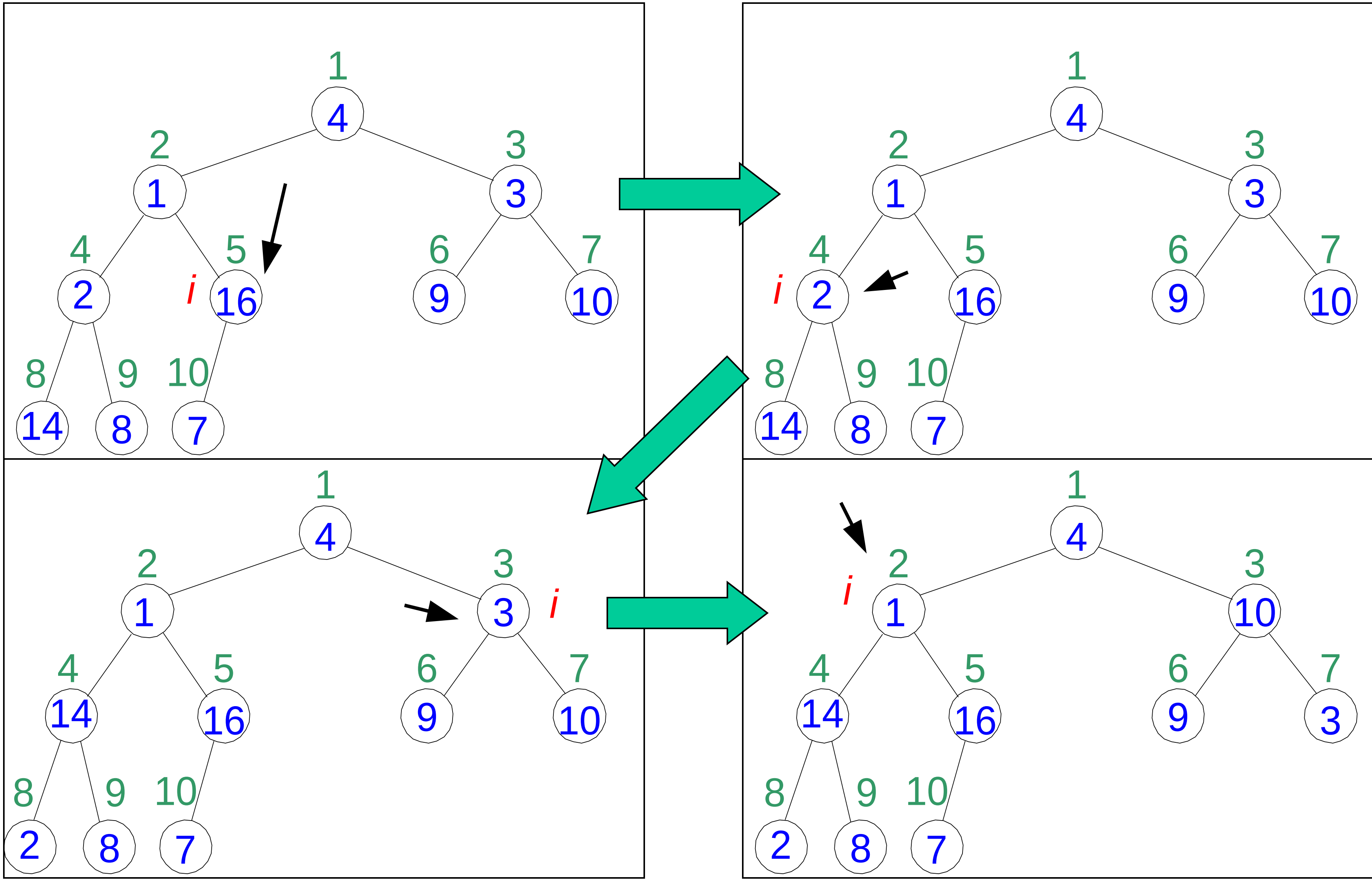
```
BUILD-HEAP(A, n)

    for i ← ⌊n/2⌋ downto 1 do
        HEAPIFY(A, i, n)
```

Running time analysis

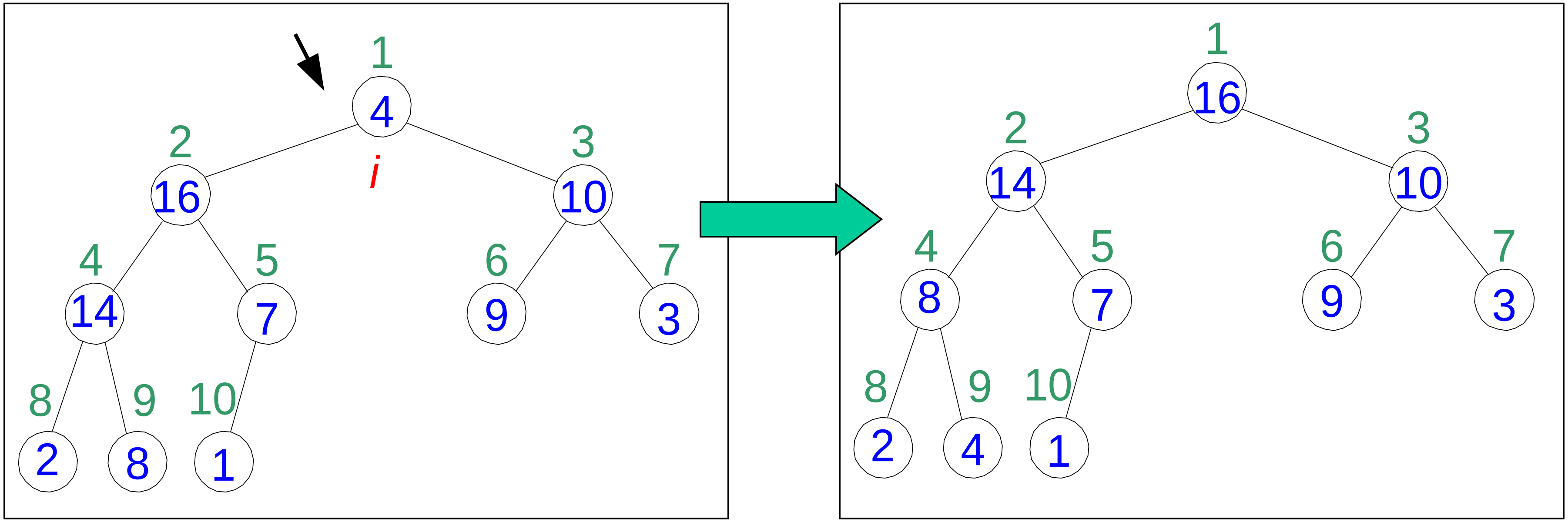
- Get simple $O(n \lg n)$ bound
  - $n$ calls to HEAPIFY each of which takes $O(\lg n)$ time
  - Loose bound
  - A good approach in general
    - Start by proving easy bound
    - Then, try to tighten it

# Build-Heap: Example

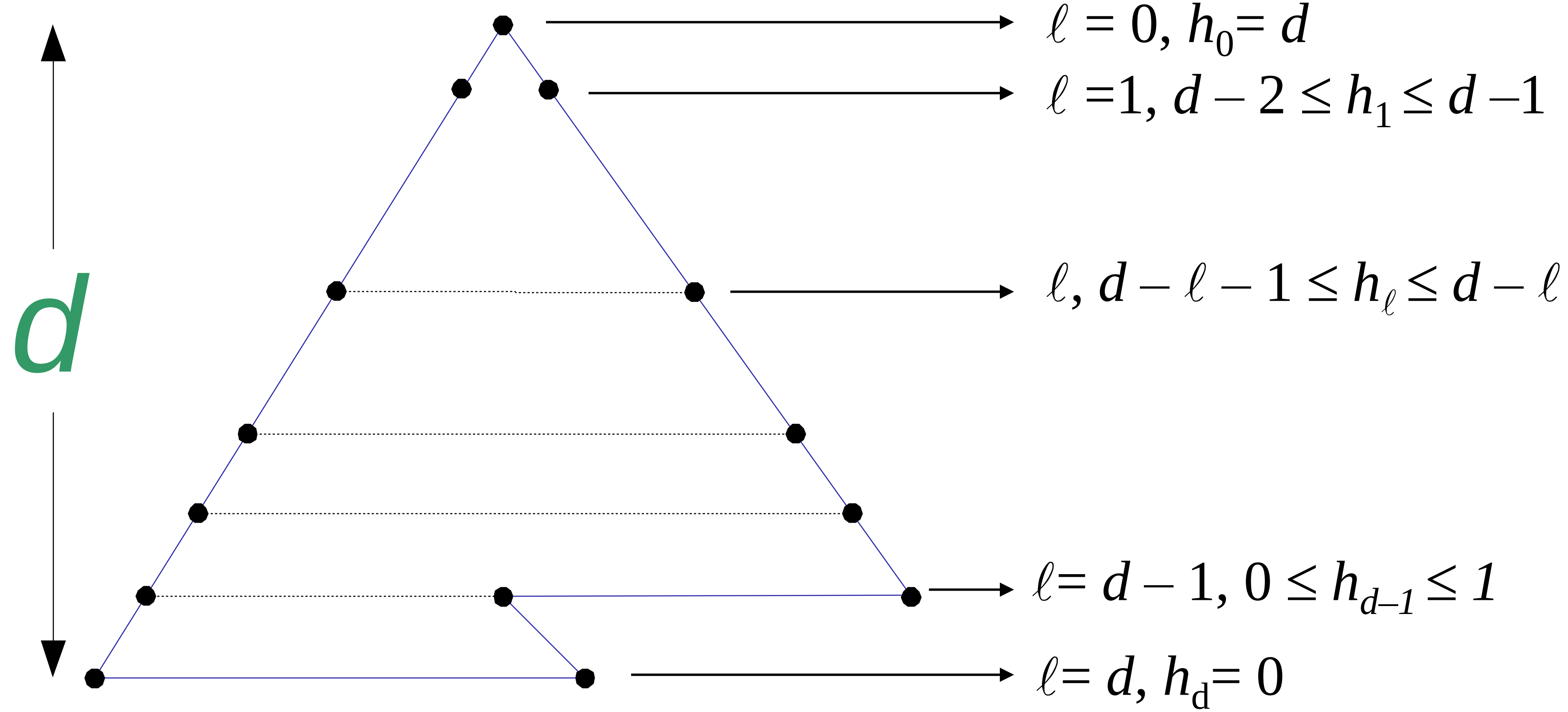
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 1 | 3 | 2 | 16 | 9 | 10 | 14 | 8 | 7 |



# Build-Heap: Example(cont')



# Build-Heap: tighter running time analysis




If the heap is complete binary tree then $h_\ell = d - \ell$

Otherwise, nodes at a given level do not all have the same height

But we have $d - \ell - 1 \le h_\ell \le d - \ell$

# Build-Heap: **tighter** running time analysis

Assume that all nodes at level $\ell = d - 1$ are processed

$$T(n) = \sum_{\ell=0}^{d-1} n_\ell O(h_\ell) = O(\sum_{\ell=0}^{d-1} n_\ell h_\ell) \quad \begin{cases} n_\ell = 2^\ell = \text{\# of nodes at level } \ell \\ h_\ell = \text{height of nodes at level } \ell \end{cases}$$

$$\therefore T(n) = O\Big(\sum_{\ell=0}^{d-1} 2^\ell (d - \ell)\Big)$$

Let $h = d - \ell \Rightarrow \ell = d - h$ (change of variables)

$$T(n) = O\Big(\sum_{h=1}^{d} h\, 2^{d-h}\Big) = O\Big(\sum_{h=1}^{d} h\, 2^d/2^h\Big) = O\Big(2^d \sum_{h=1}^{d} h\, (1/2)^h\Big)$$

but $2^d = \Theta(n) \Rightarrow T(n) = O\Big(n \sum_{h=1}^{d} h\, (1/2)^h\Big)$

# Build-Heap: **tighter** running time analysis

$$\sum_{h=1}^{d} h(1/2)^h \le \sum_{h=0}^{d} h(1/2)^h \le \sum_{h=0}^{\infty} h(1/2)^h$$

recall infinite decreasing geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ where } |x| < 1$$

differentiate both sides

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

# Build-Heap: **tighter** running time analysis

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

then, multiply both sides by $x$

$$\sum_{k=0}^{\infty} kx^{k} = \frac{x}{(1-x)^2}$$

in our case: $x = 1/2$ and $k = h$

$$\therefore \sum_{h=0}^{\infty} h(1/2)^h = \frac{1/2}{(1-1/2)^2} = 2 = \mathrm{O}(1)$$

$$\therefore T(n) = \mathrm{O}(n\sum_{h=1}^{d} h(1/2)^h) = \mathrm{O}(n)$$

# Heapsort Algorithm

The HEAPSORT algorithm

(1) Build a heap on array A[1…$n$] by calling BUILD-HEAP(A, $n$)

(2) The largest element is stored at the root A[1]

Put it into its correct final position A[$n$] by A[1] $\leftrightarrow$ A[$n$]

(3) Discard node $n$ from the heap

(4) Subtrees ($S_2$ & $S_3$) rooted at children of root remain as heaps

but the new root element may violate the heap property

Make A[1…$n$ – 1] a heap by calling HEAPIFY(A, 1, $n$ – 1)

(5) $n \leftarrow n - 1$

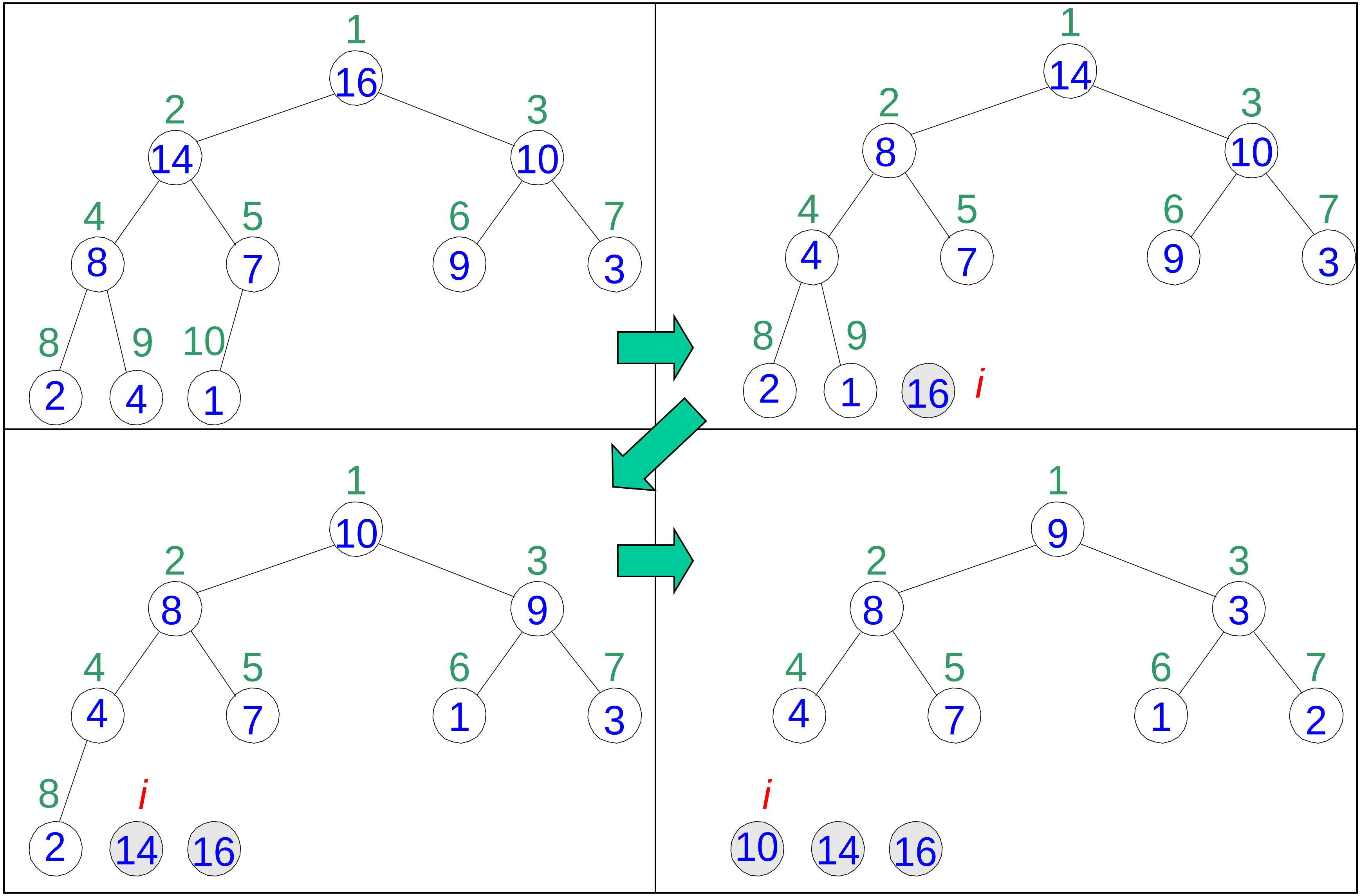
(6) Repeat steps 2–4 until $n = 2$

# Heapsort Algorithm

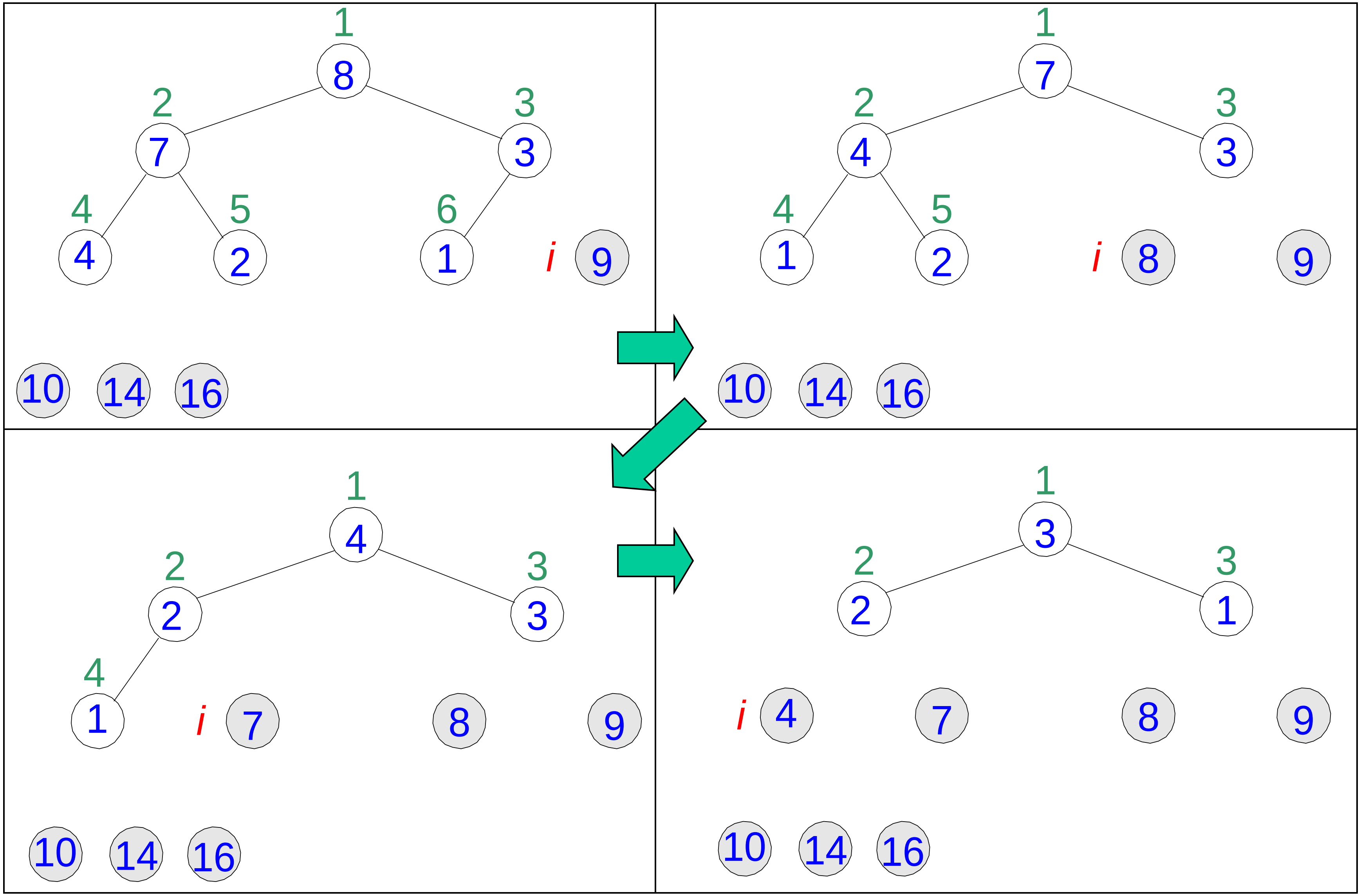
HEAPSORT(A, $n$)

    BUILD-HEAP(A, $n$)
    for $i \leftarrow n$ downto 2 do
        exchange A[1] $\leftrightarrow$ A[$i$]
        HEAPIFY(A, 1, $i$ –1)

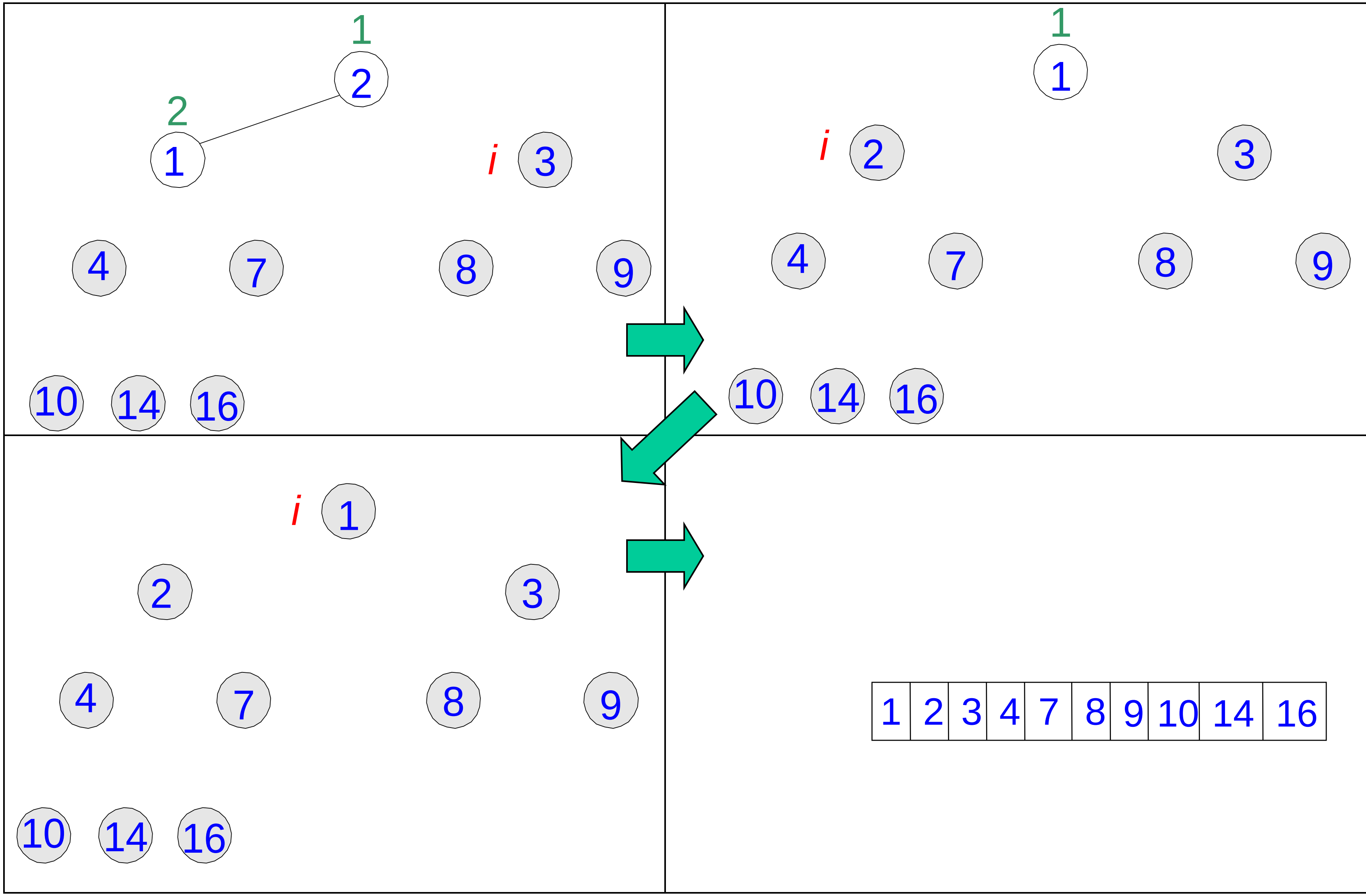
# Heapsort: Example



# Heapsort: Example



# Heapsort: Example



# Heapsort Run Time Analysis

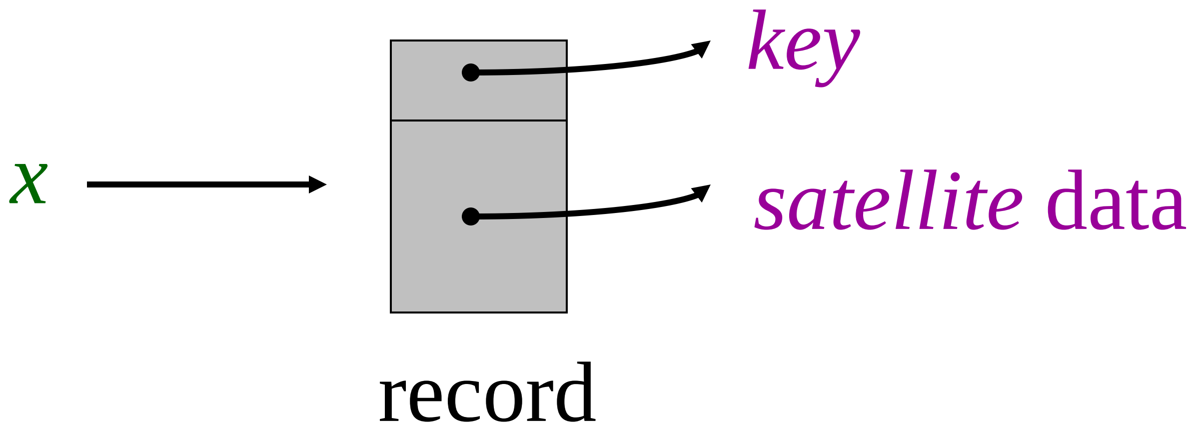
- BUILD-HEAP takes O($n$) time
- $i$-th iteration of for loop takes O(lg($n$ – $i$)) time

$$T(n) = \sum_{i=1}^{n-1} O(\lg(n-i)) = \sum_{k=1}^{n-1} O(\lg k) = O\left(\sum_{k=1}^{n-1} \lg k\right) = O(n \lg n)$$

- Heapsort is a very good algorithm but, a good implementation of quicksort always beats heapsort in practice
- However, heap data structure has many popular applications, and it can be efficiently used for implementing priority queues

# Data structures for Dynamic Sets

- Consider sets of records having *key* and *satellite* data



# Operations on Dynamic Sets

- Queries: Simply return info; Modifying operations: Change the set

– INSERT($S$, $x$): (Modifying) $S \leftarrow S \cup \{x\}$

– DELETE($S$, $x$): (Modifying) $S \leftarrow S - \{x\}$

– MAX($S$) / MIN($S$): (Query) return $x \in S$ with the largest/smallest *key*

– EXTRACT-MAX($S$) / EXTRACT-MIN($S$) : (Modifying) return and delete $x \in S$ with the largest/smallest *key*

– SEARCH($S$, $k$): (Query) return $x \in S$ with $key[x] = k$

– SUCCESSOR($S$, $x$) / PREDECESSOR($S$, $x$) : (Query) return $y \in S$ which is the next larger/smaller element after $x$

- Different data structures support/optimize different operations

# Priority Queues (*PQ*)

- Supports
  - INSERT
  - MAX / MIN
  - EXTRACT-MAX / EXTRACT-MIN
- One application: Schedule jobs on a shared resource
  - PQ keeps track of jobs and their relative priorities
  - When a job is finished or interrupted, highest priority job is selected from those pending using EXTRACT-MAX
  - A new job can be added at any time using INSERT

# Priority Queues

- Another application: Event-driven simulation
  - Events to be simulated are the items in the PQ
  - Each event is associated with a time of occurrence which serves as a *key*
  - Simulation of an event can cause other events to be simulated in the future
  - Use EXTRACT-MIN at each step to choose the next event to simulate
  - As new events are produced insert them into the PQ using INSERT

# Implementation of Priority Queue

- Sorted linked list: Simplest implementation
  - INSERT
    - O($n$) time
    - Scan the list to find place and splice in the new item
  - EXTRACT-MAX
    - O(1) time
    - Take the first element

▷ Fast extraction but slow insertion.

# Implementation of Priority Queue

- Unsorted linked list: Simplest implementation
  - INSERT
    - O(1) time
    - Put the new item at front
  - EXTRACT-MAX
    - O($n$) time
    - Scan the whole list

▹ Fast insertion but slow extraction

Sorted linked list is better on the average

- Sorted list: on the average, scans $n/2$ elem. per insertion
- Unsorted list: always scans $n$ elem. at each extraction

# Heap Implementation of PQ

- INSERT and EXTRACT-MAX are both O(lg $n$)
  - good compromise between fast insertion but slow extraction and vice versa
- EXTRACT-MAX: already discussed HEAP-EXTRACT-MAX

INSERT: Insertion is like that of Insertion-Sort.

Traverses O(lg $n$) nodes, as HEAPIFY does but makes fewer comparisons and assignments

–HEAPIFY: compares parent with both children

–HEAP-INSERT: with only one

**HEAP-INSERT**(A, *key*, *n*)

$n \leftarrow n + 1$

$i \leftarrow n$

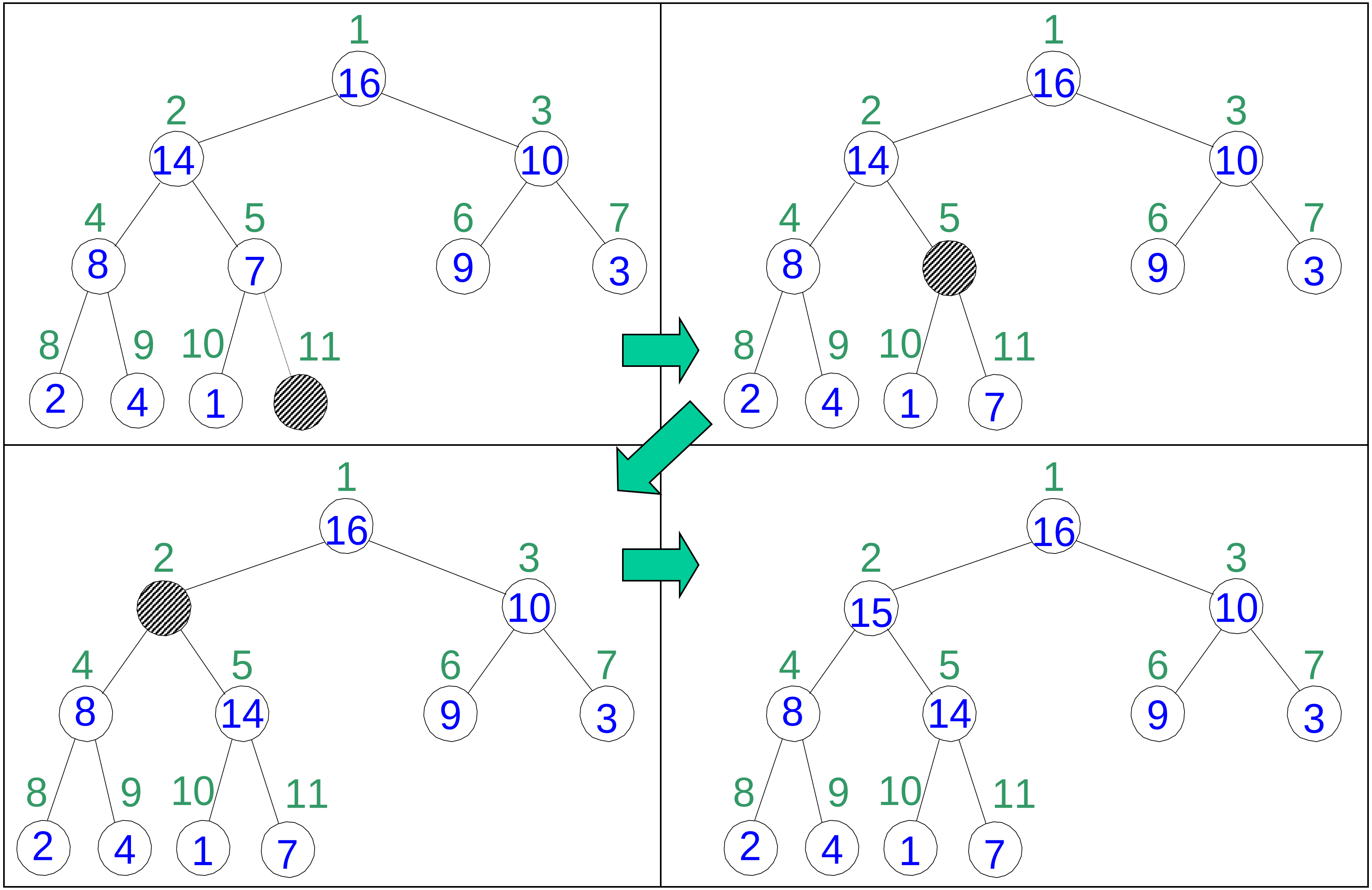
**while** $i > 1$ **and** $A[\lfloor i/2 \rfloor] < key$ **do**

$A[i] \leftarrow A[\lfloor i/2 \rfloor]$

$i \leftarrow \lfloor i/2 \rfloor$

$A[i] \leftarrow key$

# HEAP-INSERT(A, 15)

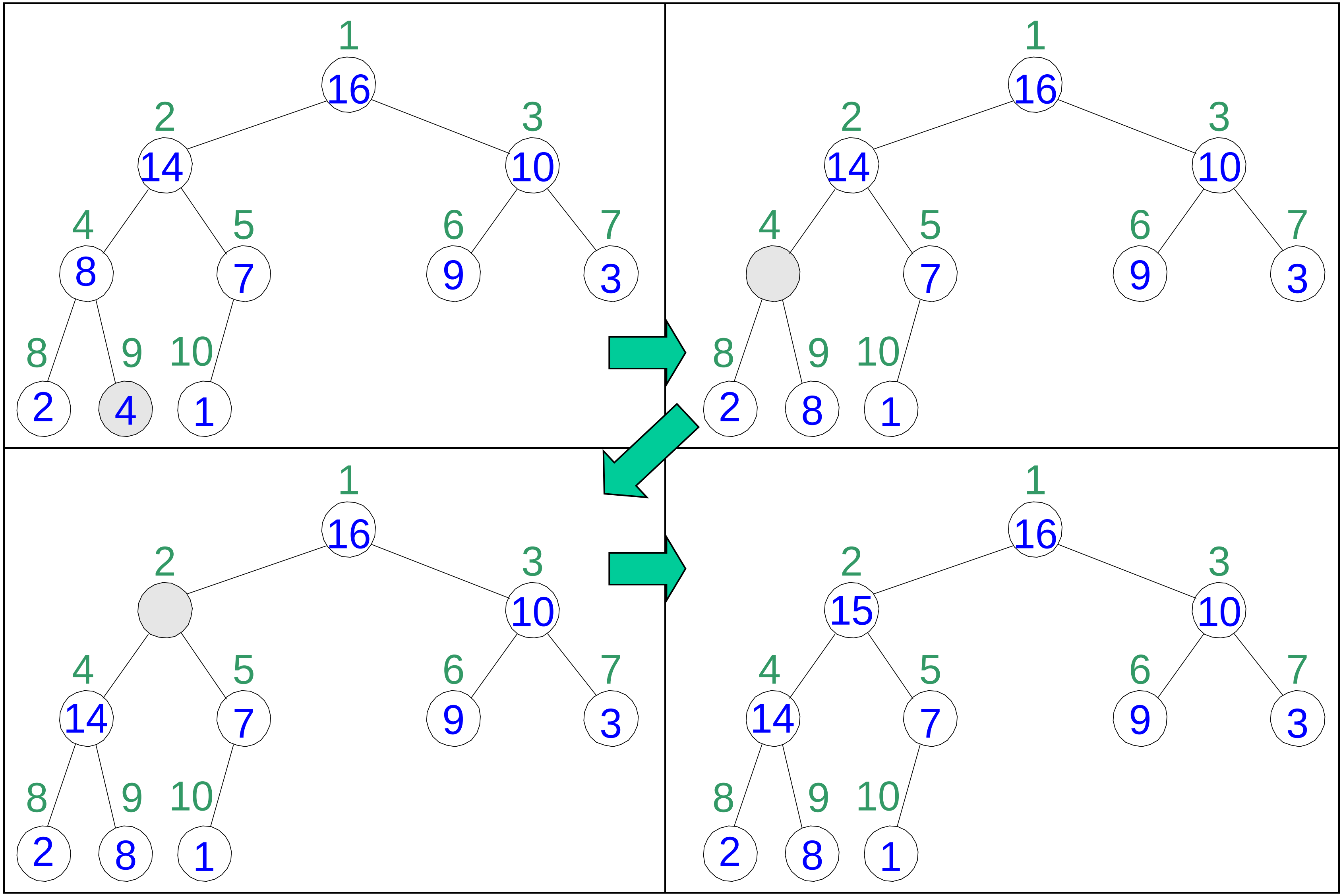


# Heap Increase Key

- Key value of *i*-th element of heap is increased from A[*i*] to *key*

```
HEAP-INCREASE-KEY(A, i, key)
    if key < A[i] then
        return error
    while i >1 and A[⌊i/2⌋] < key do
        A[i] ← A[⌊i/2⌋]
        i ← ⌊i/2⌋
    A[i] ← key
```

# HEAP-INCREASE-KEY(A, 9, 15)



# Heap Implementation of PQ

| | *key* | data | H-ptr |
|---|---|---|---|
| a | 14 | | 4 |
| b | 1 | | 10 |
| c | 10 | | 3 |
| d | 16 | | 1 |
| e | * | | ¬ |
| f | 9 | | 6 |
| g | 2 | | 8 |
| h | 15 | | 2 |
| i | * | | ¬ |
| j | 3 | | 7 |
| k | 7 | | 5 |
| l | * | | ¬ |
| m | 8 | | 9 |
| n | * | | ¬ |
| o | * | | ¬ |