CS473-Algorithms I

Lecture 1

Introduction to Analysis of Algorithms

Motivation

- Procedure vs. Algorithm
- What kind of problems are solved by Algorithms?
  - determine/compare DNA sequences
  - efficiently search (e.g. Google) web pages w/ keywords
  - route data (e.g. email) on the Internet
  - decode data (e.g. banking) for security
- Data Structures & Algorithms
- Repertoire vs. New Algorithms (Techniques)

Motivation cntd

- – Efficient (scope of course) vs. Inefficient
- – Design algorithms that are
  - fast,
  - uses as little memory as possible, and
  - correct!

Problem : Sorting (from Section 1.1)

Input : Sequence of numbers

$\langle a_1, a_2, \ldots, a_n \rangle$

Output : A permutation

$\Pi = \langle \Pi(1), \Pi(2), \ldots, \Pi(n) \rangle$

such that

$$a_{\Pi(1)} \leq a_{\Pi(2)} \leq \ldots \leq a_{\Pi(n)}$$

Algorithm: Insertion sort (from Section 1.1)

<u>Insertion-Sort</u> (A)

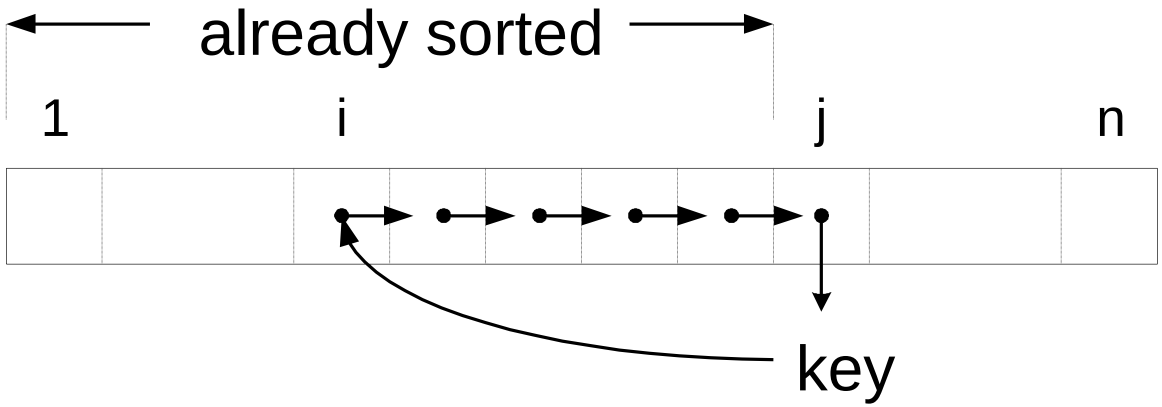
```
for j ← 2 to n do
    key ← A[j];              } Θ(1)
    i ← j - 1;               }
    while i > 0 and A[i] > key do
        A[i+1] ← A[i];       } Θ(1)
        i ← i - 1;           }
    endwhile
    A[i+1] ← key;            } Θ(1)
endfor
```

Pseudocode Notation

- Liberal use of English
- Use of indentation for block structure
- Omission of error handling and other details
  - Needed in real programs

Algorithm: Insertion sort

Idea:



- Items sorted in-place
  - Items rearranged within array
  - At most constant number of items stored outside the array at any time
  - Input array A contains sorted output sequence when Insertion-Sort is finished
- *Incremental approach*

Algorithm: Insertion sort

Example: Sample sequence

$$A = \langle 31, 42, 59, 26, 40, 35 \rangle$$

Assume first 5 items are already sorted in A[1..5]

$$A = \langle \underbrace{26, 31, 40, 42, 59}_{\text{already sorted}}, \underbrace{35}_{key} \rangle$$

26	31	40	42	59	35	35=*key*
26	31	40	42	59	59	35=*key*
26	31	40	42	42	59	35=*key*
26	31	40	40	42	59	35=*key*
26	31	35	40	42	59	

Running Time

- Depends on
  - Input size (e.g., 6 elements vs 60000 elements)
  - Input itself (e.g., partially sorted)
- Usually want *upper bound*

Kinds of running time analysis:

– Worst Case (*Usually*):

$T(n)$ = max time on any input of size n

– Average Case (*Sometimes*):

$T(n)$ = average time over all inputs of size n

Assumes statistical distribution of inputs

– Best Case (*Rarely*):

BAD*: Cheat with slow algorithm that works fast on some inputs

GOOD: Only for showing bad lower bound

*Can modify any algorithm (almost) to have a low best-case running time

- Check whether input constitutes an output at the very beginning of the algorithm

Running Time

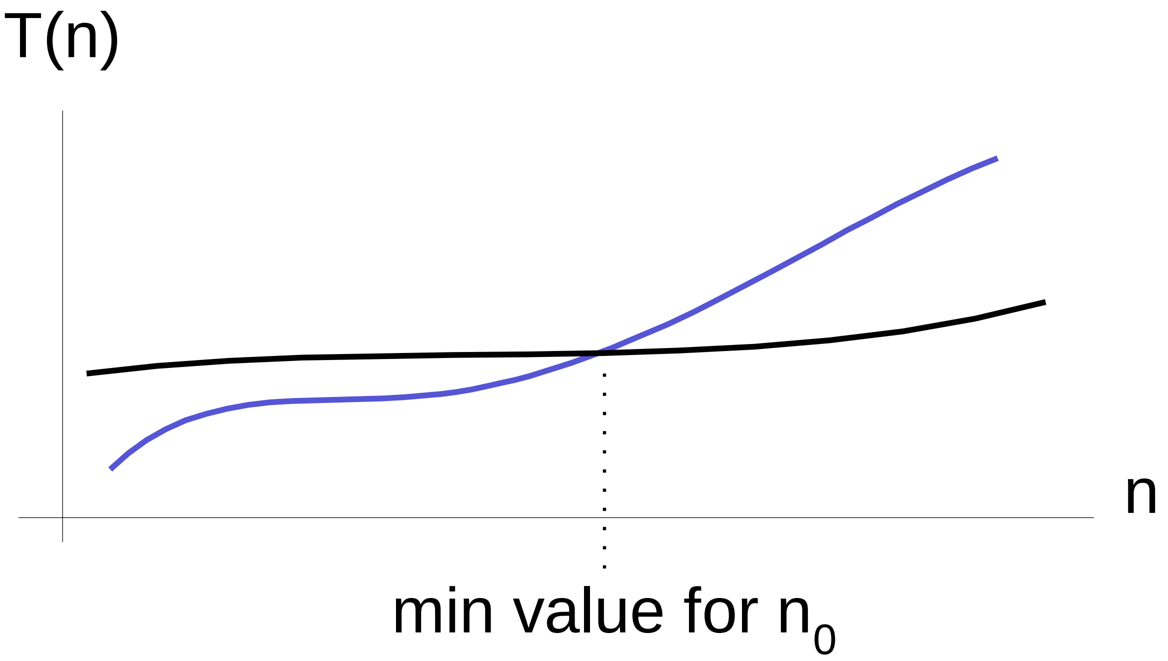
- For Insertion-Sort, what is its worst-case time
  - Depends on speed of primitive operations
    - Relative speed (on same machine)
    - Absolute speed (on different machines)
- Asymptotic analysis
  - Ignore machine-dependent constants
  - Look at growth of $T(n)$ as $n \rightarrow \infty$

Θ Notation

- Drop low order terms
- Ignore leading constants

  E.g. $3n^3+90n^2-2n+5= \Theta(n^3)$

- As n gets large a $\Theta(n^2)$ algorithm runs faster than a $\Theta(n^3)$ algorithm



Running Time Analysis of Insertion-Sort

- Sum up costs:

$$T(n) = c_1 n + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7(n-1)$$

- The best case (sorted order):

$$T(n) = (c_1 + c_2 + c_3 + c_4 + c_7)n - (c_2 + c_3 + c_4 + c_7)$$

- The worst case (reverse sorted order):

$$T(n) = \tfrac{1}{2}(c_4 + c_5 + c_6)n^2 + (c_1 + c_2 + c_3 + \tfrac{1}{2}(c_4 + c_5 + c_6) + c_7)n - (c_2 + c_3 + c_4 + c_7)$$

Running Time Analysis of Insertion-Sort

- Worst-case (input reverse sorted)
  - Inner loop is $\Theta(j)$

$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta\left(\sum_{j=2}^{n} j\right) = \Theta(n^2)$$

- Average case (all permutations equally likely)
  - Inner loop is $\Theta(j/2)$

$$T(n) = \sum_{j=2}^{n} \Theta(j/2) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2)$$

    - Often, average case not much better than worst case
- Is this a fast sorting algorithm?
  - Yes, for small *n*. No, for large *n*.

Algorithm: Merge-Sort

- Basic Step: Merge 2 sorted lists of total length n in $\Theta(n)$ time
- Example:

$$\left.\begin{array}{cccc} 2 & 3 & 7 & 8 \\ & & & \\ 1 & 4 & 5 & 6 \end{array}\right\} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \ldots$$

Recursive Algorithm:

<u>Merge-Sort</u> (A,p,r) $(T(n))$

if p = r **then return;** $(\Theta(1))$

else

q $\leftarrow \lfloor (p+r)/2 \rfloor$; : Divide $(\Theta(1))$

Merge-Sort(A,p,q); : Conquer $(T(n/2))$

Merge-Sort(A,q+1,r); : Conquer $(T(n/2))$

<u>Merge</u>(A,p,q,r); : Combine $(\Theta(n))$

endif

- Call <u>Merge-Sort</u>(A,1,n) to sort A[1..n]
- Recursion bottoms up when subsequences have length 1

Recurrence (for <u>Merge-Sort</u>) -From Section 1.3

- Describes a function recursively in terms of itself
- Describes performance of recursive algorithms
- For <u>Merge-Sort</u>

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1 \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

- How do we find a good upper bound on $T(n)$ in closed form?
- Generally, will assume $T(n)$=Constant ($\Theta(1)$) for sufficiently small n
- For <u>Merge-Sort</u> write the above recurrence as

$$T(n)=2\,T(n/2) + \Theta(n)$$

- Solution to the recurrence

$$T(n)=\Theta(n\lg n)$$

Conclusions (from Section 1.3)

- $\Theta(nlgn)$ grows more slowly than $\Theta(n^2)$

Therefore Merge-Sort beats Insertion-Sort in the worst case

- In practice, Merge-Sort beats Insertion-Sort for $n>30$ or so.