

Non-parametric Methods

Selim Aksoy

Department of Computer Engineering

Bilkent University

saksoy@cs.bilkent.edu.tr

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Introduction

- ▶ Density estimation with parametric models assumes that the forms of the underlying density functions are known.
- ▶ However, common parametric forms do not always fit the densities actually encountered in practice.
- ▶ In addition, most of the classical parametric densities are unimodal, whereas many practical problems involve multimodal densities.
- ▶ Non-parametric methods can be used with arbitrary distributions and without the assumption that the forms of the underlying densities are known.



Non-parametric Density Estimation

- ▶ Suppose that n samples $\mathbf{x}_1, \dots, \mathbf{x}_n$ are drawn i.i.d. according to the distribution $p(\mathbf{x})$.
- ▶ The probability P that a vector \mathbf{x} will fall in a region \mathcal{R} is given by

$$P = \int_{\mathcal{R}} p(\mathbf{x}') d\mathbf{x}'.$$

- ▶ The probability that k of the n will fall in \mathcal{R} is given by the binomial law

$$P_k = \binom{n}{k} P^k (1 - P)^{n-k}.$$

- ▶ The expected value of k is $E[k] = nP$ and the MLE for P is $\hat{P} = \frac{k}{n}$.



Non-parametric Density Estimation

- ▶ If we assume that $p(\mathbf{x})$ is continuous and \mathcal{R} is small enough so that $p(\mathbf{x})$ does not vary significantly in it, we can get the approximation

$$\int_{\mathcal{R}} p(\mathbf{x}') d\mathbf{x}' \simeq p(\mathbf{x})V$$

where \mathbf{x} is a point in \mathcal{R} and V is the volume of \mathcal{R} .

- ▶ Then, the density estimate becomes

$$p(\mathbf{x}) \simeq \frac{k/n}{V}.$$



Non-parametric Density Estimation

- ▶ Let n be the number of samples used, \mathcal{R}_n be the region used with n samples, V_n be the volume of \mathcal{R}_n , k_n be the number of samples falling in \mathcal{R}_n , and $p_n(\mathbf{x}) = \frac{k_n/n}{V_n}$ be the estimate for $p(\mathbf{x})$.
- ▶ If $p_n(\mathbf{x})$ is to converge to $p(\mathbf{x})$, three conditions are required:

$$\lim_{n \rightarrow \infty} V_n = 0$$

$$\lim_{n \rightarrow \infty} k_n = \infty$$

$$\lim_{n \rightarrow \infty} \frac{k_n}{n} = 0.$$



Histogram Method

- ▶ A very simple method is to partition the space into a number of equally-sized cells (*bins*) and compute a *histogram*.

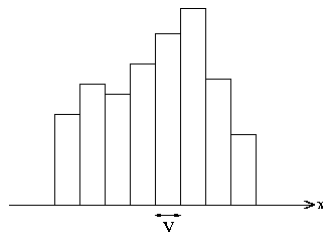


Figure 1: Histogram in one dimension.

- ▶ The estimate of the density at a point \mathbf{x} becomes

$$p(\mathbf{x}) = \frac{k}{nV}$$

where n is the total number of samples, k is the number of samples in the cell that includes \mathbf{x} , and V is the volume of that cell.



Histogram Method

- ▶ Although the histogram method is very easy to implement, it is usually not practical in high-dimensional spaces due to the number of cells.
- ▶ Many observations are required to prevent the estimate being zero over a large region.
- ▶ Modifications for overcoming these difficulties:
 - ▶ Data-adaptive histograms,
 - ▶ Independence assumption (naive Bayes),
 - ▶ Dependence trees.



Non-parametric Density Estimation

- ▶ Other methods for obtaining the regions for estimation:
 - ▶ Shrink regions as some function of n , such as $V_n = 1/\sqrt{n}$. This is the *Parzen window* estimation.
 - ▶ Specify k_n as some function of n , such as $k_n = \sqrt{n}$. This is the *k-nearest neighbor* estimation.

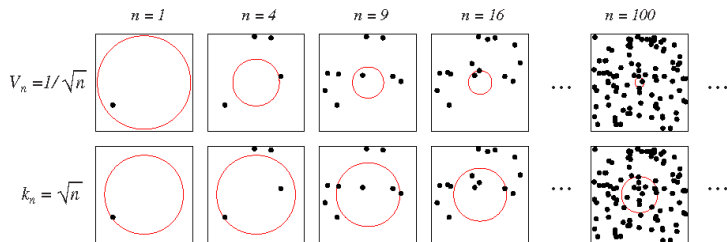


Figure 2: Methods for estimating the density at a point, here at the center of each square.

Parzen Windows

- Suppose that φ is a d -dimensional window function that satisfies the properties of a density function, i.e.,

$$\varphi(\mathbf{u}) \geq 0 \quad \text{and} \quad \int \varphi(\mathbf{u}) d\mathbf{u} = 1.$$

- A density estimate can be obtained as

$$p_n(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n \frac{1}{V_n} \varphi\left(\frac{\mathbf{x} - \mathbf{x}_i}{h_n}\right)$$

where h_n is the window width and $V_n = h_n^d$.



Parzen Windows

- The density estimate can also be written as

$$p_n(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n \delta_n(\mathbf{x} - \mathbf{x}_i) \quad \text{where} \quad \delta_n(\mathbf{x}) = \frac{1}{V_n} \varphi\left(\frac{\mathbf{x}}{h_n}\right).$$

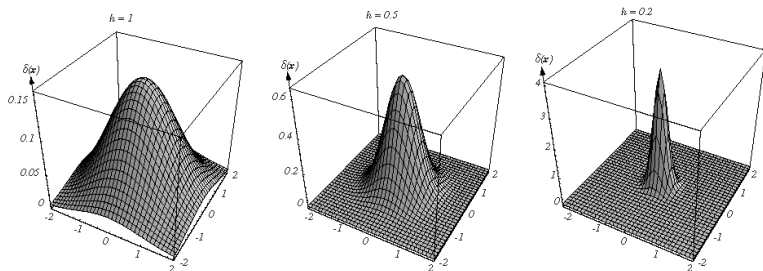


Figure 3: Examples of two-dimensional circularly symmetric Parzen window functions for three different values of h_n . The value of h_n affects both the amplitude and the width of $\delta_n(\mathbf{x})$.

Parzen Windows

- ▶ If h_n is very large, $p_n(\mathbf{x})$ is the superposition of n broad functions, and is a smooth “out-of-focus” estimate of $p(\mathbf{x})$.
- ▶ If h_n is very small, $p_n(\mathbf{x})$ is the superposition of n sharp pulses centered at the samples, and is a “noisy” estimate of $p(\mathbf{x})$.
- ▶ As h_n approaches zero, $\delta_n(\mathbf{x} - \mathbf{x}_i)$ approaches a Dirac delta function centered at \mathbf{x}_i , and $p_n(\mathbf{x})$ is a superposition of delta functions.

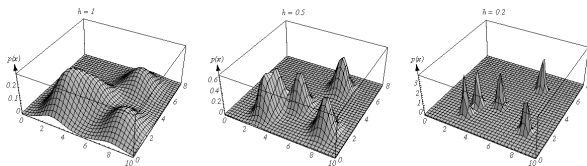


Figure 4: Parzen window density estimates based on the same set of five samples using the window functions in the previous figure.

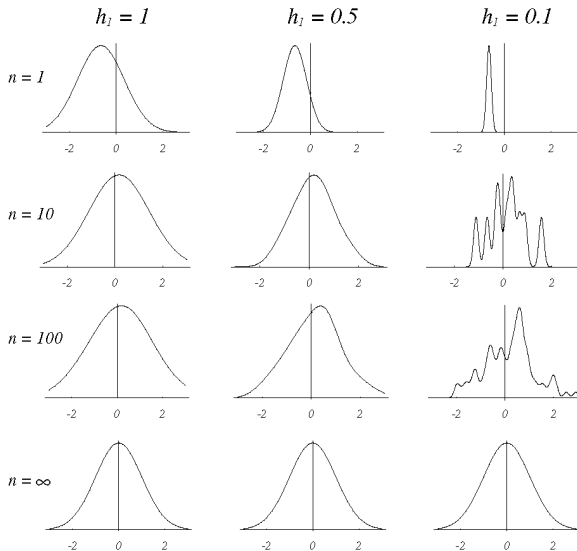


Figure 5: Parzen window estimates of a univariate Gaussian density using different window widths and numbers of samples where $\varphi(u) = N(0, 1)$ and $h_n = h_1/\sqrt{n}$.

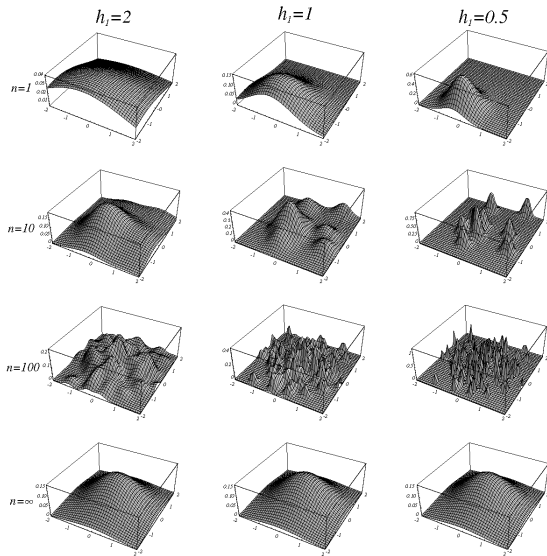


Figure 6: Parzen window estimates of a bivariate Gaussian density using different window widths and numbers of samples where $\varphi(\mathbf{u}) = N(\mathbf{0}, \mathbf{I})$ and $h_n = h_1/\sqrt{n}$.

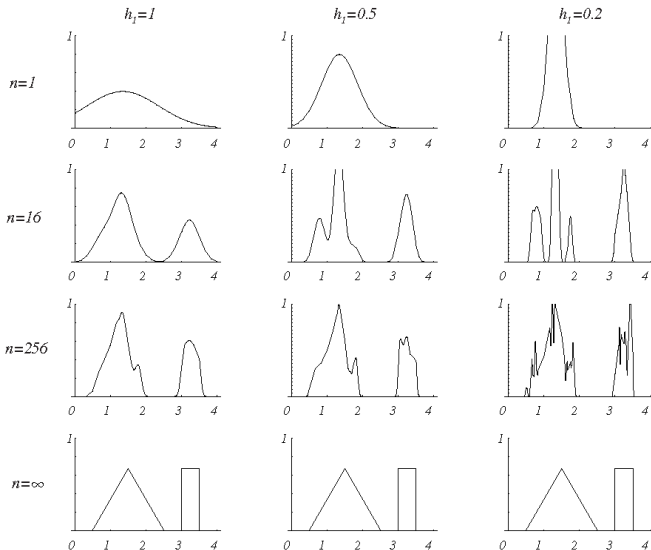


Figure 7: Estimates of a mixture of a uniform and a triangle density using different window widths and numbers of samples where $\varphi(u) = N(0, 1)$ and $h_n = h_1/\sqrt{n}$.

Parzen Windows

- ▶ Densities estimated using Parzen windows can be used with the Bayesian decision rule for classification.
- ▶ The training error can be made arbitrarily low by making the window width sufficiently small.
- ▶ However, the goal is to classify novel patterns so the window width cannot be made too small.

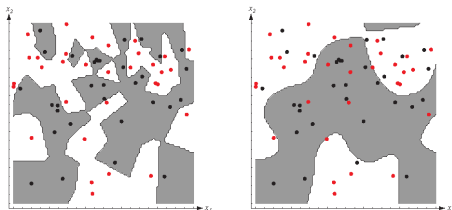


Figure 8: Decision boundaries in 2-D. The left figure uses a small window width and the right figure uses a larger window width.

k -Nearest Neighbors

- ▶ A potential remedy for the problem of the unknown “best” window function is to let the estimation volume be a function of the training data, rather than some arbitrary function of the overall number of samples.
- ▶ To estimate $p(\mathbf{x})$ from n samples, we can center a volume about \mathbf{x} and let it grow until it captures k_n samples, where k_n is some function of n .
- ▶ These samples are called the k -nearest neighbors of \mathbf{x} .
- ▶ If the density is high near \mathbf{x} , the volume will be relatively small. If the density is low, the volume will grow large.



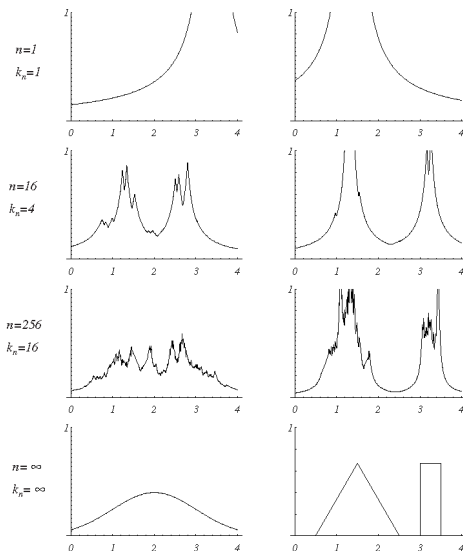


Figure 9: k -nearest neighbor estimates of two 1-D densities: a Gaussian and a bimodal distribution.

k -Nearest Neighbors

- ▶ Posterior probabilities can be estimated from a set of n labeled samples and can be used with the Bayesian decision rule for classification.
- ▶ Suppose that a volume V around \mathbf{x} includes k samples, k_i of which are labeled as belonging to class w_i .
- ▶ As estimate for the joint probability $p(\mathbf{x}, w_i)$ becomes

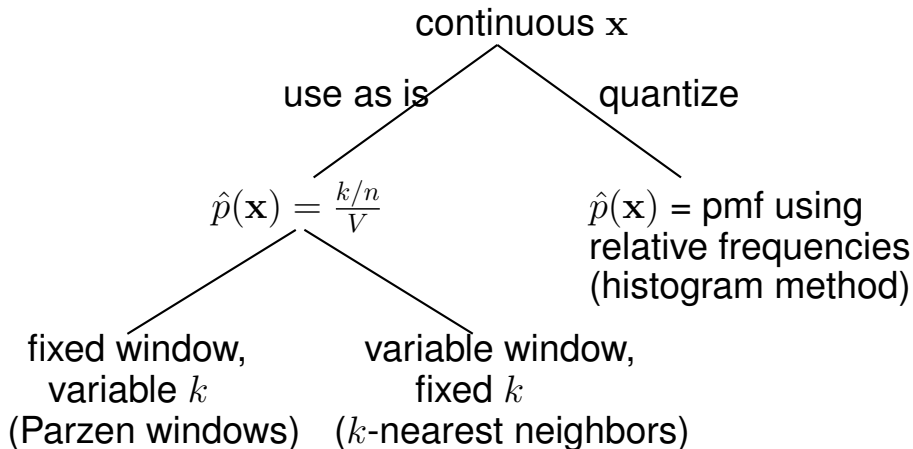
$$p_n(\mathbf{x}, w_i) = \frac{k_i/n}{V}$$

and gives an estimate for the posterior probability

$$P_n(w_i|\mathbf{x}) = \frac{p_n(\mathbf{x}, w_i)}{\sum_{j=1}^c p_n(\mathbf{x}, w_j)} = \frac{k_i}{k}.$$



Non-parametric Methods



Non-parametric Methods

- ▶ Advantages:

- ▶ No assumptions are needed about the distributions ahead of time (generality).
- ▶ With enough samples, convergence to an arbitrarily complicated target density can be obtained.

- ▶ Disadvantages:

- ▶ The number of samples needed may be very large (number grows exponentially with the dimensionality of the feature space).
- ▶ There may be severe requirements for computation time and storage.



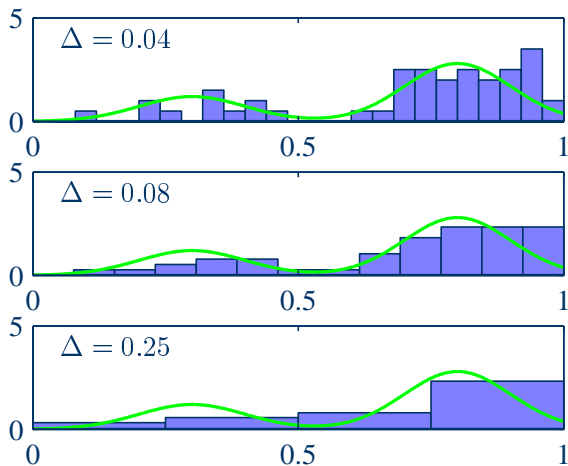


Figure 10: An illustration of the histogram approach to density estimation, in which a data set of 50 points is generated from the distribution shown by the green curve. Histogram density estimates are shown for various values of the cell volume (Δ).

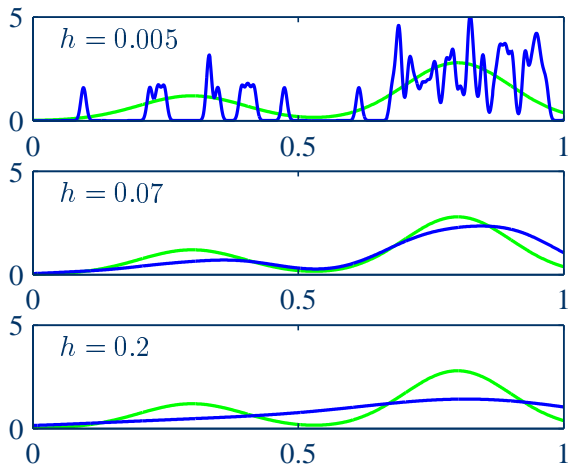


Figure 11: Illustration of the Parzen density model. The window width (h) acts as a smoothing parameter. If it is set too small (top), the result is a very noisy density model. If it is set too large (bottom), the bimodal nature of the underlying distribution is washed out. An intermediate value (middle) gives a good estimate.

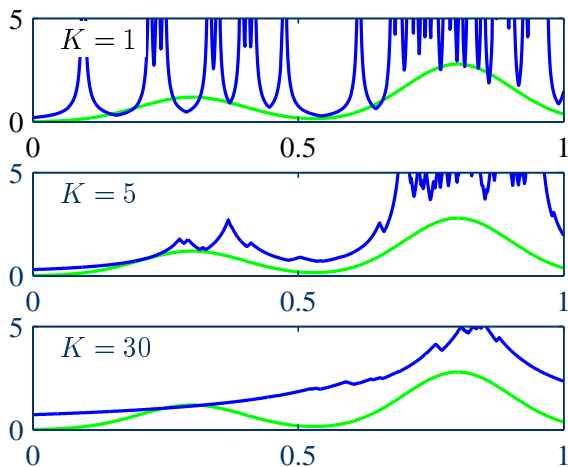


Figure 12: Illustration of the k -nearest neighbor density model. The parameter k governs the degree of smoothing. A small value of k (top) leads to a very noisy density model. A large value (bottom) smooths out the bimodal nature of the true distribution.

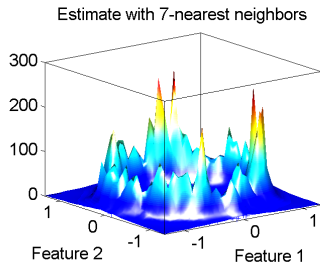
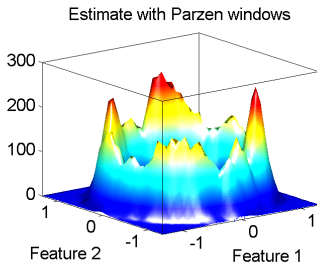
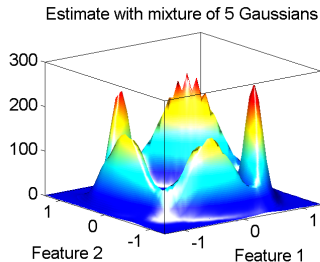
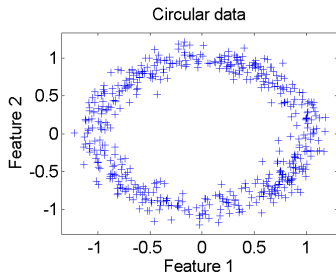


Figure 13: Density estimation examples for 2-D circular data.

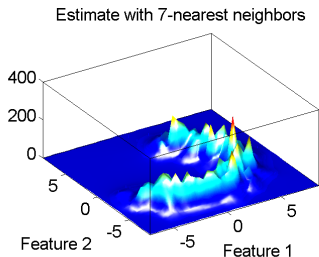
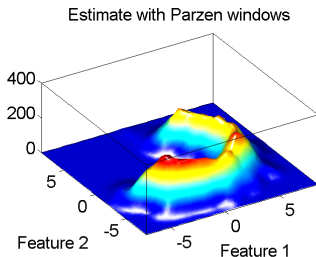
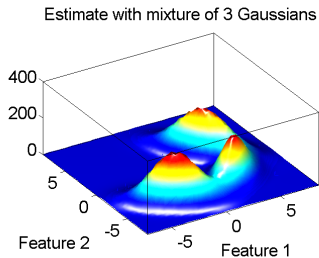
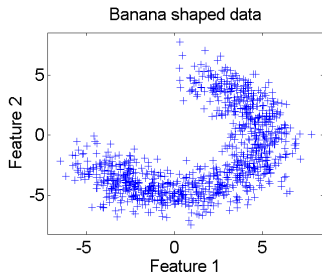


Figure 14: Density estimation examples for 2-D banana shaped data.