Non-Bayesian Classifiers
Part I: $k$-Nearest Neighbor Classifier and Distance Functions

Selim Aksoy

Department of Computer Engineering
Bilkent University
saksoy@cs.bilkent.edu.tr

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Non-Bayesian Classifiers

- We have been using Bayesian classifiers that make decisions according to the posterior probabilities.
- We have discussed parametric and non-parametric methods for learning classifiers by estimating the probabilities using training data.
- We will study new techniques that use training data to learn the classifiers directly without estimating any probabilistic structure.
- In particular, we will study the $k$-nearest neighbor classifier, linear discriminant functions, and support vector machines.
The Nearest Neighbor Classifier

- Given the training data $\mathcal{D} = \{x_1, \ldots, x_n\}$ as a set of $n$ labeled examples, the nearest neighbor classifier assigns a test point $x$ the label associated with its closest neighbor in $\mathcal{D}$.

- Closeness is defined using a distance function.

- Given the distance function, the nearest neighbor classifier partitions the feature space into cells consisting of all points closer to a given training point than to any other training points.
The Nearest Neighbor Classifier

- All points in such a cell are labeled by the class of the training point, forming a *Voronoi tessellation* of the feature space.

![Voronoi diagram](image)

**Figure 1:** In two dimensions, the nearest neighbor algorithm leads to a partitioning of the input space into Voronoi cells, each labeled by the class of the training point it contains. In three dimensions, the cells are three-dimensional, and the decision boundary resembles the surface of a crystal.
The $k$-Nearest Neighbor Classifier

- The $k$-nearest neighbor classifier classifies $x$ by assigning it the label most frequently represented among the $k$ nearest samples.
- In other words, a decision is made by examining the labels on the $k$-nearest neighbors and taking a vote.

Figure 2: The $k$-nearest neighbor query forms a spherical region around the test point $x$ until it encloses $k$ training samples, and it labels the test point by a majority vote of these samples. In the case for $k = 5$, the test point will be labeled as black.
The $k$-Nearest Neighbor Classifier

- The computational complexity of the nearest neighbor algorithm — both in space (storage) and time (search) — has received a great deal of analysis.
- In the most straightforward approach, we inspect each stored training point one by one, calculate its distance to $x$, and keep a list of the $k$ closest ones.
- There are some parallel implementations and algorithmic techniques for reducing the computational load in nearest neighbor searches.
The $k$-Nearest Neighbor Classifier

- Examples of algorithmic techniques include
  - computing partial distances using a subset of dimensions, and eliminating the points with partial distances greater than the full distance of the current closest points,
  - using search trees that are hierarchically structured so that only a subset of the training points are considered during search,
  - editing the training set by eliminating the points that are surrounded by other training points with the same class label.
Distance Functions

- The nearest neighbor classifier relies on a metric or a distance function between points.
- For all points $x$, $y$ and $z$, a metric $D(\cdot, \cdot)$ must satisfy the following properties:
  - Nonnegativity: $D(x, y) \geq 0$.
  - Reflexivity: $D(x, y) = 0$ if and only if $x = y$.
  - Symmetry: $D(x, y) = D(y, x)$.
  - Triangle inequality: $D(x, y) + D(y, z) \geq D(x, z)$.
- If the second property is not satisfied, $D(\cdot, \cdot)$ is called a pseudometric.
A general class of metrics for $d$-dimensional patterns is the *Minkowski metric*

$$L_p(x, y) = \left( \sum_{i=1}^{d} |x_i - y_i|^p \right)^{1/p}$$

also referred to as the $L_p$ *norm*.

The *Euclidean distance* is the $L_2$ norm

$$L_2(x, y) = \left( \sum_{i=1}^{d} |x_i - y_i|^2 \right)^{1/2}.$$ 

The *Manhattan* or *city block distance* is the $L_1$ norm

$$L_1(x, y) = \sum_{i=1}^{d} |x_i - y_i|.$$
The $L_\infty$ norm is the maximum of the distances along individual coordinate axes

$$L_\infty(x, y) = \max_{i=1}^{d} |x_i - y_i|.$$ 

Figure 3: Each colored shape consists of points at a distance 1.0 from the origin, measured using different values of $p$ in the Minkowski $L_p$ metric.
Feature Normalization

- We should be careful about scaling of the coordinate axes when we compute these metrics.
- When there is great difference in the range of the data along different axes in a multidimensional space, these metrics implicitly assign more weighting to features with large ranges than those with small ranges.
- Feature normalization can be used to approximately equalize ranges of the features and make them have approximately the same effect in the distance computation.
- The following methods can be used to independently normalize each feature.
Feature Normalization

- **Linear scaling to unit range:**
  Given a lower bound $l$ and an upper bound $u$ for a feature $x \in \mathbb{R}$,
  \[
  \tilde{x} = \frac{x - l}{u - l}
  \]
  results in $\tilde{x}$ being in the $[0, 1]$ range.

- **Linear scaling to unit variance:**
  A feature $x \in \mathbb{R}$ can be transformed to a random variable with zero mean and unit variance as
  \[
  \tilde{x} = \frac{x - \mu}{\sigma}
  \]
  where $\mu$ and $\sigma$ are the sample mean and the sample standard deviation of that feature, respectively.
Feature Normalization

- **Normalization using the cumulative distribution function:**
  Given a random variable \( x \in \mathbb{R} \) with cumulative distribution function \( F_x(x) \), the random variable \( \tilde{x} \) resulting from the transformation \( \tilde{x} = F_x(x) \) will be uniformly distributed in \([0, 1]\).

- **Rank normalization:**
  Given the sample for a feature as \( x_1, \ldots, x_n \in \mathbb{R} \), first we find the order statistics \( x^{(1)}, \ldots, x^{(n)} \) and then replace each pattern’s feature value by its corresponding normalized rank as

  \[
  \tilde{x}_i = \frac{\text{rank}(x_i) - 1}{n - 1}
  \]

  where \( x_i \) is the feature value for the \( i \)’th pattern. This procedure uniformly maps all feature values to the \([0, 1]\) range.