Lecture 4
The Divide-and-Conquer Design Paradigm

View in slide-show mode
Reminder: Merge Sort

Input array $A$

Divide

sort this half

Conquer

sort this half

Combine

merge two sorted halves
The Divide-and-Conquer Design Paradigm

1. *Divide* the problem (instance) into subproblems.

2. *Conquer* the subproblems by solving them recursively.

3. *Combine* subproblem solutions.
Example: Merge Sort

1. **Divide**: Trivial.

2. **Conquer**: Recursively sort 2 subarrays.

3. **Combine**: Linear-time merge.

\[ T(n) = 2 \cdot T(n/2) + \Theta(n) \]

- # subproblems
- subproblem size
- work dividing and combining
Master Theorem: Reminder

\[ T(n) = aT(n/b) + f(n) \]

**Case 1:**

\[ \frac{n^{\log_b a}}{f(n)} = (n) \]

\[ T(n) = (n^{\log_b a}) \]

**Case 2:**

\[ \frac{f(n)}{n^{\log_b a}} = (\log^k n) \]

\[ T(n) = (n^{\log_b a} \log^{k+1} n) \]

**Case 3:**

\[ \frac{f(n)}{n^{\log_b a}} = \Omega(n^\varepsilon) \]

\[ T(n) = (f(n)) \]

and \( af(n/b) \leq cf(n) \) for \( c < 1 \)
Merge Sort: Solving the Recurrence

\[ T(n) = 2 \ T(n/2) + \Theta(n) \]

\[ a = 2, \quad b = 2, \quad f(n) = \Theta(n), \quad n^{\log_b a} = n \]

**Case 2:**

\[ \frac{f(n)}{n^{\log_b a}} = (\lg^k n) \]

\[ T(n) = (n^{\log_b a} \lg^{k+1} n) \]

holds for \( k = 0 \)

\[ T(n) = \Theta(n \lg n) \]
Binary Search

Find an element in a sorted array:

1. **Divide**: Check middle element.
2. **Conquer**: Recursively search 1 subarray.
3. **Combine**: Trivial.

**Example**: Find 9
Binary Search

Find an element in a sorted array:

1. **Divide**: Check middle element.
2. **Conquer**: Recursively search 1 subarray.
3. **Combine**: Trivial.

**Example**: Find 9
Binary Search

Find an element in a sorted array:

1. **Divide:** Check middle element.
2. **Conquer:** Recursively search 1 subarray.
3. **Combine:** Trivial.

*Example:* Find 9

3 5 7 8 9 12 15
Binary Search

Find an element in a sorted array:

1. **Divide**: Check middle element.
2. **Conquer**: Recursively search 1 subarray.
3. **Combine**: Trivial.

**Example**: Find 9

```
3   5   7   8   9   12   15
```
Binary Search

Find an element in a sorted array:

1. **Divide**: Check middle element.

2. **Conquer**: Recursively search 1 subarray.

3. **Combine**: Trivial.

**Example**: Find 9

3 5 7 8 9 12 15
Binary Search

Find an element in a sorted array:

1. **Divide**: Check middle element.
2. **Conquer**: Recursively search 1 subarray.
3. **Combine**: Trivial.

**Example**: Find 9

3  5  7  8  9  12  15
Recurrence for Binary Search

\[ T(n) = 1 \cdot T(n/2) + \Theta(1) \]

- # subproblems
- subproblem size
- work dividing and combining
Binary Search: Solving the Recurrence

\[ T(n) = T(n/2) + \Theta(1) \]

Case 2:

\[ \frac{f(n)}{n^{\log_b a}} = (\lg^k n) \]

\[ T(n) = (n^{\log_b a} \lg^{k+1} n) \]

holds for \( k = 0 \)

\[ T(n) = \Theta(\lg n) \]
Powering a Number

- Problem: Compute $a^n$, where $n$ is a natural number

**Naive-Power** ($a, n$)

```
powerVal ← 1
for i ← 1 to n
    powerVal ← powerVal . a
return powerVal
```

- What is the complexity? $T(n) = \Theta(n)$
Powering a Number: Divide & Conquer

Basic idea:

\[ a^n = \begin{cases} 
  a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even} \\
  a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd}
\end{cases} \]
POWER \ (a, \ n) \\
\quad \textbf{if} \ n = 0 \ \textbf{then return} \ 1 \\
\quad \textbf{else if} \ n \ \textbf{is even then} \\
\quad \quad \text{val} \leftarrow \text{POWER} \ (a, \ n/2) \\
\quad \quad \text{return} \ \text{val} \ast \text{val} \\
\quad \textbf{else if} \ n \ \textbf{is odd then} \\
\quad \quad \text{val} \leftarrow \text{POWER} \ (a, \ (n-1)/2) \\
\quad \quad \text{return} \ \text{val} \ast \text{val} \ast \ a
Powering a Number: Solving the Recurrence

\[ T(n) = T(n/2) + \Theta(1) \]

\[ a = 1, \quad b = 2, \quad f(n) = \Theta(1), \quad n^{\log_b a} = n^0 = 1 \]

Case 2:

\[ \frac{f(n)}{n^{\log_b a}} = (\lg^k n) \]

\[ T(n) = (n^{\log_b a} \lg^{k+1} n) \]

holds for \( k = 0 \)

\[ T(n) = \Theta (\lg n) \]
Matrix Multiplication

Input: $A = [a_{ij}]$, $B = [b_{ij}]$. \[ i, j = 1, 2, \ldots, n. \]

Output: $C = [c_{ij}] = A \cdot B.$

\[
\begin{pmatrix}
  c_{11} & c_{12} & \ldots & c_{1n} \\
  c_{21} & c_{22} & \ldots & c_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{n1} & c_{n2} & \ldots & c_{nn}
\end{pmatrix}
= 
\begin{pmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \ldots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
  b_{11} & b_{12} & \ldots & b_{1n} \\
  b_{21} & b_{22} & \ldots & b_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{n1} & b_{n2} & \ldots & b_{nn}
\end{pmatrix}
\]

\[
c_{ij} = \sum_{1 \leq k \leq n} a_{ik} \cdot b_{kj}
\]
Standard Algorithm

for $i \leftarrow 1$ to $n$
   do for $j \leftarrow 1$ to $n$
      do $c_{ij} \leftarrow 0$
         for $k \leftarrow 1$ to $n$
            do $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$

Running time $= \Theta(n^3)$
Matrix Multiplication: Divide & Conquer

IDEA: **Divide** the $n \times n$ matrix into

2x2 matrix of $(n/2) \times (n/2)$ submatrices

$$
\begin{bmatrix}
    c_{11} & c_{12} \\
    c_{21} & c_{22}
\end{bmatrix}
= \begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix}
\cdot \begin{bmatrix}
    b_{11} & b_{12} \\
    b_{21} & b_{22}
\end{bmatrix}
$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21}$$
Matrix Multiplication: Divide & Conquer

IDEA: **Divide** the \( n \times n \) matrix into

\[ 2 \times 2 \] matrix of \( (n/2) \times (n/2) \) submatrices

\[
\begin{bmatrix}
  c_{11} & c_{12} \\
  c_{21} & c_{22}
\end{bmatrix}
=\begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix}
\cdot\begin{bmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{bmatrix}
\]

\[
c_{12} = a_{11}b_{12} + a_{12}b_{22}
\]
Matrix Multiplication: Divide & Conquer

IDEA: **Divide** the $n \times n$ matrix into

2x2 matrix of $(n/2) \times (n/2)$ submatrices

\[
\begin{pmatrix}
  c_{11} & c_{12} \\
  c_{21} & c_{22}
\end{pmatrix} =
\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix} \cdot
\begin{pmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{pmatrix}
\]

\[c_{21} = a_{21} b_{11} + a_{22} b_{21}\]
Matrix Multiplication: Divide & Conquer

**IDEA:** **Divide** the $n \times n$ matrix into

2x2 matrix of $(n/2) \times (n/2)$ submatrices

$$
\begin{bmatrix}
  c_{11} & c_{12} \\
  c_{21} & c_{22}
\end{bmatrix}
= \begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix}
\times
\begin{bmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{bmatrix}
$$

$$c_{22} = a_{21} b_{12} + a_{22} b_{22}$$
Matrix Multiplication: Divide & Conquer

\[
\begin{align*}
C &= \begin{pmatrix}
c_{11} & c_{12} \\
\end{align*}
= \begin{pmatrix}
a_{11} & a_{12} \\
\end{pmatrix}
\cdot
\begin{pmatrix}
b_{11} & b_{12} \\
\end{pmatrix}
\begin{pmatrix}
c_{21} & c_{22} \\
\end{pmatrix}
\begin{pmatrix}
a_{21} & a_{22} \\
\end{pmatrix}
\begin{pmatrix}
b_{21} & b_{22} \\
\end{pmatrix}
\]

\[
c_{11} = a_{11}b_{11} + a_{12}b_{21}
\]
\[
c_{12} = a_{11}b_{12} + a_{12}b_{22}
\]
\[
c_{21} = a_{21}b_{11} + a_{22}b_{21}
\]
\[
c_{22} = a_{21}b_{12} + a_{22}b_{22}
\]

8 mults of \((n/2)\times(n/2)\) submatrices
4 adds of \((n/2)\times(n/2)\) submatrices
**Matrix Multiplication: Divide & Conquer**

**MATRIX-MULTIPLY** $(A, B)$

// Assuming that both $A$ and $B$ are $n \times n$ matrices

```plaintext
if n = 1 then return A * B
else
    partition $A$, $B$, and $C$ as shown before
    $c_{11} = \text{MATRIX-MULTIPLY} (a_{11}, b_{11}) + \text{MATRIX-MULTIPLY} (a_{12}, b_{21})$
    $c_{12} = \text{MATRIX-MULTIPLY} (a_{11}, b_{12}) + \text{MATRIX-MULTIPLY} (a_{12}, b_{22})$
    $c_{21} = \text{MATRIX-MULTIPLY} (a_{21}, b_{11}) + \text{MATRIX-MULTIPLY} (a_{22}, b_{21})$
    $c_{22} = \text{MATRIX-MULTIPLY} (a_{21}, b_{12}) + \text{MATRIX-MULTIPLY} (a_{22}, b_{22})$
    return $C$
```
Matrix Multiplication: Divide & Conquer

Analysis

$$T(n) = 8 \ T(n/2) + \Theta(n^2)$$

- 8 recursive calls
- Each subproblem has size $n/2$
- Submatrix addition
Matrix Multiplication: Solving the Recurrence

\[ T(n) = 8 \cdot T(n/2) + \Theta(n^2) \]

Case 1:

\[ \frac{n^{\log_b a}}{f(n)} = (n) \]

\[ T(n) = (n^{\log_b a}) \]

\[ T(n) = \Theta(n^3) \]

*No better than the ordinary algorithm!*
Matrix Multiplication: Strassen’s Idea

\[
\begin{pmatrix}
  c_{11} & c_{12} \\
  c_{21} & c_{22}
\end{pmatrix}
= \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix}
\cdot
\begin{pmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{pmatrix}
\]

Compute \(c_{11}, c_{12}, c_{21},\) and \(c_{22}\) using 7 recursive multiplications
Matrix Multiplication: Strassen’s Idea

\[ P_1 = a_{11} \times (b_{12} - b_{22}) \]
\[ P_2 = (a_{11} + a_{12}) \times b_{22} \]
\[ P_3 = (a_{21} + a_{22}) \times b_{11} \]
\[ P_4 = a_{22} \times (b_{21} - b_{11}) \]
\[ P_5 = (a_{11} + a_{22}) \times (b_{11} + b_{22}) \]
\[ P_6 = (a_{11} - a_{22}) \times (b_{21} + b_{22}) \]
\[ P_7 = (a_{11} - a_{21}) \times (b_{11} + b_{12}) \]

**Reminder**: Each submatrix is of size \((n/2)\times(n/2)\)

Each add/sub operation takes \(\Theta(n^2)\) time

Compute \(P_1..P_7\) using 7 recursive calls to matrix-multiply

**How to compute \(c_{ij}\) using \(P_1..P_7\) ?**
Matrix Multiplication: Strassen’s Idea

\[ P_1 = a_{11} \times (b_{12} - b_{22}) \]
\[ P_2 = (a_{11} + a_{12}) \times b_{22} \]
\[ P_3 = (a_{21} + a_{22}) \times b_{11} \]
\[ P_4 = a_{22} \times (b_{21} - b_{11}) \]
\[ P_5 = (a_{11} + a_{22}) \times (b_{11} + b_{22}) \]
\[ P_6 = (a_{11} - a_{22}) \times (b_{21} + b_{22}) \]
\[ P_7 = (a_{11} - a_{21}) \times (b_{11} + b_{12}) \]

\[ c_{11} = P_5 + P_4 - P_2 + P_6 \]
\[ c_{12} = P_1 + P_2 \]
\[ c_{21} = P_3 + P_4 \]
\[ c_{22} = P_5 + P_1 - P_3 - P_7 \]

7 recursive multiply calls
18 add/sub operations

Does not rely on commutativity of multiplication
Matrix Multiplication: Strassen’s Idea

\[ P_1 = a_{11} \times (b_{12} - b_{22}) \]
\[ P_2 = (a_{11} + a_{12}) \times b_{22} \]
\[ P_3 = (a_{21} + a_{22}) \times b_{11} \]
\[ P_4 = a_{22} \times (b_{21} - b_{11}) \]
\[ P_5 = (a_{11} + a_{22}) \times (b_{11} + b_{22}) \]
\[ P_6 = (a_{11} - a_{22}) \times (b_{21} + b_{22}) \]
\[ P_7 = (a_{11} - a_{21}) \times (b_{11} + b_{12}) \]

e.g. Show that \( c_{12} = P_1 + P_2 \)

\[ c_{12} = P_1 + P_2 \\
= a_{11}(b_{12} - b_{22}) + (a_{11} + a_{12})b_{22} \\
= a_{11}b_{12} - a_{11}b_{22} + a_{11}b_{22} + a_{12}b_{22} \\
= a_{11}b_{12} + a_{12}b_{22} \]
Strassen’s Algorithm

1. **Divide**: Partition A and B into \((n/2) \times (n/2)\) submatrices. Form terms to be multiplied using + and –.

2. **Conquer**: Perform 7 multiplications of \((n/2) \times (n/2)\) submatrices recursively.

3. **Combine**: Form C using + and – on \((n/2) \times (n/2)\) submatrices.

**Recurrence**: 
\[
T(n) = 7 T(n/2) + \Theta(n^2)
\]
Strassen’s Algorithm: Solving the Recurrence

\[ T(n) = 7 \ T(n/2) + \Theta(n^2) \]

\[ a = 7, \quad b = 2, \quad f(n) = \Theta(n^2), \quad \frac{n^\log_b a}{f(n)} = \frac{n^\log_b a}{\Theta(n^2)} = (n) \]

\[ T(n) = (n^\log_b a) \]

\[ T(n) = \Theta(n^{\log_7 7}) \]

Note: \( \log_7 7 \approx 2.81 \)
Strassen’s Algorithm

- The number 2.81 may not seem much smaller than 3.
- But, it is significant because the difference is in the exponent.
- Strassen’s algorithm beats the ordinary algorithm on today’s machines for \( n \geq 30 \) or so.
- Best to date: \( \Theta(n^{2.376...}) \) (of theoretical interest only)
VLSI Layout: Binary Tree Embedding

- **Problem**: Embed a complete binary tree with n leaves into a 2D grid with minimum area.

- **Example**: 

![Diagram of a binary tree and its embedding into a 2D grid.](image)
Binary Tree Embedding

- Use divide and conquer

1. Embed the root node
2. Embed the left subtree
3. Embed the right subtree

What is the min-area required for n leaves?
Binary Tree Embedding

\[ W(n) = 2W(n/2) + 1 \]

\[ H(n) = H(n/2) + 1 \]

\[ W(n/2) \]

\[ H(n/2) \]

\[ W(n) = 2W(n/2) + 1 \]
Binary Tree Embedding

- Solve the recurrences:
  \[ W(n) = 2W(n/2) + 1 \]
  \[ H(n) = H(n/2) + 1 \]

- \[ W(n) = \Theta(n) \]
- \[ H(n) = \Theta(lgn) \]

- \[ \text{Area}(n) = \Theta(n lgn) \]
Binary Tree Embedding

Example:

\[ W(n) \]

\[ H(n) \]
Binary Tree Embedding: H-Tree

- Use a different divide and conquer method

1. Embed root, left, right nodes
2. Embed subtree 1
3. Embed subtree 2
4. Embed subtree 3
5. Embed subtree 4

What is the min-area required for n leaves?
Binary Tree Embedding: H-Tree

\[
H(n) = 2H(n/4) + 1
\]

\[
W(n) = 2W(n/4) + 1
\]
Binary Tree Embedding: H-Tree

- Solve the recurrences:
  \[ W(n) = 2W(n/4) + 1 \]
  \[ H(n) = 2H(n/4) + 1 \]

  \[ \Rightarrow W(n) = \Theta(\sqrt{n}) \]
  \[ \Rightarrow H(n) = \Theta(\sqrt{n}) \]

- \[ \text{Area}(n) = \Theta(n) \]
Binary Tree Embedding: H-Tree

Example:

![Diagram of a grid representing H-tree with labels W(n) and H(n)]
Correctness Proofs

- **Proof by induction** commonly used for D&C algorithms

- **Base case**: Show that the algorithm is correct when the recursion bottoms out (i.e., for sufficiently small \( n \))

- **Inductive hypothesis**: Assume the alg. is correct for any recursive call on any smaller subproblem of size \( k \) (\( k < n \))

- **General case**: Based on the inductive hypothesis, prove that the alg. is correct for any input of size \( n \)
Example Correctness Proof: Powering a Number

\[
\text{POWER} \ (a, \ n) \\
\quad \text{if} \ n = 0 \ \text{then} \ \text{return} \ 1 \\
\quad \text{else if} \ n \ \text{is even} \ \text{then} \\
\quad \quad \text{val} \leftarrow \ \text{POWER} \ (a, \ n/2) \\
\quad \quad \text{return} \ \text{val} \ * \ \text{val} \\
\quad \text{else if} \ n \ \text{is odd} \ \text{then} \\
\quad \quad \text{val} \leftarrow \ \text{POWER} \ (a, \ (n-1)/2) \\
\quad \quad \text{return} \ \text{val} \ * \ \text{val} \ * \ a
\]
Example Correctness Proof: Powering a Number

- **Base case**: POWER \((a, 0)\) is correct, because it returns 1
- **Ind. hyp**: Assume POWER \((a, k)\) is correct for any \(k < n\)
- **General case**:
  In POWER \((a, n)\) function:
  
  If \(n\) is even:
  
  \[
  \text{val} = a^{n/2} \quad \text{(due to ind. hyp.)}
  \]
  
  it returns \(\text{val} \cdot \text{val} = a^n\)

  If \(n\) is odd:

  \[
  \text{val} = a^{(n-1)/2} \quad \text{(due to ind. hyp.)}
  \]

  it returns \(\text{val} \cdot \text{val} \cdot a = a^n\)

*The correctness proof is complete*
Maximum Subarray Problem

- **Input**: An array of values
- **Output**: The contiguous subarray that has the largest sum of elements

Input array:

| 13 | -3 | -25 | 20 | -3 | -16 | -23 | 18 | 20 | -7 | 12 | -22 | -4 | 7 |

the maximum contiguous subarray
Maximum Subarray Problem: Divide & Conquer

- **Basic idea:**
  - Divide the input array into 2 from the middle
  - Pick the **best** solution among the following:
    1. The max subarray of the **left half**
    2. The max subarray of the **right half**
    3. The max subarray **crossing the mid-point**
Maximum Subarray Problem: Divide & Conquer

- **Divide**: Trivial (divide the array from the middle)
- **Conquer**: Recursively compute the max subarrays of the left and right halves
- **Combine**: Compute the max-subarray crossing the mid-point (*can be done in $\Theta(n)$ time*). Return the max among the following:
  1. the max subarray of the left subarray
  2. the max subarray of the right subarray
  3. the max subarray crossing the mid-point

See textbook for the detailed solution.
Conclusion

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- Can lead to more efficient algorithms