CS473 - Algorithms I

Lecture 12-b
Dynamic Tables

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Why Dynamic Tables?

- Assume we need a data structure that needs to reside in contiguous memory (e.g. linear array, etc.).
- But, we don't know how many objects will be stored in the table ahead of time.
- We may allocate space for a table, but later find out that it is not enough.
  - Then, the table must be reallocated with a larger size.
  - All the objects stored in the original table must be copied over into the new table.

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Why Dynamic Tables?

- Similarly, if many objects are deleted from the table:
  - It may be worthwhile to reallocate the table with a smaller size.

- This problem is called:

  dynamically expanding and contracting a table

Why Dynamic Tables?

Using amortized analysis we will show that,

- The amortized cost of insertion and deletion is $O(1)$.
- Even though the actual cost of an operation is large when it triggers an expansion or a contraction.

We will also show how to guarantee that

- The unused space in a dynamic table never exceeds a constant fraction of the total space.

Operations

TABLE-INSERT:

Inserts into the table an item that occupies a single slot.

TABLE-DELETE:

Removes an item from the table & frees its slot.

Load Factor

Load Factor of a Dynamic Table T

$$\alpha(T) = \frac{\text{Number of items stored in the table}}{size(number\ of\ slots)\ \text{of the table}}$$

For an empty table

$$\alpha(T) = \frac{0}{0} = 1$$

by definition

Insertion-Only Dynamic Tables

Table-Expansion:

- Assumption:
  - Table is allocated as an array of slots
- A table fills up when
  - all slots have been used
  - equivalently, when its load factor becomes 1
- Table-Expansion occurs when
  - An item is to be inserted into a full table

Insertion-Only Dynamic Tables

- A Common Heuristic
  - Allocate a new table that has twice as many slots as the old one.
- Hence, we have:

$$1 / 2 \leq \alpha(\mathrm{T}) \leq 1$$

Table Insert

```
TABLE-INSERT (T, x)
    if size[T] = 0 then
            allocate table[T] with 1 slot
            size[T] ← 1
    if num[T] = size[T] then
            allocate new-table with 2.size[T] slots
            copy all items in table[T] into new-table
            free table[T]
            table[T] ← new-table[T]
            size[T] ← 2.size[T]
    insert x into table[T]
    num[T] ← num[T] + 1
end
```

table[T] : pointer to block of table storage
num[T] : number of items in the table
size[T] : total number of slots in the table
Initially, table is empty, so num[T] = size[T] = 0

Example: Dynamic Table Insertion

T

d_1

INSERT(d_1)

INSERT(d_2)

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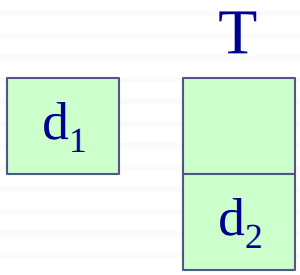
Example: Dynamic Table Insertion

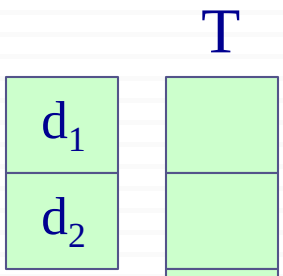


INSERT(d_1)

INSERT(d_2)

INSERT(d_3)

Example: Dynamic Table Insertion



INSERT(d_1)

INSERT(d_2)

INSERT(d_3)

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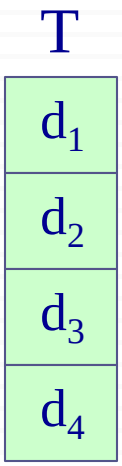
Example: Dynamic Table Insertion



INSERT(d_1)

INSERT(d_2)

INSERT(d_3)

INSERT(d_4)

INSERT(d_5)

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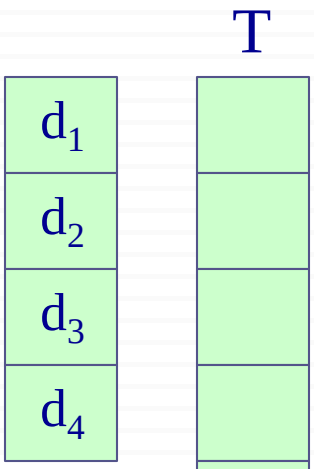
Example: Dynamic Table Insertion



INSERT(d_1)

INSERT(d_2)

INSERT(d_3)

INSERT(d_4)

INSERT(d_5)

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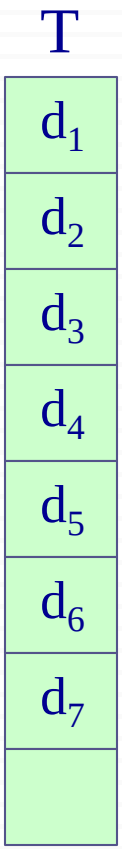
Example: Dynamic Table Insertion



INSERT(d_1)

INSERT(d_2)

INSERT(d_3)

INSERT(d_4)

INSERT(d_5)

INSERT(d_6)

INSERT(d_7)

Table Expansion: Runtime Analysis

- The actual running time of TABLE-INSERT is linear in the time to insert individual items.
- Assume that allocating and freeing storage is dominated by the cost of transferring items.
- Assign a cost of 1 to each elementary insertion.
- Analyze a sequence of n TABLE-INSERT operations on an initially empty table.

Cost of Table Expansion

- What is the cost c_i of the i^{th} operation if there is room in the current table?

  $c_i = 1$ *(only one elementary insert operation)*

- What is the cost c_i of the i^{th} operation if the current table is full?

  $c_i = i$

  $i-1$ for the items that must be copied from the old table to the new table.

  1 for the elementary insertion of the new item

Cost of Table Expansion

- What is the worst-case runtime of n INSERT operations?

  The worst case cost of 1 INSERT operation is O(n)

  Therefore, the total running time is $O(n^2)$

- **This bound is not tight!**

  Expansion does not occur so often in the course of n INSERT operations

Amortized Analysis of INSERT: Aggregate Method

- Table is initially empty.
- Compute the total cost of n INSERT operations.

- When does the i^{th} operation require an expansion?

  only when i-1 is a power of 2

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	...
c_i	1	2	3	1	5	1	1	1	9	1	1	1	1	1	1	1	17	1	1	1	...
elem. ins	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	...
Expansion cost		1	2		4				8								16				

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Amortized Analysis of INSERT: Aggregate Method

Reminder: c_i *is the actual cost of the* i^{th} *INSERT operation*

$$c_i = \begin{cases} i & \text{if } i-1 \text{ is an exact power of 2} \\ 1 & \text{otherwise} \end{cases}$$

Therefore the total cost of n TABLE-INSERT operations is:

$$\sum_{i=1}^{n} c_i \le n + \sum_{j=0}^{\lfloor \lg n \rfloor} 2^j < n + 2n = 3n$$

The amortized cost of a single operation is $3n/n = 3 = O(1)$

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The Accounting Method

Assign the following amortized costs

- Table-Expansion : $0
- Insertion of a new item : $3

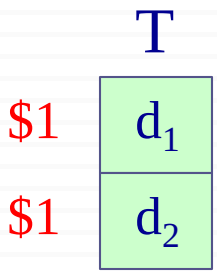
Insertion of a new item:

$1 (as an actual cost) for inserting itself into the table

+ $1 (as a credit) for moving itself in the next expansion

+ $1 (as a credit) for moving another item (in the next expansion) that has already moved in the last expansion

Accounting Method Example



INSERT(d_2) $3

Note: Amortized cost of INSERT(d_2): $3

$1 spent for the actual cost of inserting d_2

$1 credit for moving d_2 in the next expansion

$1 credit for moving d_1 in the next expansion

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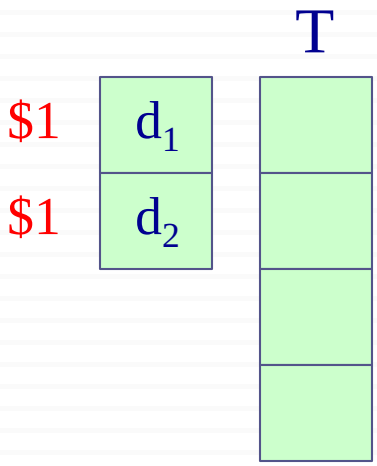
Accounting Method Example

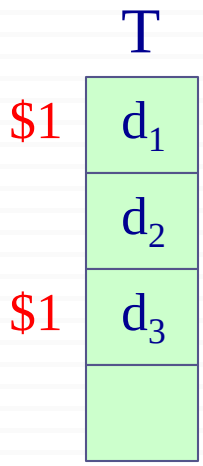


INSERT(d_2) $3

INSERT(d_3)

Note: When expansion is needed for the next INSERT operation, we have $1 stored credit for each item to move it to the new memory location.

Accounting Method Example



INSERT(d_2) $3

INSERT(d_3) $3

Note: Amortized cost of INSERT(d_3): $3

$1 spent for the actual cost of inserting d_3

$1 credit for moving d_3 in the next expansion

$1 credit for moving d_1 in the next expansion

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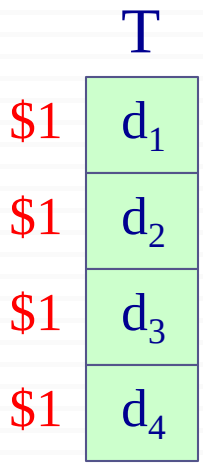
Accounting Method Example



INSERT(d_2) $3

INSERT(d_3) $3

INSERT(d_4) $3

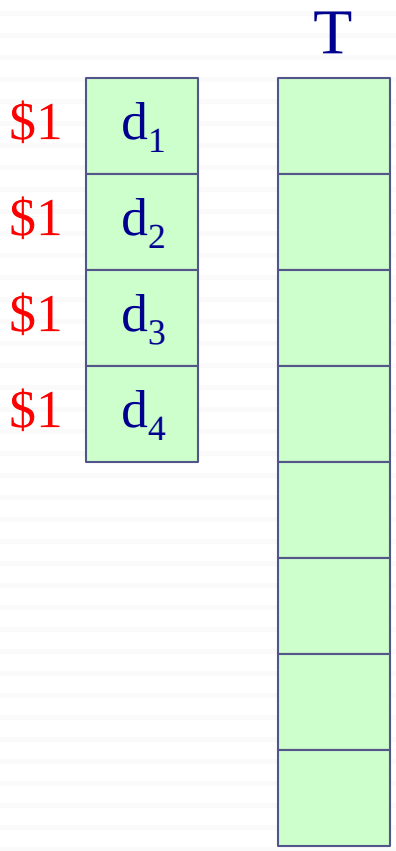
Note: Amortized cost of INSERT(d_4): $3

$1 spent for the actual cost of inserting d_4

$1 credit for moving d_4 in the next expansion

$1 credit for moving d_2 in the next expansion

Accounting Method Example



INSERT(d_2) $3

INSERT(d_3) $3

INSERT(d_4) $3

INSERT(d_5)

Note: When expansion is needed for the next INSERT operation, we have $1 stored credit for each item to move it.

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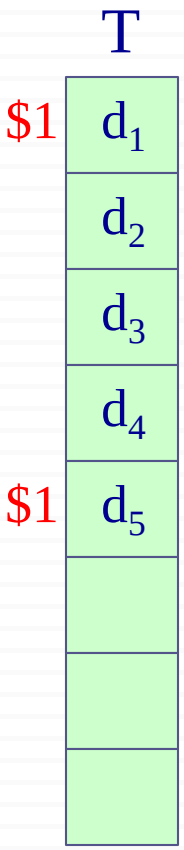
Accounting Method Example



INSERT(d_2) $3

INSERT(d_3) $3

INSERT(d_4) $3

INSERT(d_5) $3

Amortized cost of INSERT(d_5): $3

$1 spent for the actual cost of inserting d_5

$1 credit for moving d_5 later

$1 credit for moving d_1 later

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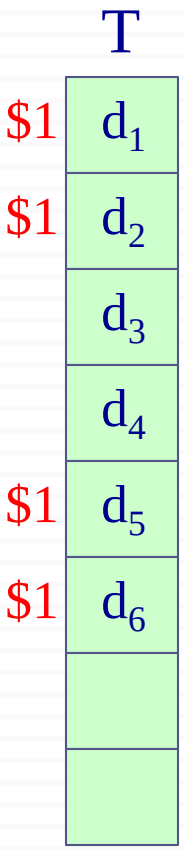
Accounting Method Example



INSERT(d_2) $3

INSERT(d_3) $3

INSERT(d_4) $3

INSERT(d_5) $3

INSERT(d_6) $3

Amortized cost of INSERT(d_6): $3

$1 spent for the actual cost of inserting d_6

$1 credit for moving d_6 later

$1 credit for moving d_2 later

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Accounting Method Example

Size of the table: *M*

Immediately after an expansion (just before the insertion)

num[T] = *M*/2 and size[T] = *M* where *M* is a power of 2.

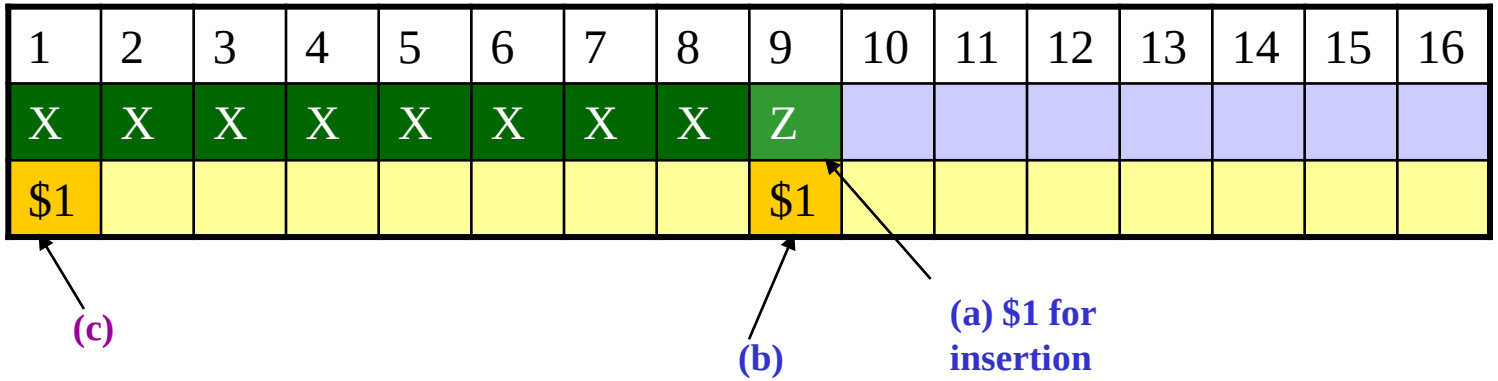
Table contains no credits

.

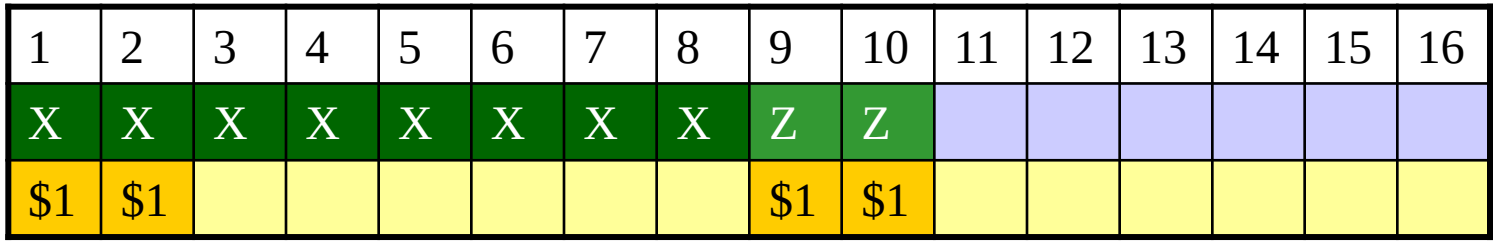
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
X	X	X	X	X	X	X	X								

Accounting Method Example

1st insertion



2nd insertion



Accounting Method Example

M/2th Insertion

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
X	X	X	X	X	X	X	X	Z	Z	Z	Z	Z	Z	Z	Z
$1	$1	$1	$1	$1	$1	$1	$1	$1	$1	$1	$1	$1	$1	$1	$1

Thus, by the time the table contains *M* items and is full

– each item in the table has $1 of credit to pay for its move during the next expansion

Amortized Analysis of INSERT: Potential Method

Practical guideline reminder:

Choose a potential function that increases a little after every cheap operation, and decreases a lot after an expensive operation.

- Define a potential function Φ

  that is 0 immediately after an expansion, and

  that builds to the table size by the time table becomes full.

- This way the next expansion can be paid for by the potential.

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Definition of Potential

- One possible potential function Φ can be defined as:

  $\Phi(T) = 2*num[T] - size[T]$

  where:

  num[T]: the # of entries stored in table T

  size[T]: the size allocated for table T

- What is the potential value immediately after an expansion?

  $\Phi(T) = 0$ *because size[T] = 2*num[T]*

- What is the potential value immediately before an expansion?

  $\Phi(T) = num[T]$ *because size[T] = num[T]*

- The initial value of the potential is 0.

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Definition of Potential

Potential function: $\Phi(T) = 2*num[T] - size[T]$

- Can the potential be ever negative?

  No, because the table is always at least half full.

  i.e. $num[T] \geq size[T] / 2$

- Since $\Phi(T)$ is always nonnegative:

  The sum of the amortized costs of n INSERT operations is an upper bound on the sum of the actual costs.

Analysis of *i*-th Table Insert

n_i : num[T] after the *i*-th operation

s_i : size[T] after the *i*-th operation

Φ_i : Potential after the *i*-th operation

Initially we have $n_i = s_i = \Phi_i = 0$

Note that, $n_i = n_{i-1} + 1$ always holds.

Amortized Cost of TABLE-INSERT

Potential function: $\Phi(T) = 2*num[T] - size[T]$

If the i^{th} TABLE-INSERT does not trigger an expansion:

Intuitively:

size[T] remains the same

num[T] increases by 1

$\Longrightarrow$ potential change = 2

amortized cost = real cost + potential change

= 1 + 2 = 3

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Amortized Cost of TABLE-INSERT

Potential function: $\Phi(T) = 2*num[T] - size[T]$

If the i^{th} TABLE-INSERT does not trigger an expansion:

Formally:

$s_i = s_{i-1}$ and $c_i = 1$

$$\hat{c}_i = c_i + \phi_i - \phi_{i-1} = 1 + (2n_i - s_i) - (2n_{i-1} - s_{i-1})$$
$$= 1 + (2(n_{i-1} + 1) - s_{i-1}) - (2n_{i-1} - s_{i-1})$$
$$= 1 + 2 - 2 + 2 = 3$$

Amortized Cost of TABLE-INSERT

Potential function: $\Phi(T) = 2*num[T] - size[T]$

If the i^{th} TABLE-INSERT triggers an expansion:

Intuitively:

size[T] is doubled, i.e. increases by n_{i-1}

num[T] increases by 1

$\Longrightarrow$ potential change = $2 - n_{i-1}$

real cost: $n_{i-1} + 1$ *(copy n_{i-1} entries to new memory + insert the new element)*

amortized cost = real cost + potential change

$= n_i - 1 + 1 + 2 - n_i - 1 = 3$

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Amortized Cost of TABLE-INSERT

Potential function: $\Phi(T) = 2*num[T] - size[T]$

If the i[th] TABLE-INSERT triggers an expansion:

Formally:

$$n_{i-1} = s_{i-1}; \qquad s_i = 2s_{i-1}; \qquad c_i = n_i = n_{i-1} + 1$$

$$\hat{c}_i = c_i + \phi_i - \phi_{i-1} = n_i + (2n_i - s_i) - (2n_{i-1} - s_{i-1})$$

$$= (n_{i-1} + 1) + (2(n_{i-1} + 1) + 2s_{i-1}) - (2n_{i-1} - s_{i-1})$$

$$= n_{i-1} + 1 + 2n_{i-1} + 2 - 2n_{i-1} - 2n_{i-1} + n_{i-1} = 3$$

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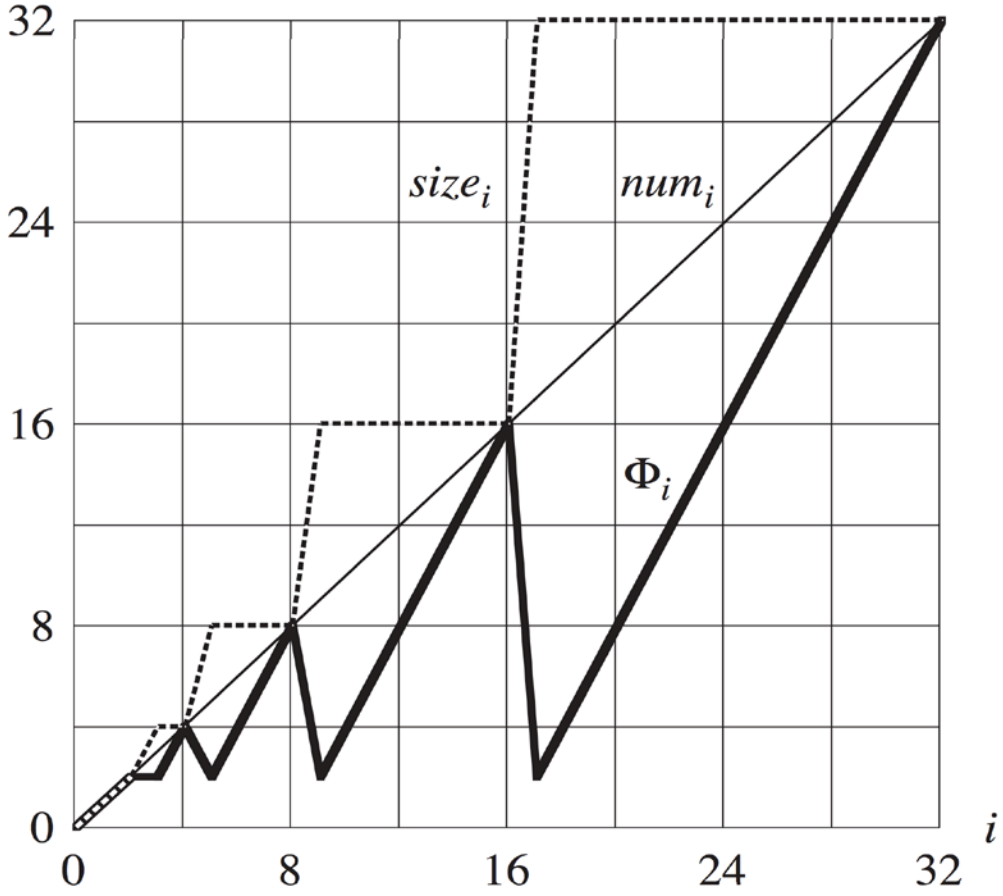
A Sequence of TABLE-INSERT Operations



Size of the table is doubled when i-1 is a power of 2.

The potential function increases gradually after every INSERT that doesn't require table expansion.

The potential function drops to 2 after every INSERT that requires table expansion.

Supporting Insertions and Deletions

- So far, we have assumed that we only INSERT elements into the table. Now, we want to support DELETE operations as well.

- TABLE-DELETE: Remove the specified item from the table. Contract the table if needed.

- In table contraction, we want to preserve two properties:
  - The load factor of the table is bounded below by a constant.
  - Amortized cost of an operation is bounded above by a constant.

- As before, we assume that the cost can be measured in terms of elementary insertions and deletions.

Expansion and Contraction

Load factor reminder: $\alpha(T) = \dfrac{\text{Number of items stored in the table}}{\textit{size}(\textit{number of slots}) \text{ of the table}}$

- An intuitive strategy for expansion and contraction:
  - Double the table size when an item is to be inserted into a full table.
  - Halve the size when a deletion would cause $\alpha(T) < ½$

- What is the problem with this strategy?
  - **Good**: It guarantees $½ \leq \alpha(T) \leq 1.0$
  - **Bad**: Amortized cost of an operation can be quite large.

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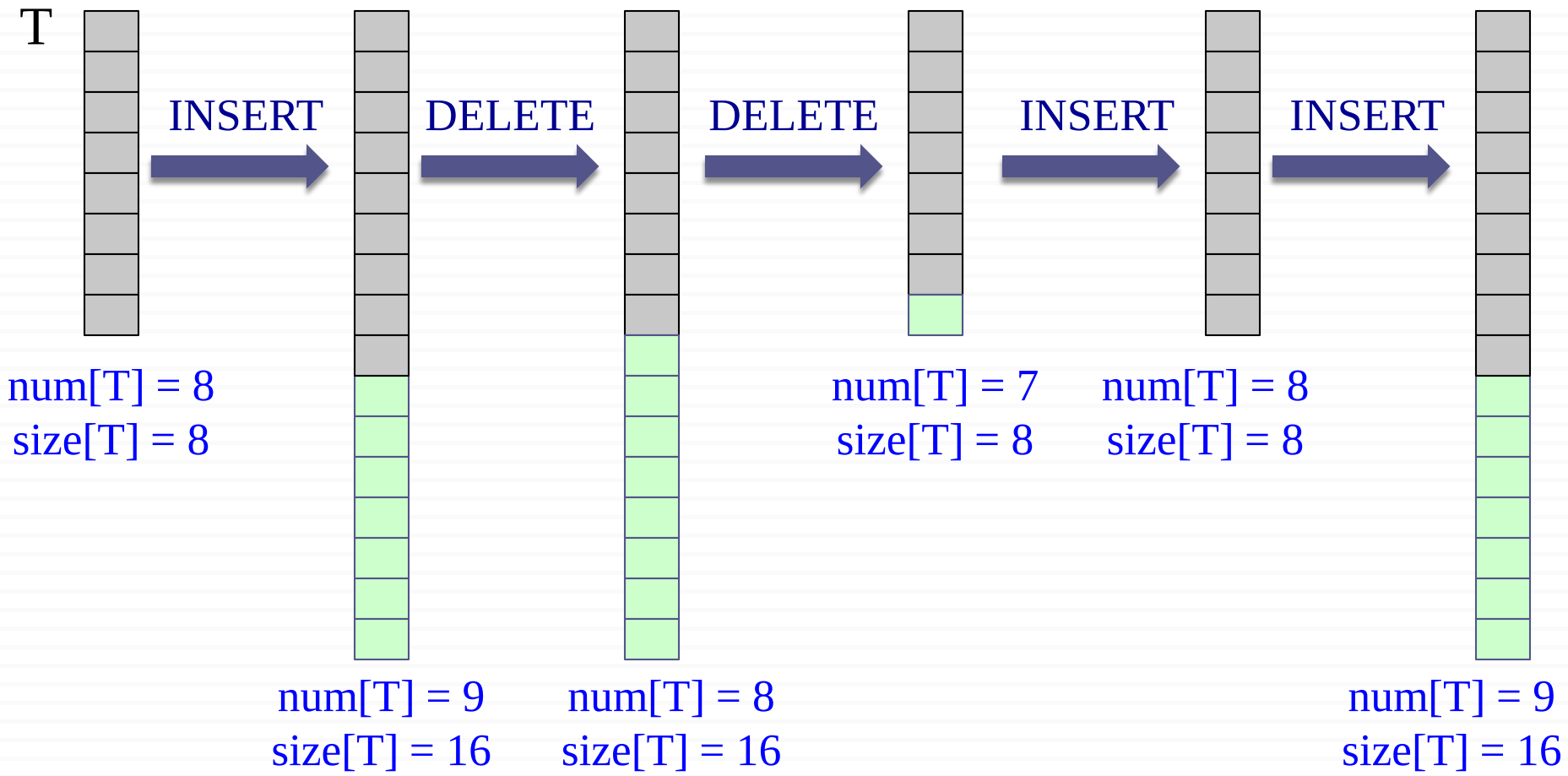
Worst-Case Behavior for $\alpha(T) \geq ½$



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Worst-Case for $\alpha(T) \geq ½$

Consider the following worst case scenario

- We perform n operations on an empty table where n is a power of 2
- First $n/2$ operations are all insertions , cost a total of $\Theta(n)$

  at the end: we have num[T] = size[T] = $n/2$
- Second $n/2$ operations repeat the sequence

  I D D I

  that is **I D D I I D D I I D D I** ...

Worst-Case for $\alpha(T) \geq ½$

Example: $n=16$

i:	1	2	...	7	8	9	10	11	12	13	14	15	16
oper:	I	I	...	I	I	I	D	D	I	I	D	D	I
n_i	1	2	...	7	8	9	8	7	8	9	8	7	8
s_i	1	2	...	8	8	16	16	8	8	16	16	8	8
						E		C		E		C	

In the second $n/2$ operations

- The first INSERT cause an expansion
- Two further DELETEs cause contraction
- Two further INSERTs cause expansion ... and so on

Hence there are $n/8$ expansions and $n/8$ contractions

The cost of each expansion and contraction is $\approx n/2$

Worst-Case for $\alpha(T) \geq ½$

Thus the total cost of n operations is $\Theta(n^2)$ since

- First $n/2$ operations : $3n/2$
- Second $n/2$ operations : $(n/4)*(n/2)=n^2/8$

The amortized cost of an operation is $\Theta(n)$

The problem with this strategy is

- After an expansion, we do not perform enough deletions to pay for a contraction
- After a contraction, we do not perform enough insertions to pay for an expansion

Improving Amortized Time of Expansion and Contraction

- We saw that if we enforce $\frac{1}{2} \leq \alpha(T) \leq 1$, the amortized time becomes $O(n)$ in the worst case.
- To improve the amortized cost:

  Allow $\alpha(T)$ to drop below $\frac{1}{2}$.

- Basic idea:
  - Expansion: Double the table size when an item is inserted into a full table (same as before).
  - Contraction: Halve the table size when a deletion causes:

    $\alpha(T) < \frac{1}{4}$

Improving Amortized Time of Expansion and Contraction

- In other words, we enforce: $\frac{1}{4} \leq \alpha(T) \leq 1$

- Intuition:
  - Immediately after an expansion, we have $\alpha(T) = \frac{1}{2}$

    $\Longrightarrow$ At least half of the items in the table must be deleted before a contraction can occur (i.e. when $\alpha(T) < \frac{1}{4}$)

  - Immediately after a contraction, we have $\alpha(T) = \frac{1}{2}$

    $\Longrightarrow$ The number of items in the table must be doubled before an expansion can occur (i.e. when $\alpha(T)=1$).

Potential Method for INSERT & DELETE

- We want to define the potential function $\Phi(T)$ as follows:
  - Immediately after an expansion or contraction:

    $\Phi(T) = 0$

  - Immediately before an expansion or contraction:

    $\Phi(T) = num[T]$

    because we need to copy over $num[T]$ elements, and the cost of expansion or contraction should be paid by the decrease in potential.

Potential Method for INSERT & DELETE

- *Reminder*: Immediately after an expansion or contraction,

  $\alpha(T) = ½$

- So, we want to define a potential function $\Phi(T)$ such that:

  $\Phi(T)$ **starts at** 0 when $\alpha(T) = ½$

  $\Phi(T)$ **gradually increases to** num[T],

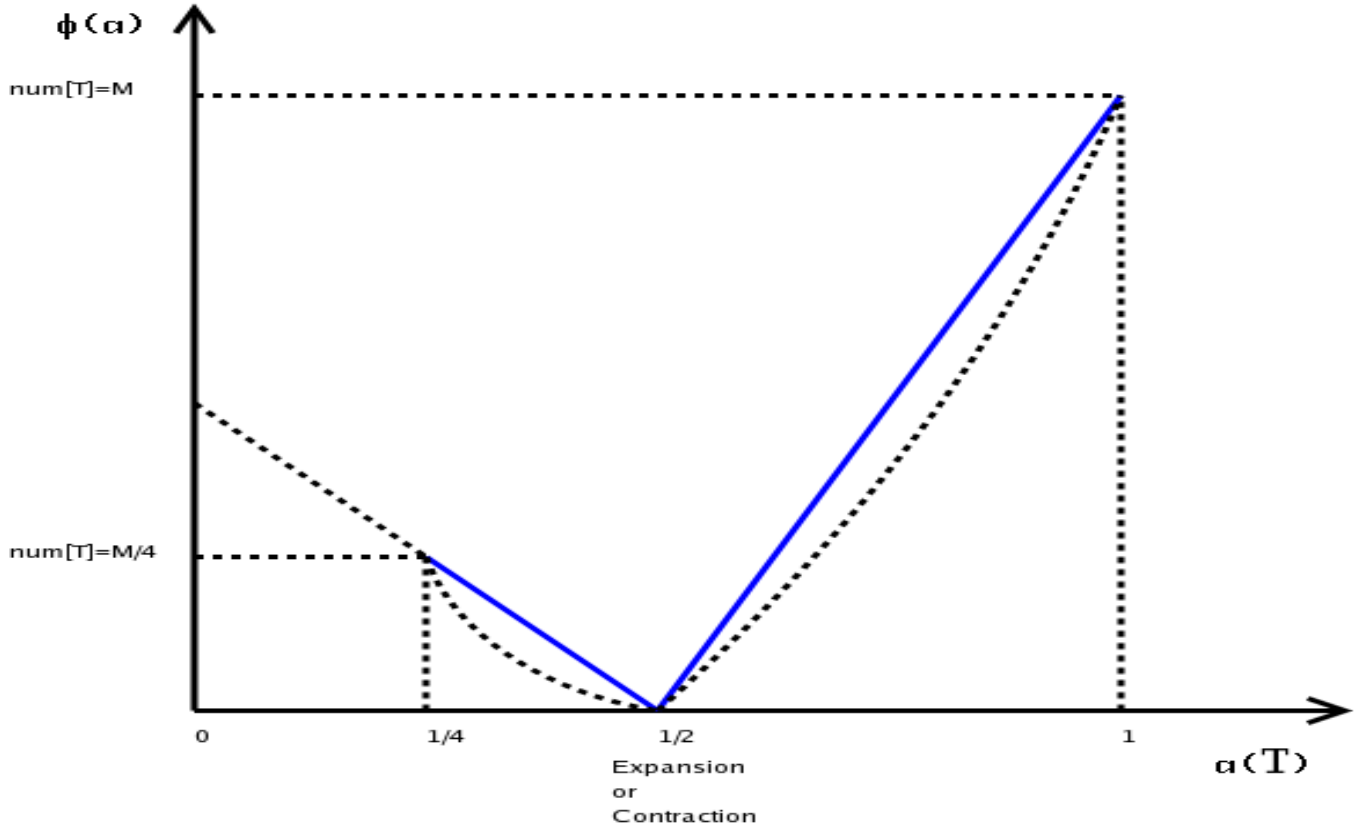
  when $\alpha(T)$ **increases to** 1, or

  when $\alpha(T)$ **decreases to** ¼

- This way, the next expansion or contraction can be paid by the decrease in potential.

$\Phi(\alpha)$ w.r.t. $\alpha(T)$

M=num[T] when an expansion or contraction occurs



Definition of New Φ

One such Φ is

$$\Phi(T) = \begin{cases} 2num[T] - size[T] & \text{if } \alpha(\mathrm{T}) \geq \frac{1}{2} \\ \frac{\mathrm{size[T]}}{2} - num[T] & \text{if } \alpha(\mathrm{T}) < \frac{1}{2} \end{cases}$$

or

$$\Phi(T) = \begin{cases} num[T](2 - 1/\alpha) & \text{if } \alpha(\mathrm{T}) \geq \frac{1}{2} \\ num[T](1/2\alpha - 1) & \text{if } \alpha(\mathrm{T}) < \frac{1}{2} \end{cases}$$

Description of New Φ

$$\Phi(T) = \begin{cases} 2num[T] - size[T] & \text{if } \alpha(T) \geq \frac{1}{2} \\ \frac{size[T]}{2} - num[T] & \text{if } \alpha(T) < \frac{1}{2} \end{cases}$$

$$\alpha = \frac{num[T]}{size[T]}$$

- ✓ Φ = 0 **when** α = ½
- ✓ Φ = num[T] **when** α = ¼
- ✓ Φ = num[T] **when** α = 1
- ✓ Φ = 0 **when** the table is empty

  (num[T] = size[T] = 0, α(T) = 0)

- ✓ Φ is always nonnegative

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Amortized Analysis

We need to analyze the operations:

TABLE-INSERT and TABLE-DELETE

Notations:

c_i: Actual cost of the i^{th} operation

$\hat{c}_i$: Amortized cost of the i^{th} operation

Φ_i: Potential $\Phi(T)$ after the i^{th} operation

n_i: Number of elements num[T] after the i^{th} operation

s_i: Table size size[T] after the i^{th} operation

α_i: Load factor $\alpha(T)$ after the i^{th} operation

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Amortized Analysis: Table Insert – Case 1

- There is no possibility of contraction in any case.
- In all cases: $n_i = n_{i-1} + 1$

Case 1: $\alpha_{i-1} \geq ½$

Analysis is identical to the one we did before for only TABLE-INSERT operation.

$\Longrightarrow$ Amortized cost $\hat{c}_i = 3$ whether the table expands or not.

Amortized Analysis: Table Insert – Case 2

$$\Phi(T) = \begin{cases} 2num[T] - size[T] & \text{if } \alpha(T) \geq \frac{1}{2} \\ \frac{size[T]}{2} - num[T] & \text{if } \alpha(T) < \frac{1}{2} \end{cases}$$

Case 2: $\alpha_{i-1} < ½$ and $\alpha_i < ½$

There is no possibility of expansion.

Intuitively:

Potential change: -1

Real cost: 1

Amortized cost = 1 – 1 = 0

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Amortized Analysis: Table Insert – Case 2

$$\Phi(T) = \begin{cases} 2num[T] - size[T] & \text{if } \alpha(T) \geq \frac{1}{2} \\ \frac{size[T]}{2} - num[T] & \text{if } \alpha(T) < \frac{1}{2} \end{cases}$$

Case 2: $\alpha_{i-1} < ½$ and $\alpha_i < ½$

There is no possibility of expansion.

Formally: $c_i = 1$; $s_i = s_{i-1}$; $n_i = n_{i-1} + 1$

$$\hat{c}_i = c_i + \Phi_i - \Phi_{i-1} = 1 + (s_i / 2 - n_i) - (s_{i-1} / 2 - n_{i-1})$$

$$= 1 + \frac{s_i}{2} - n_i - \frac{s_i}{2} + (n_i - 1) = 0$$

Amortized Analysis: Table Insert – Case 3

$$\Phi(T) = \begin{cases} 2num[T] - size[T] & \text{if } \alpha(T) \geq \frac{1}{2} \\ \frac{size[T]}{2} - num[T] & \text{if } \alpha(T) < \frac{1}{2} \end{cases}$$

Case 3: $\alpha_{i-1} < ½$ and $\alpha_i \geq ½$ *(which means $\alpha_i = ½$ because size[T] is even)*

There is no possibility of expansion.

Intuitively: $n_i = s_i/2$; $n_{i-1} = s_i/2 - 1$

Old potential: 1

New potential: 0

Real cost: 1

Amortized cost = 1 – 1 = 0

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Amortized Analysis: Table Insert – Case 3

$$\Phi(T) = \begin{cases} 2num[T] - size[T] & \text{if } \alpha(T) \geq \frac{1}{2} \\ \frac{size[T]}{2} - num[T] & \text{if } \alpha(T) < \frac{1}{2} \end{cases}$$

Case 3: $\alpha_{i-1} < ½$ and $\alpha_i \geq ½$ *(which means $\alpha_i = ½$ because size[T] is even)*

There is no possibility of expansion.

Formally: $c_i = 1;$ $n_i = s_i/2;$ $n_{i-1} = s_i/2 - 1;$ $s_i = s_{i-1}$

$$\begin{aligned} \hat{c}_i &= c_i + \Phi_i - \Phi_{i-1} = 1 + (2n_i - s_i) - (s_{i-1}/2 - n_{i-1}) \\ &= 1 + (2(s_i/2) - s_i) - (s_i/2 - (s_i/2 - 1)) \\ &= 0 \end{aligned}$$

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Amortized Analysis: Table Insert - Summary

Case 1: $\alpha_{i-1} \geq ½$

Amortized cost of TABLE-INSERT = 3

Case 2: $\alpha_{i-1} < ½$ and $\alpha_i < ½$

Amortized cost of TABLE-INSERT = 0

Case 3: $\alpha_{i-1} < ½$ and $\alpha_i \geq ½$

Amortized cost of TABLE-INSERT = 0

So, the amortized cost of TABLE-INSERT is at most 3

Table Delete

$$n_i = n_{i-1} - 1 \Rightarrow n_{i-1} = n_i + 1$$

Table expansion cannot occur.

- $\alpha_{i-1} \leq ½$ and $¼ \leq \alpha_i < ½$ (It does not trigger a contraction)

$s_i = s_{i-1}$ and $c_i = 1$ and $\alpha_i < ½$

$$\hat{c}_i = c_i + \Phi_i - \Phi_{i-1} = 1 + (s_i / 2 - n_i) - (s_{i-1} / 2 - n_{i-1})$$

$$= 1 + s_i / 2 - n_i - s_i / 2 + (n_i + 1) = 2$$

Table Delete

- $\alpha_{i-1} = ¼$ (It does trigger a contraction)

  $s_i = s_{i-1}/2$; $n_i = s_{i-1}/2$; and $c_i = n_i + 1$

  $$\hat{c}_i = c_i + \Phi_i - \Phi_{i-1} = (n_i + 1) + (s_i / 2 - n_i) - (s_{i-1} / 2 - n_{i-1})$$
  $$= n_i + 1 + s_i / 2 - n_i - s_i + s_i / 2 = 1$$

- $\alpha_{i-1} > ½$ ($\alpha_i \geq ½$)

  Contraction cannot occur (c_i=1 ; $s_i = s_{i-1}$)

  $$\hat{c}_i = c_i + \Phi_i - \Phi_{i-1} = 1 + (2n_i - s_i) - (2n_{i-1} - s_{i-1})$$
  $$= 1 + 2n_i - s_i - 2(n_i + 1) + s_i = -1$$

Table Delete

- $\alpha_{i\text{-}1} = ½ \;\; (\alpha_i < ½)$

  Contraction cannot occur

  $c_i = 1$; $s_i = s_{i\text{-}1}$; $n_i = s_{i\text{-}1}/2$; and $\Phi_{i\text{-}1}=0)$

$$\hat{c}_i = c_i + \Phi_i - \Phi_{i-1} = 1 + (s_i / 2 - n_i) - 0$$
$$= 1 + s_i / 2 - n_i \quad \text{but} \quad n_{i+1} = s_i / 2$$
$$= 1 + (n_i + 1) - n_i = 2$$

Table Delete

Thus, the amortized cost of a TABLE-DELETE operation is at most 2

Since the amortized cost of each operation is bounded above by a constant

The actual time for any sequence of n operations on a Dynamic Table is $O(n)$