CS473-Algorithms I

Lecture 13-A

Graphs

Graphs

A directed graph (or digraph) G is a pair (V, E), where

V is a finite set, and

E is a binary relation on V

The set V: Vertex set of G

The set E: Edge set of G

Note that, self-loops -edges from a vertex to itself- are possible

In an undirected graph $G=(V, E)$

- the edge set E consists of unordered pairs of vertices
  rather than ordered pairs,
  that is, (u, v) & (v, u) denote the same edge
- self-loops are forbidden, so
  every edge consists of two distinct vertices

Graphs

Many definitions for directed and undirected graphs are the same although certain terms have slightly different meanings

If $(u, v) \in E$ in a directed graph $G=(V, E)$, we say that
(u, v) is incident from or leaves vertex u and
is incident to or enters vertex v

If $(u, v) \in E$ in an undirected graph $G=(V, E)$, we say that
(u, v) is incident on vertices u and v

If (u, v) is an edge in a graph $G=(V, E)$, we say that
vertex v is adjacent to vertex u

When the graph is undirected,
the adjacency relation is symmetric

When the graph is directed
the adjacency relation is not necessarily symmetric
if v is adjacent to u, we sometimes write $u \to v$

Graphs

The degree of a vertex in an undirected graph is the number of edges incident on it

In a directed graph,

out-degree of a vertex: number of edges leaving it

in-degree of a vertex : number of edges entering it

degree of a vertex : its in-degree + its out-degree

A path of length k from a vertex u to a vertex u' in a graph $G=(V, E)$ is a sequence $\langle v_0, v_1, v_2, \ldots, v_k\rangle$ of vertices such that $v_0=u$, $v_k=u'$ and $(v_{i-1}, v_i) \in E$, for $i = 1, 2, \ldots, k$

The length of a path is the number of edges in the path

Graphs

If there is a path p from u to u', we say that u' is reachable from u via p: $u \xrightarrow{p} u'$

A path is simple if all vertices in the path are distinct

A subpath of path $p = \langle v_0, v_1, v_2, \ldots, v_k \rangle$ is a contiguous subsequence of its vertices

That is, for any $0 \le i \le j \le k$, the subsequence of vertices $\langle v_i, v_{i+1}, \ldots, v_j \rangle$ is a subpath of p

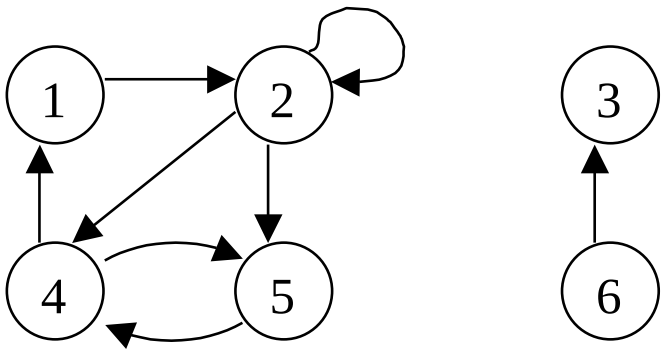
In a directed graph, a path $\langle v_0, v_1, \ldots, v_k \rangle$ forms a cycle if $v_0 = v_k$ and the path contains at least one edge

The cycle is simple if, in addition, $v_0, v_1, \ldots, v_k$ are distinct

A self-loop is a cycle of length 1

Graphs

Two paths $\langle v_0,v_1,v_2,\ldots,v_k\rangle$ & $\langle v_0',v_1',v_2',\ldots,v_k'\rangle$ form the same cycle if there is an integer j such that $v_i' = v_{(i+j) \bmod k}$ for $i = 0, 1,\ldots, k-1$



The path $p_1 = \langle 1, 2, 4, 1\rangle$ forms the same cycles as the paths $p_2 = \langle 2, 4, 1, 2\rangle$ and $p_3 = \langle 4, 1, 2, 4\rangle$

A directed graph with no self-loops is simple

In an undirected graph a path $\langle v_0,v_1,\ldots,v_k\rangle$ forms a cycle if $v_0=v_k$ and $v_1,v_2,\ldots,v_k$ are distinct

A graph with no cycles is acyclic

Graphs

An undirected graph is connected
if every pair of vertices is connected by a path

The connected components of a graph are the
equivalence classes of vertices under the
"is reachable from" relation

An undirected graph is connected if it has exactly one component,
i.e., if every vertex is reachable from every other vertex

A directed graph is strongly-connected
if every two vertices are reachable from each other

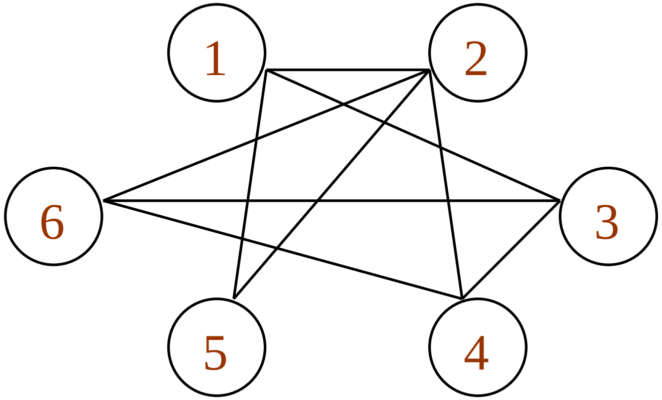
The strongly-connected components of a digraph are the
equivalence classes of vertices under the
"are mutually reachable" relation

A directed graph is strongly-connected
if it has only one strongly-connected component

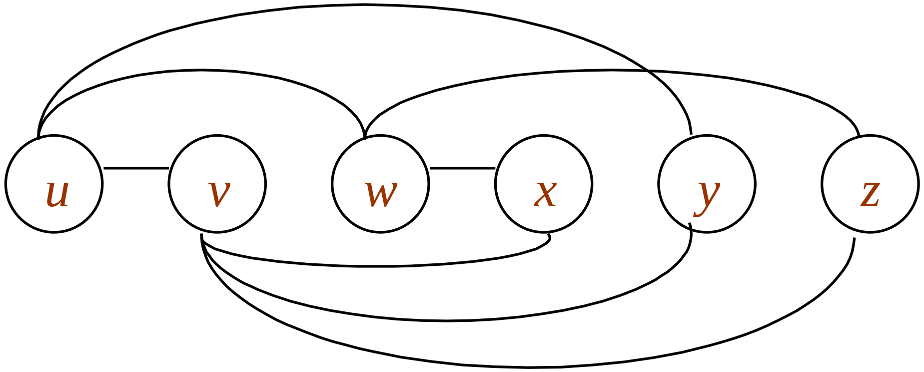
Graphs

Two graphs $G=(V, E)$ and $G'=(V', E')$ are isomorphic if there exists a bijection $f: V \rightarrow V'$ such that $(u, v) \in E$ iff $(f(u), f(v)) \in E'$

That is, we can relabel the vertices of G to be vertices of G' maintaining the corresponding edges in G and G'



$G=(V, E)$
$V=\{1,2,3,4,5,6\}$



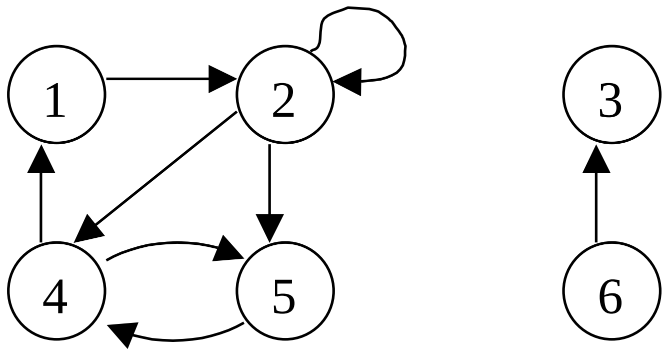
$G'=(V', E')$
$V'=\{u,v,w,x,y,z\}$

Map from $V \rightarrow V'$: $f(1)=u$, $f(2)=v$, $f(3)=w$, $f(4)=x$, $f(5)=y$, $f(6)=z$

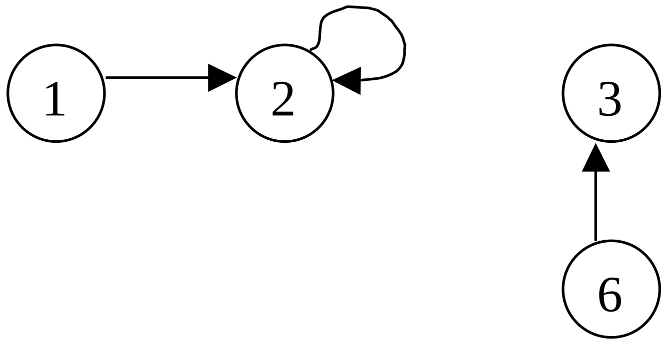
Graphs

A graph $G'=(V', E')$ is a subgraph of $G=(V, E)$ if
$V' \subseteq V$ and $E' \subseteq E$

Given a set $V' \subseteq V$, the subgraph of G induced by V' is the graph
$G'=(V', E')$ where $E'=\{(u,v) \in E: u,v \in V'\}$



$G=(V, E)$



$G'=(V', E')$, the subgraph of G induced by the vertex set $V'=\{1,2,3,6\}$

Graphs

Given an undirected graph $G=(V, E)$, the directed version of G is the directed graph $G'=(V', E')$, where $(u,v)\in E'$ and $(v,u)\in E' \Leftrightarrow (u,v)\in E$

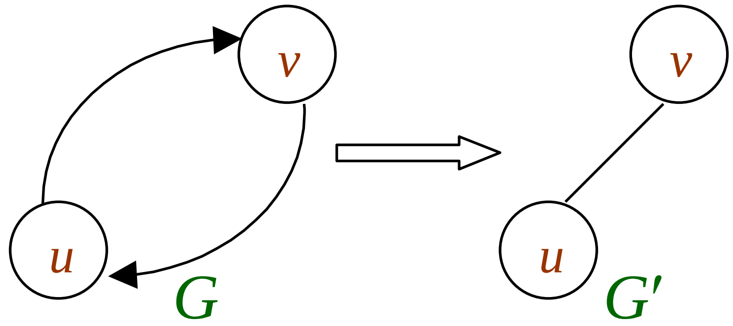
That is, each undirected edge (u,v) in G is replaced in G' by two directed edges (u,v) and (v,u)

Given a directed graph $G=(V, E)$, the undirected version of G is the undirected graph $G'=(V', E')$, where $(u,v)\in E' \Leftrightarrow u\neq v$ and $(u,v)\in E$

That is the undirected version contains the edges of G "with their directions removed" and with self-loops eliminated

Graphs

Note:



i.e., (u,v) and (v,u) in G are replaced in G' by the same edge (u,v)

In a directed graph G=(V, E), a neighbor of a vertex u is any vertex that is adjacent to u in the undirected version of G

That, is v is a neighbor of u iff either $(u,v)\in E$ or $(v,u)\in E$

v is a neighbor of u in both cases

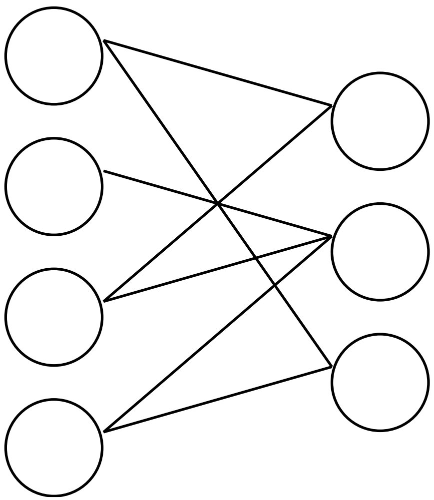
In an undirected graph, u and v are neighbors if they are adjacent

Graphs

Several kinds of graphs are given special names

Complete graph: undirected graph in which every pair of vertices is adjacent

Bipartite graph: undirected graph $G=(V, E)$ in which V can be partitioned into two disjoint sets V_1 and V_2 such that $(u,v) \in E$ implies either $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$



Graphs

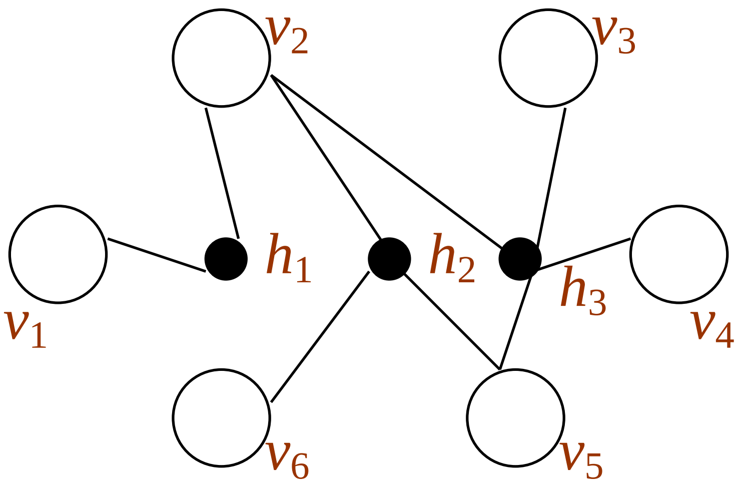
Forest: acyclic, undirected graph

Tree: connected, acyclic, undirected graph

Dag: directed acyclic graph

Multigraph: undirected graph with multiple edges between vertices and self-loops

Hypergraph: like an undirected graph, but each hyperedge, rather than connecting two vertices, connects an arbitrary subset of vertices



$$h_1 = (v_1, v_2)$$

$$h_2 = (v_2, v_5, v_6)$$

$$h_3 = (v_2, v_3, v_4, v_5)$$

Free Trees

- A free tree is a connected, acyclic, undirected graph
- We often omit the adjective “free” when we say that a graph is a tree
- If an undirected graph is acyclic but possibly disconnected it is a forest

Theorem (Properties of Free Trees)

The following are equivalent for an undirected graph $G=(V,E)$

1. G is a free tree
2. Any two vertices in G are connected by a unique simple-path
3. G is connected, but if any edge is removed from E the resulting graph is disconnected
4. G is connected, and $|E| = |V|-1$
5. G is acyclic, and $|E| = |V| - 1$
6. G is acyclic, but if any edge is added to E, the resulting graph contains a cycle

Properties of Free Trees (1⇒2)

(1) G is a free tree

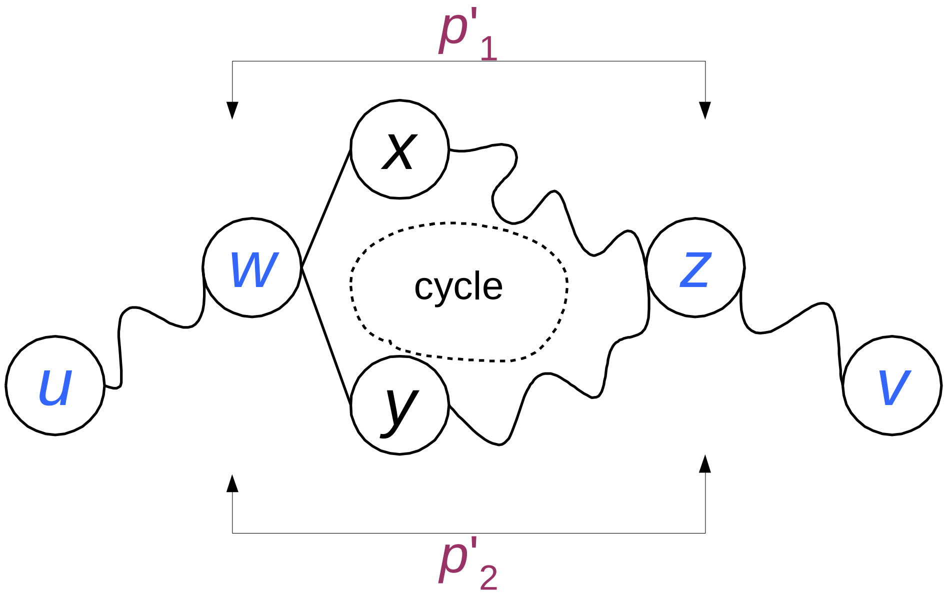
(2) Any two vertices in G are connected by a unique simple-path

Properties of Free Trees ($1 \Rightarrow 2$)

Since a tree is connected, any two vertices in G are connected by a simple path

- Let two vertices $u,v \in V$ are connected by two simple paths p_1 and p_2
- Let w and z be the first vertices at which p_1 and p_2 diverge and re-converge
- Let p'_1 be the subpath of p_1 from w to z
- Let p'_2 be the subpath of p_2 from w to z
- p'_1 and p'_2 share no vertices except their end points
- The path $p'_1 \| p'_2$ is a cycle (contradiction)

Properties of Free Trees (1⇒2)



- p'_1 and p'_2 share no vertices except their end points
- $p'_1 \| p'_2$ is a cycle (contradiction)
- Thus, if G is a tree, there can be at most one path between two vertices

Properties of Free Trees ($2 \Rightarrow 3$)

(2) Any two vertices in G are connected by a unique simple-path

(3) G is connected, but if any edge is removed from E the resulting graph is disconnected

Properties of Free Trees ($2 \Rightarrow 3$)

If any two vertices in G are connected by a unique simple path, then G is connected

- Let (u,v) be any edge in E. This edge is a path from u to v. So it must be the unique path from u to v
- Thus, if we remove (u,v) from G, there is no path from u to v
- Hence, its removal disconnects G

Properties of Free Trees ($3 \Rightarrow 4$)

Before proving $3 \Rightarrow 4$ consider the following

Lemma: any connected, undirected graph $G=(V,E)$ satisfies $|E| \geq |V|-1$

Proof: Consider a graph G' with $|V|$ vertices and no edges. Thus initially there are $|C|=|V|$ connected components

- Each isolated vertex is a connected component

Consider an edge (u,v) and let C_u and C_v denote the connected-components of u and v

Properties of Free Trees (Lemma)

If $C_u \neq C_v$ then (u,v) connects C_u and C_v into a connected component C_{uv}

Otherwise (u,v) adds an extra edge to the connected component $C_u = C_v$

Hence, each edge added to the graph reduces the number of connected components by at most 1

Thus, at least $|V|-1$ edges are required to reduce the number of components to 1   Q.E.D

Properties of Free Trees ($3 \Rightarrow 4$)

(3) G is connected, but if any edge is removed from E the resulting graph is disconnected

(4) G is connected, and $|E| = |V|–1$

Properties of Free Trees ($3 \Rightarrow 4$)

By assuming (3), the graph G is connected

We need to show both $|E| \geq |V|-1$ and $|E| \leq |V|-1$ in order to show that $|E| = |V|-1$

$|E| \geq |V|-1$: valid due previous lemma

$|E| \leq |V|-1$: (proof by induction)

<u>Basis</u>: a connected graph with $n = 1$ or $n = 2$ vertices has $n-1$ edges

<u>IH</u>: suppose that all graphs $G' = (V', E')$ satisfying (3) also satisfy $|E'| \leq |V'|-1$

Properties of Free Trees ($3 \Rightarrow 4$)

Consider $G=(V,E)$ that satisfies (3) with $|V| = n \geq 3$

Removing an arbitrary edge (u,v) from G separates the graph into 2 connected graphs $G_u=(V_u,E_u)$ and $G_v=(V_v,E_v)$ such that $V = V_u \cup V_v$ and $E = E_u \cup E_v$

Hence, connected graphs G_u and G_v both satisfy (3) else G would not satisfy (3)

Note that $|V_u|$ and $|V_v| < n$ since $|V_u| + |V_v| = n$

Hence, $|E_u| \leq |V_u|-1$ and $|E_v| \leq |V_v|-1$ (by IH)

Thus, $|E| = |E_u| + |E_v| + 1 \leq (|V_u|-1) + (|V_v|-1) + 1$

$\Rightarrow \quad |E| \leq |V|-1$

Q.E.D

Properties of Free Trees ($4 \Rightarrow 5$)

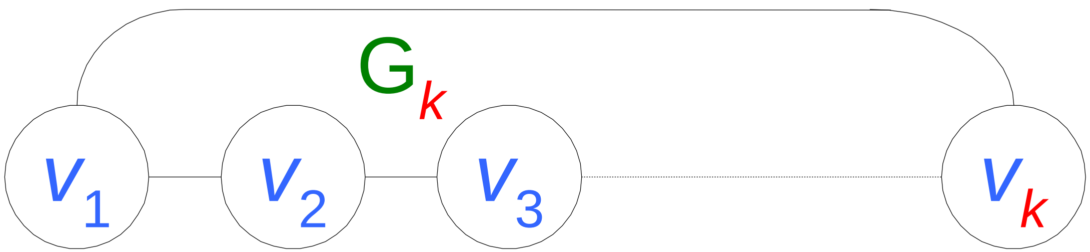
(4) G is connected, and $|E| = |V| - 1$

(5) G is acyclic, and $|E| = |V| - 1$

Properties of Free Trees (4⇒5)

Suppose that G is connected, and $|E| = |V|-1$, we must show that G is acyclic

- Suppose G has a cycle containing k vertices $v_1, v_2, \ldots, v_k$
- Let $G_k=(V_k,E_k)$ be subgraph of G consisting of the cycle

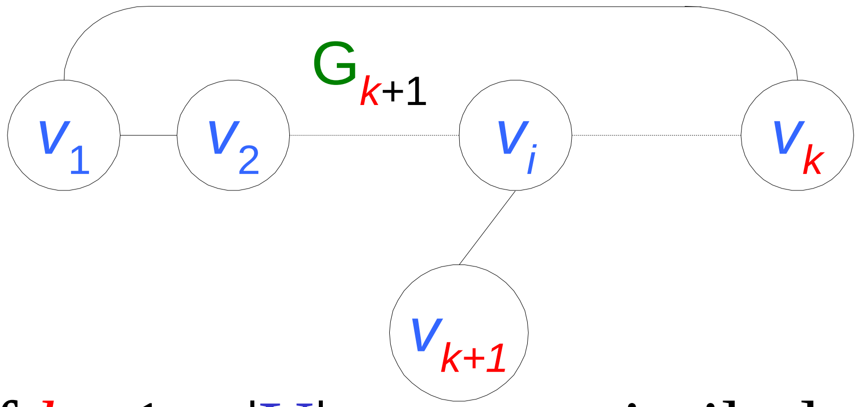


Note: $|V_k| = |E_k| = k$

If $k < |V|$, there must be a vertex $v_{k+1} \in V-V_k$ that is adjacent to some vertex $v_i \in V_k$, since G is connected

Properties of Free Trees (4⇒5)

Define $G_{k+1}=(V_{k+1},E_{k+1})$ to be subgraph of G with $V_{k+1}=V_k \cup v_{k+1}$ and $E_{k+1} = E_k \cup (v_{k+1},v_i)$



Note: $|V_{k+1}|=|E_{k+1}|$

If $k+1 < |V|$, we can similarly define $G_{k+2}=(V_{k+2},E_{k+2})$ to be the subgraph of G with $V_{k+2}=V_{k+1} \cup v_{k+2}$ and $E_{k+2} = E_{k+1} \cup (v_{k+2},v_j)$ for some $v_j \in V_{k+1}$ where $|V_{k+2}|=|E_{k+2}|$

Properties of Free Trees ($4 \Rightarrow 5$)

We can continue defining G_{k+m} with $|V_{k+m}|=|E_{k+m}|$ until we obtain $G_n=(V_n,E_n)$ where

$n=|V|$ and $V_n=|V|$ and $|V_n|=|E_n|=|V|$

- Since G_n is a subgraph of G, we have

  $E_n \subseteq E \Rightarrow |E| \geq |E_n|=|V|$ which contradicts the assumption $|E|=|V|-1$

Hence G is acyclic

Q.E.D

Properties of Free Trees (5⇒6)

(5) G is acyclic, and $|E| = |V| - 1$

(6) G is acyclic, but if any edge is added to E, the resulting graph contains a cycle

Properties of Free Trees (5⇒6)

Suppose that G is acyclic and $|E| = |V| - 1$

- Let k be the number of connected components of G

$G_1=(V_1,E_1)$, $G_2=(V_2,E_2)$,…, $G_k=(V_k,E_k)$ such that

$$\bigcup_{i=1}^{k} V_i = V; \quad V_i \cap V_j = \varnothing; \quad 1 \le i, j \le k \text{ and } i \ne j$$

$$\bigcup_{i=1}^{k} E_i = E; \quad E_i \cap E_j = \varnothing; \quad 1 \le i, j \le k \text{ and } i \ne j$$

Each connected component G_i is a tree by definition

Properties of Free Trees (5⇒6)

Since (1⇒5) each component G_i is satisfies

$|E_i| = |V_i| - 1$ for $i = 1, 2, \ldots, k$

- Thus

$$\sum_{i=1}^{k} |E_i| = \sum_{i=1}^{k} |V_i| - \sum_{i=1}^{k} 1$$

$$|E| = |V| - k$$

- Therefore, we must have $k = 1$

Properties of Free Trees (5$\Rightarrow$6)

That is (5) $\Rightarrow$ G is connected $\Rightarrow$ G is a tree

Since (1$\Rightarrow$2)

any two vertices in G are connected by a unique simple path

Thus,

adding any edge to G creates a cycle

Properties of Free Trees (6$\Rightarrow$1)

(6) G is acyclic, but if any edge is added to E, the resulting graph contains a cycle

(1) G is a free tree

Properties of Free Trees (6$\Rightarrow$1)

Suppose that G is acyclic but if any edge is added to E a cycle is created

We must show that G is connected due to the definition

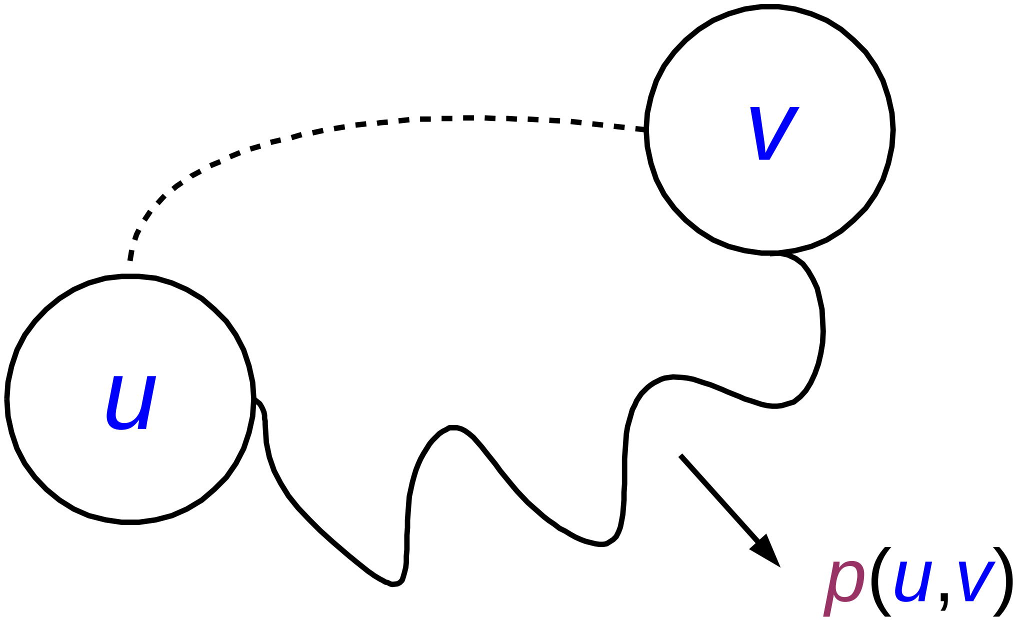
Let u and v be two arbitrary vertices in G

If u and v are not already adjacent

adding the edge (u,v) creates a cycle in which all edges but (u,v) belong to G

Properties of Free Trees ($6 \Rightarrow 1$)

Thus there is a path from u to v, and since u and v are chosen arbitrarily G is connected



Representations of Graphs

- The standard two ways to represent a graph $G=(V,E)$
  - As a collection of adjacency-lists
  - As an adjacency-matrix
- Adjacency-list representation is usually preferred
  - Provides a compact way to represent sparse graphs
    - Those graphs for which $|E| << |V|^2$

Representations of Graphs

- Adjacency-matrix representation may be preferred
  - for dense graphs for which $|E|$ is close to $|V|^2$
  - when we need to be able to tell quickly if there is an edge connecting two given vertices

Adjacency-List Representation

- An array Adj of |V| lists, one for each vertex $u \in V$
- For each $u \in V$ the adjacency-list Adj[u] contains (pointers to) all vertices v such that $(u,v) \in E$
- That is, Adj[u] consists of all vertices adjacent to u in G
- The vertices in each adjacency-list are stored in an arbitrary order

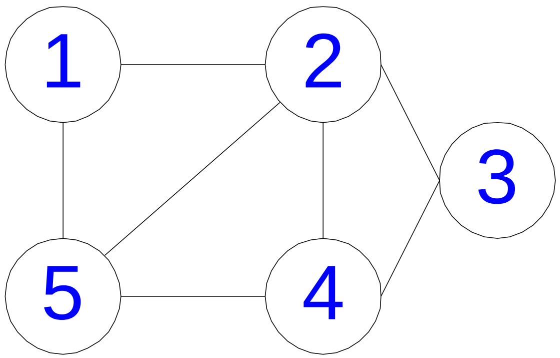
Adjacency-List Representation

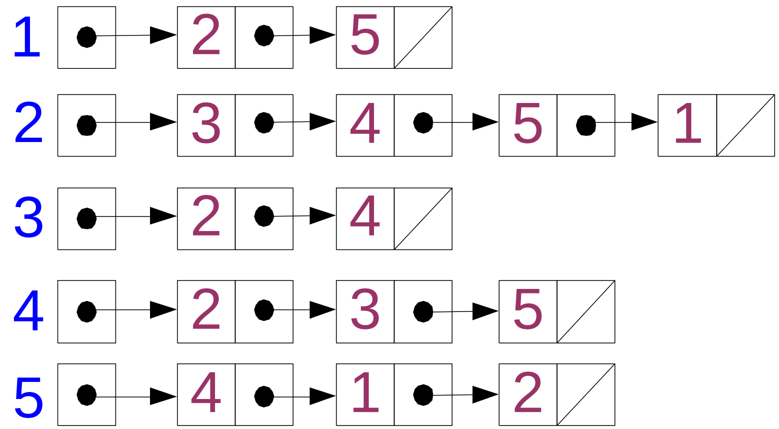
- If G is a directed graph
  - The sum of the lengths of the adjacency lists = | E |
- If G is an undirected graph
  - The sum of the lengths of the adjacency lists = 2| E |

  since an edge (u,v) appears in both Adj[u] and Adj[v]

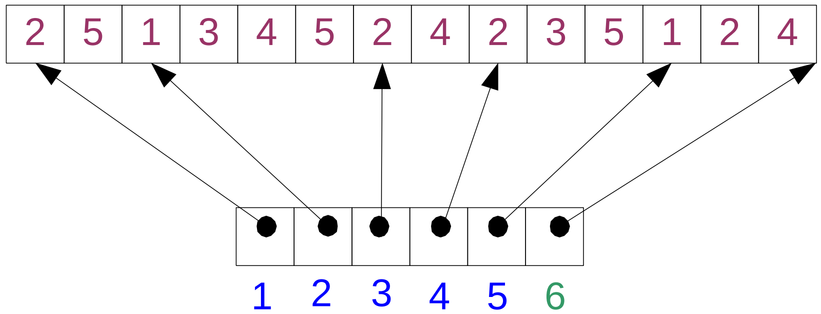
Representations of Graphs

Undirected Graphs



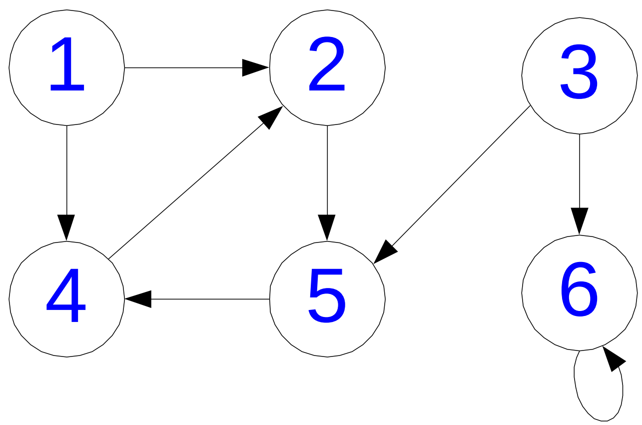


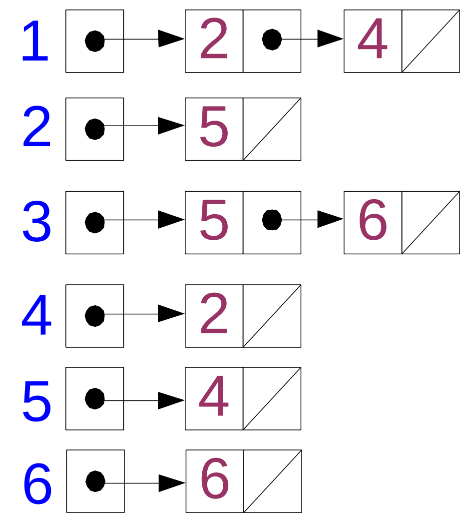
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0



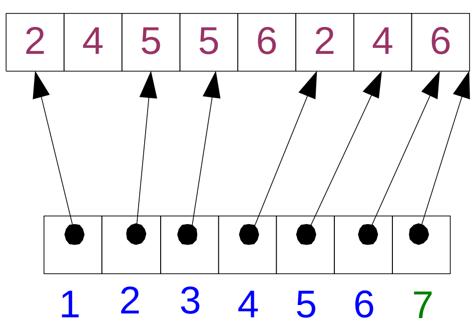
Representations of Graphs

Directed Graphs





1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1



Adjacency List Representation (continued)

Adjacency list representation has the desirable property
 it requires $O(\max(V, E)) = O(V+E)$ memory
 for both undirected and directed graphs

Adjacency lists can be adopted to represent weighted graphs
 each edge has an associated weight typically given by a
 weight function $w: E \rightarrow R$

The weight $w(u, v)$ of an edge $(u, v) \in E$ is simply stored with
 vertex v in Adj[u] or with
 vertex u in Adj[v] or both

Adjacency List Representation (continued)

A potential disadvantage of adjacency list representation
there is no quicker way to determine if a given edge (u, v) is present in G than to search v in Adj[u] or u in Adj[v]

This disadvantage can be remedied by an adjacency matrix representation at the cost of using asymptotically more memory

Adjacency Matrix Representation

Assume that, the vertices of G=(V, E) are numbered as 1,2,…,$|V|$

Adjacency matrix rep. consists of a $|V| \times |V|$ matrix A=(a_{ij}) ∍

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Requires $\Theta(V^2)$ memory independent of the number of edges $|E|$

We define the transpose of a matrix A=(a_{ij}) to be the matrix $A^T = (a_{ij})^T$ given by $a_{ij}{}^T = a_{ji}$

Since in an undirected graph, (u,v) and (v,u) represent the same edge $A = A^T$ for an undirected graph

That is, adjacency matrix of an undirected graph is symmetric

Hence, in some applications, only upper triangular part is stored

Adjacency Matrix Representation

Adjacency matrix representation can also be used for weighted graphs

$$a_{ij} = \begin{cases} w(i,j) & \text{if } (i,j) \in E \\ \text{NIL or 0 or } \infty & \text{otherwise} \end{cases}$$

Adjacency matrix may also be preferable for reasonably small graphs

Moreover, if the graph is unweighted
rather than using one word of memory for each matrix entry
adjacency matrix representation uses one bit per entry