Linear Filtering – Part II

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Fourier theory

Jean Baptiste Joseph Fourier had a crazy idea:

- Any periodic function can be written as a weighted sum of sines and cosines of different frequencies (1807).
- Don't believe it?
 - Neither did Lagrange, Laplace, Poisson, ...
- But it is true!
 - \rightarrow Fourier series
- Even functions that are not periodic (but whose area under the curve is finite) can be expressed as the integral of sines and cosines multiplied by a weighing function.
 - \rightarrow Fourier transform

Fourier theory

- The Fourier theory shows how most real functions can be represented in terms of a basis of sinusoids.
- The building block:
 - A sin(ωx + Φ)
- Add enough of them to get any signal you want.



Adapted from Alexei Efros, CMU

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• The *Fourier transform*, F(u), of a single variable, continuous function, f(x), is defined by

$$F(u) = \int_{-\infty}^{\infty} f(x) \ e^{-j2\pi ux} \ dx.$$

• Given F(u), we can obtain f(x) using the *inverse* Fourier transform

$$f(x) = \int_{-\infty}^{\infty} F(u) \ e^{j2\pi ux} \ du.$$

• The *discrete Fourier transform (DFT)*, F(u), of a discrete function of one variable, f(x), x = 0, 1, 2, ..., M - 1, is defined by

$$F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x) \ e^{-j2\pi u x/M}$$

for $u = 0, 1, 2, \dots, M - 1$.

• Given F(u), we can obtain the original function back using the *inverse DFT*

$$f(x) = \sum_{u=0}^{M-1} F(u) \ e^{j2\pi u x/M}$$
 for $x = 0, 1, 2, \dots, M-1$.

- These formulas can be extended for functions of two variables.
- Fourier transform:

$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \ e^{-j2\pi(ux+vy)} \ dx \ dy.$$

• Inverse Fourier transform:

$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) \ e^{j2\pi(ux+vy)} \ du \ dv.$$

• Discrete Fourier transform:

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \ e^{-j2\pi(ux/M + vy/N)}$$

for $u = 0, 1, 2, \dots, M - 1, v = 0, 1, 2, \dots, N - 1$.

• Inverse discrete Fourier transform:

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) \ e^{j2\pi(ux/M + vy/N)}$$

for $x = 0, 1, 2, \dots, M - 1, y = 0, 1, 2, \dots, N - 1$.

• F(u,v) can also be expressed in polar coordinates as

$$F(u,v) = |F(u,v)| e^{j\phi(u,v)}$$

where

$$|F(u,v)| = \left(\Re^2 \{F(u,v)\} + \Im^2 \{F(u,v)\}\right)^{1/2}$$

is called the *magnitude* or *spectrum* of the Fourier transform, and

$$\phi(u,v) = \tan^{-1}\left(\frac{\Im\{F(u,v)\}}{\Re\{F(u,v)\}}\right)$$

is called the *phase angle* or *phase spectrum*.

• $\Re\{F(u,v)\}\$ and $\Im\{F(u,v)\}\$ are the real and imaginary parts of F(u,v), respectively.

- The spectrum need not be interpreted as an image, but rather as a 2D display of the power in the original image versus the frequency components u and v.
- The value F(0,0) is the average of f(x,y).
- Fourier transform is conjugate symmetric $(F(u, v) = F^*(-u, -v))$ and its spectrum is symmetric about the origin (|F(u, v)| = |F(-u, -v)|) (when f(x, y) is real).
- Usually the input image function is multiplied by $(-1)^{x+y}$ prior to computing the Fourier transform so that

$$\mathfrak{F}[f(x,y) \ (-1)^{x+y}] = F(u - M/2, v - N/2).$$

The origin of the Fourier transform is located at u = M/2 and v = N/2.







FIGURE 4.3 (a) Image of a 20×40 white rectangle on a black background of size 512×512 pixels. (b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.



Adapted from Gonzales and Woods

• The *power spectrum* is defined as the square of the Fourier spectrum:

$$P(u, v) = |F(u, v)|^2$$

= $\Re^2 \{F(u, v)\} + \Im^2 \{F(u, v)\}.$

- The radial distribution of values in the Fourier spectrum of an image is sensitive to texture coarseness in that image.
 - A coarse texture will have high values concentrated near the origin of the spectrum.
 - ► A fine texture will cause the values to be spread out.

- The angular distribution of values in the spectrum is sensitive to the directionality of the texture in the image.
 - A texture with many edges in a given direction θ will have high values of the spectrum concentrated around the perpendicular direction $\theta + \pi/2$.
 - For a non-directional texture, the spectrum is also non-directional.
- We will come back to this when we talk about texture.



Figure 5.42: Four images (above) and their power spectrums (below). The power spectrum of the brick texture shows energy in many sinusoids of many frequencies, but the dominant direction is perpendicular to the 6 dark seams running about 45 degrees with the X-axis. There is noticable energy at 0 degrees with the X axis, due to the several short vertical seams. The power spectrum of the building shows high frequency energy in waves along the X-direction and the Y-direction. The third image is an aerial image of an orchard: the power spectrum shows the rows and columns of the orchard and also the "diagnonal rows". The far right image, taken from a phone book, shows high frequency power at about 60° with the X-axis, which represents the texture in the lines of text. Energy is spread more broadly in the perpendicular direction also in order to model the characters and their spacing.

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Adapted from Shapiro and Stockman





Example building patterns in a satellite image and their Fourier spectrum.

Convolution theorem

• The discrete *convolution* of two functions f(x, y) and h(x, y) of size $M \times N$ is defined as

$$f(x,y) \star h(x,y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) h(x-m,y-n).$$

- This is equivalent to the *correlation* of f(x, y) with h(x, y) flipped about the origin.
- Convolution theorem:

 $\begin{aligned} f(x,y) \star h(x,y) \Leftrightarrow F(u,v) \; H(u,v) \\ f(x,y) \; h(x,y) \Leftrightarrow F(u,v) \star H(u,v) \end{aligned}$

where " \Leftrightarrow " indicates a Fourier transform pair. ^{CS 484, Spring 2007} ©2007, Selim Aksoy</sup>

Frequency domain filtering

Filter image f(x, y) with mask h(x, y)

- (1) Fourier transform the image f(x, y) to obtain its frequency rep. F(u, v).
- (2) Fourier transform the mask h(x, y) to obtain its frequency rep. H(u, v)
- (3) multiply F(u, v) and H(u, v) pointwise to obtain F'(u, v)
- (4) apply the inverse Fourier transform to F'(u, v) to obtain the filtered image f'(x, y).

Algorithm 3: Filtering image f(x, y) with mask h(x, y) using the Fourier transform



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Frequency domain filtering



 Since the discrete Fourier transform is periodic, padding is needed in the implementation to avoid aliasing (see section 4.6 in the Gonzales-Woods book for implementation details).

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Frequency domain filtering



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abc

FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.



a b

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FIGURE 4.11 (a) An image of size 500×500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.



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a b

c d

e f

FIGURE 4.12 (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.



abc

FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0 .



 FIGURE 4.18 (a) Original image. (b)-(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.
 a b

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a b c

FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

Adapted from Gonzales and Woods



a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).



a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

Frequency domain processing





An image and its Fourier spectrum.

Frequency domain processing



Results of modifying the spectrum and reconstructing the image.

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Frequency domain processing



Results of modifying the spectrum and reconstructing the image.

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- Correlation can also be used for matching.
- If we want to determine whether an image f contains a particular object, we let h be that object (also called a template) and compute the correlation between f and h.
- If there is a match, the correlation will be maximum at the location where h finds a correspondence in f.
- Preprocessing such as scaling and alignment is necessary in most practical applications.





Face detection using template matching: face templates.



Face detection using template matching: detected faces.



How can we generate a half-sized version of a large image?

Adapted from Steve Seitz, U of Washington

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1/4

Throw away every other row and column to create a 1/2 size image (also called sub-sampling).

Adapted from Steve Seitz, U of Washington

1/8



1/2

1/4 (2x zoom) Does this look nice?

1/8 (4x zoom)

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We cannot shrink an image by simply taking every k'th pixel.Solution: smooth the image, then sub-sample.







Gaussian 1/8

Gaussian 1/4

Gaussian 1/2 CS 484, Spring 2007 ©20

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Gaussian 1/2

Gaussian 1/4 (2x zoom) Gaussian 1/8 (4x zoom)

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Sampling and aliasing



Examples of GOOD sampling

Examples of BAD sampling -> Aliasing

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Sampling and aliasing

- Errors appear if we do not sample properly.
- Common phenomenon:
 - High spatial frequency components of the image appear as low spatial frequency components.
- Examples:
 - Wagon wheels rolling the wrong way in movies.
 - Checkerboards misrepresented in ray tracing.
 - Striped shirts look funny on color television.



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512 256 128 64 32 16 8



A bar in the big images is a hair on the zebra's nose; in smaller images, a stripe; in the smallest, the animal's nose

Ponce & Forsyth

Adapted from David Forsyth, UC Berkeley 45

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Irani & Basri

Adapted from Michael Black, Brown University