The Distribution of Queuing Network States at Input and Output Instants

K. C. SEVCIK

University of Toronto, Toronto, Ontario, Canada

AND

I. MITRANI

University of Newcastle-upon-Tyne, England

ABSTRACT. Queuing networks are studied at selected points in the steady state, namely, at the moments when jobs of a given class arrive into a given node (either from the outside or from other nodes) and at the moments when jobs of a given class leave a given node (either for the outside or for other nodes). The processes defined by these points are known to be, in general, non-Poisson, interdependent, and serially correlated; therefore the relation between the distribution of the system state embedded at those moments and the steady-state (or random point) distribution is not obvious a priori. For a large class of networks having product-form equilibrium distributions it is shown that (a) if the given job class belongs to an open subchain, the state distributions at input points, output points, and random points are identical, and (b) if the job class belongs to a closed subchain, the distribution at input and output points is the same as the steady-state distribution of a network with one less job in that subchain.

KEY WORDS AND PHRASES: queuing theory, network of queues, product form, waiting time distributions

CR CATEGORIES: 4 32, 4.35, 5.5, 6.20

1. *Introduction*

Ever since the discovery by Jackson [14] that in some cases the steady-state distribution of a network of queues can be factored into a product of the marginal distributions of the states of individual nodes, there has been some uncertainty as to the exact meaning of this result. If a node has c parallel exponential servers, to what extent can it be treated as an isolated $M/M/c$ queue? In particular, can the distribution of the sojourn times at that node (and not just the mean sojourn time) be obtained by such a treatment? The answer to this last question depends on whether customers arriving into the node see the steady-state distribution of the node state, and that, in turn, has prompted several investigations of the properties of input streams.

Permission to copy without fee all or part of this material is granted provided that the copies are not made or distributed for direct commercial advantage, the ACM copyright notice and the title of the publication and its date appear, and notice is given that copying is by permission of the Association for Computing Machinery. To copy otherwise, or to republish, requires a fee and/or specific permission.

Earlier versions of this paper appeared as Research Report 307, IRIA Laboria, May 1978, and in the Proceedings of the 4th International Symposium on Modelling and Performance Evaluation of Computer Systems, Vienna, Austria, February 1979

Authors' addresses: K.C Sevcik, Computer Systems Research Group, University of Toronto, 121 St. Joseph Street, Toronto, Ontario, Canada M5S 1A1; I. Mitrani, University of Newcastle-upon-Tyne, England NE1 7RU.

© 1981 ACM 0004-5411/81/0400-0358 $00 75

Burke [4] has shown, by examining the simplest nontrivial network (a single M/M/1 node with Bernoulli feedbacks), that the input process composed of the external arrivals and the fed-back customers is not Poisson in general. Disney and McNickle [12] have strengthened this result by showing that the input process is not even renewal. On the basis of this and other evidence of the complexity and interdependence of the flow processes in a queuing network, these two authors expressed a strong doubt about the possibility of decomposing the network and treating individual nodes in isolation.

Yet in the one-node example which Burke considered, it emerged that although the input customers do not form a Poisson stream, they still see the steady-state distribution of the queue size (where fed-back customers do not include themselves in the queue that they see). Input instant and steady-state distributions were shown to be closely related also in three finite-state models: an M/M/1 queue with a finite customer population, an M/M/1 queue with finite waiting room, and a two-node closed network with a feedback loop around one of the nodes [9, 23]. The first two models can be viewed as two-node closed networks in which the service rate at the second node is, respectively, proportional to the number of customers there or constant. In all three cases the following result holds:

$$p_k^{\mathrm{i}}(n) = \frac{p_K^{\mathrm{e}}(n)}{1 - p_K^{\mathrm{e}}(K)} = p_{K-1}^{\mathrm{e}}(n), \quad n = 0, 1, \ldots, K - 1,$$

where $p_K^{\mathrm{i}}(n)$ and $p_K^{\mathrm{e}}(n)$ are the input instant and the equilibrium probabilities of n customers at the server, given that there are K customers in the network.

We demonstrate that these results generalize to a large class of networks, including open, closed, and mixed, with many customer classes, where scheduling disciplines and service requirement distributions are of certain types. The probability that a class-r customer joining node m sees network state S, $\xi(S)$, depends on whether the network is open or closed with respect to class r. If it is open, then $\xi(S)$ is equal to the probability that a class-r customer arriving from the outside into node m sees state S (which is equal to the steady-state probability of S if the external arrivals are state-independent). If it is closed, then $\xi(S)$ is equal to the steady-state probability of S in a network with one less class-r customer. Similar statements can be made concerning the state left behind by a class-r customer leaving node m.

These results are of practical as well as theoretical interest. Input-instant and output-instant distributions arise in the analyses of some queuing systems, and determination of a job's residence time at a service center depends on knowing the distribution of system state at job arrival instants. Reiser and Lavenberg developed and Bard extended mean-value analysis, which consists of recursive methods of calculating mean queue lengths and mean response time [2, 25, 26]. The relationship between input-instant and equilibrium distributions is stated (without proof) as the intuitive interpretation of one of the fundamental formulas of mean-value analysis.

Since this paper was submitted for publication, several related papers have appeared. Melamed has applied general results on Markov jump processes to queuing networks, obtaining characterizations of the network state distribution at arrival and departure instants [22]. His results require that the arrival process be independent of system state. Lavenberg and Reiser produced a proof of their earlier conjecture [20]. Their results apply only to the states of individual service centers at transition instants rather than to the state of the whole network. In conjunction with an operational derivation of mean-value analysis, Buzen and Denning have provided a simple proof of the Reiser–Lavenberg conjecture in the framework of operational analysis [5]. Like that of Lavenberg and Reiser, their proof applies only to the states of individual queues rather than to total network state.

Section 2 contains a statement of the problem in the context of BCMP networks [3] with Poisson arrivals and state-independent service rates. The main results are derived in Section 3, using concepts and tools from Markov renewal theory. The model is then generalized in Section 4 to allow state-dependent arrival and service rates, more general routing, lost and triggered arrivals, and absolutely continuous service-time distributions. (This last generalization involves a conjecture.)

2. *Definition of the Problem*

We are concerned mainly with the class of separable queuing networks, which are generally known in the computing literature as BCMP networks [3, 16, 17, 21]. The reader is assumed to be familiar with the concepts and terminology used in [3].

Consider a BCMP network with M nodes and K job classes. The path followed by a job in the network is governed by a routing matrix P. A job of class r completing service at node i goes to node j as a job of class s with probability $P_{ir,js}$ $(i, j = 1, 2, \ldots, M;\ r, s = 1, 2, \ldots, K)$; that job leaves the network with probability

$$P_{ir,0} = 1 - \sum_{j,s} p_{ir,js}.$$

To simplify notation, we reclassify the jobs, using the pair (i, r) as the new class index. Thus, from now on there will be $R = KM$ job classes, and "class r" will specify both the current affiliation and the current location of a job. The elements of the routing matrix become p_{rs} (the probability that a class-r job completing service turns into a class-s job for $r, s = 1, 2, \ldots, R$).

Two job classes r and s are said to be connected if either $p_{rs} > 0$ or $p_{sr} > 0$. Define the relation "communicate" recursively as follows: r and s communicate if either they are connected or there exists class r' such that r and r' are connected and r' and s communicate. That relation partitions the set of job classes into one or more nonintersecting subsets or "subchains," $E_1, E_2, \ldots, E_D$; two job classes belong to the same subchain if and only if they communicate. If, for example, in the original notation we had $p_{ir,js} = 0$ for all $i, j = 1, 2, \ldots, M$ and $r \neq s$, there would now be at least K subchains.

Assume for the moment that if there are external arrivals into the network, these form a Poisson stream with rate λ (we shall see in Section 4 that this assumption can be relaxed to allow certain state-dependent arrival processes). An external arrival joins class r with probability p_{0r} $(r = 1, 2, \ldots, R, \sum_r p_{0r} = 1)$. The flow of jobs around the network is described by D (the number of subchains) systems of linear equations by relating the frequencies with which the various classes are entered:

$$e_r = \lambda p_{0r} + \sum_{s \in E_l} e_s p_{sr}, \qquad r \in E_l, \quad l = 1, 2, \ldots, D. \tag{1}$$

A subchain E_l is open if external jobs arrive into it and jobs depart from it ($p_{0r} > 0$ for some $r \in E_l$ and $p_{s0} > 0$ for some $s \in E_l$). The system (1) corresponding to an open subchain has a unique solution $\{e_r\}$, which can be interpreted as the average number of times a job joins class r during its life in the network ($r \in E_l$). A subchain E_l is closed if the number of jobs circulating in it is constant ($p_{0r} = p_{r0} = 0$ for all $r \in E_l$); the corresponding system (1) has infinitely many solutions (differing from one another by multiplicative constants), which can be interpreted as the relative frequency of joining class r. Any job in a closed subchain will eventually join every class in that subchain with probability one.

The state of the network at time t is defined as the vector $S(t) = (S_1(t), S_2(t), \ldots, S_R(t))$, where $S_r(t)$ describes the state of the class-r jobs at time t ($r = 1, 2, \ldots, R$). (For notational convenience, we shall drop the t.) The definition of S_r depends on the type of the node associated with class r:

(1) If that node is of type 1 (FCFS), then all required service times there are distributed exponentially with the same mean; S_r is the set of integers $\{h_1, h_2, \ldots, h_{k_r}\}$ indicating the position of each class-r job in the corresponding FCFS queue (k_r is the number of class-r jobs in it).

(2) If the node type is 2 or 3 (processor-sharing or infinitely many servers), then the class-r required service times can have an arbitrary Coxian distribution (see [3, 10]) with Q_r stages; S_r is the vector $(k_{r1}, k_{r2}, \ldots, k_{rQ_r})$, where k_{rj} is the number of class-r jobs in stage j of their service ($k_r = k_{r1} + k_{r2} + \cdots + k_{rQ_r}$ is the total number of class-r jobs).

(3) If the node type is 4 (LCFS preemptive resume), then again the required service times can have Coxian distributions; S_r is the set of pairs $\{(h_1, j_1), (h_2, j_2), \ldots, (h_{k_r}, j_{k_r})\}$ indicating, for each class-r job, its position in the LCFS queue and its stage of service.

The set of states that the network can be in is determined completely by the network configuration C. We define C as the triple $\{R, P, \mathbf{N}\}$, where R is the set $\{(i_1, Q_1), (i_2, Q_2), \ldots, (i_R, Q_R)\}$ indicating, for each job class, the type of node associated with it and the number of stages in the corresponding service-time distribution; P is the $R \times R$ routing matrix; $\mathbf{N}$ is a vector of integers $(N_1, N_2, \ldots, N_J)$, where N_j is the number of jobs circulating in the jth closed subchain; that vector is empty if there are no closed subchains.

Given a configuration C of the network, $\{S(t)\,; t > 0\}$ is an irreducible Markov process with a denumerable state space. We assume that the arrival and service parameters are such that the process is recurrent nonnull, that is, the limiting probabilities

$$\pi_C(S) = \lim_{t \to \infty} P(S(t) = S) \tag{2}$$

exist and are nonzero for all states S of configuration C.

Let us now examine the changes of state as jobs join, or leave, job classes. Suppose that the network is in state S of configuration C, and that a class-r job completes service. Denote by S_{-r} the state that would result if that job departed from the network. Note that if class r is in a closed subchain, S_{-r} is not attainable in configuration C; in that case, S_{-r} belongs to a configuration that we denote by C_{-r} and which differs from C by having one less job in that subchain. Note further that if class r is associated with a type-1 or type-4 node, S_{-r} is defined only for states S with a class-r job at the head of the queue. S_{-r} is the state that the completing job "leaves behind." It is also the state that the job would "see" if it joined another class.

More formally, let $t_1^r, t_2^r, \ldots, t_n^r, \ldots$ be the consecutive moments when jobs join class r; we shall call these moments "class-r input instants." If the job joining class r came from outside the network, the input instant is "external"; otherwise it is "internal." Define the random variable S_n^r as follows:

$$S_n^r = \begin{cases} S(t_n^r-) & \text{if } t_n^r \text{ is external;} \\ S_{-r}(t_n^r-) & \text{if } t_n^r \text{ is internal,} \quad n = 1, 2, \ldots. \end{cases} \tag{3}$$

S_n^r is the network state seen at the nth class-r input instant. We choose definition (3) because we wish to exclude the job joining class r from the state that it sees. Similarly, let $t_1^o, t_2^o, \ldots, t_n^o, \ldots$ be the consecutive moments when class-r jobs complete service; these are the "class-r output instants." An output instant is external if the completing job departs from the network; otherwise it is internal. Define the random variable S_n^o as

$$S_n^o = \begin{cases} S(t_n^o+) & \text{if } t_n^o \text{ is external;} \\ S_{-r}(t_n^o-) & \text{if } t_n^o \text{ is internal,} \quad n = 1, 2, \ldots. \end{cases} \tag{4}$$

S_n^o is the network state left behind at the nth output instant.

Clearly, $\{S_n^i; n = 1, 2, \ldots\}$ and $\{S_n^o; n = 1, 2, \ldots\}$ are irreducible and recurrent Markov chains with the same state space (configuration C if the subchain to which class r belongs is open, C_{-r} if it is closed). Let $\xi_r^i(S)$ and $\xi_r^o(S)$ be the limiting distributions,

$$\xi_r^i(S) = \lim_{n\to\infty} P(S_n^i = S) \tag{5}$$

and

$$\xi_r^o(S) = \lim_{n\to\infty} P(S_n^o = S). \tag{6}$$

Our aim is to express these stationary distributions of embedded Markov chains in terms of the stationary distribution (2) of the original Markov process.

3. *Main Results*

We cannot find the distributions (5) and (6) in the traditional way (by solving systems of linear equations) because we do not have the transition probability matrices of Markov chains (3) and (4). We use, instead, the properties of the continuous parameter process $\{S(t); t > 0\}$, together with some results from Markov renewal theory. Most of what follows concerns the network state seen at input instants; the treatment of the output instants is very similar.

The stochastic process $\{S_n^i, t_n^i; n = 1, 2, \ldots\}$ is a Markov renewal process (see [8]); the joint distribution of S_{n+1}^i and $(t_{n+1}^i - t_n^i)$ depends only on S_n^i and not on the process history prior to t_n^i. Let $\gamma_r^i(S, t)$ be the expected number of times that $S_n^i = S$ on the interval $(0, t)$ for $n = 1, 2, \ldots$. Denote

$$\eta_r^i(S) = \lim_{t\to\infty} \frac{\gamma_r^i(S, t)}{t}. \tag{7}$$

Since the embedded Markov chain is irreducible, aperiodic, and recurrent, the limits (7) exist for all states in the appropriate configuration and are independent of the initial state. The reciprocal $1/\eta_r^i(S)$ is the expected interval between two consecutive class-r input instants at which the network is in state S. The desired probabilities (5) are proportional to the quantities $\eta_r^i(S)$ ([8, eq. 10.4.5]. Therefore, denoting

$$\eta_r^i = \sum_S \eta_r^i(S),$$

we can write

$$\xi_r^i(S) = \frac{\eta_r^i(S)}{\eta_r^i}. \tag{8}$$

In other words, the steady-state probability of seeing state S at a class-r input instant is equal to the ratio of two expectations: the expected number of class-r input instants at which the network is seen in state S per unit time, and the total expected number of class-r input instants per unit time.

Next, we express the above expectations in terms of the instantaneous transition rates of the Markov process $\{S(t);\ t > 0\}$. The connection is provided by the fact that, if S' and S'' are two states of configuration C, with $a(S', S'')$ the instantaneous transition rate from S' to S'', and if $\gamma(S', S'', t)$ is the expected number of transitions from S' to S'' in the interval $(0, t)$, then

$$\lim_{t\to\infty} \frac{\gamma(S', S'', t)}{t} = \pi_C(S')a(S', S'') \tag{9}$$

where $\pi_C(S')$ is the steady-state probability (2) of state S'. Equation (9), which is valid for any recurrent Markov process with a denumerable state space, is derived in the appendix.

The expectation $\gamma_r^1(S, t)$, which appears in the right-hand side of (7), has two components:

(i) The expected number of external input instants at which the network is seen in state S. According to (3) these instants correspond to transitions of $S(t)$ from state S to the state S_{+r} resulting from the addition of a new class-r job. The instantaneous transition rate is $a(S, S_{+r}) = \lambda p_{0r}$ (that rate is zero if r belongs to a closed subchain).

(ii) The expected number of internal input instants at which the network is seen in state S. These are the movements when jobs of other classes in the same subchain complete service and join class r, seeing state S. The transitions involved are from states S', such that $S'_{-r'} = S$ (for some r' in the same subchain as r), to state S_{+r}. The corresponding instantaneous transition rate is $a(S', S_{+r}) = \nu_{r'}(S')p_{r',r}$, where $\nu_r(S')$ is the service completion rate for class r in state S'.

The above observations, together with (9) and (7), allow us to write

$$\eta_r^1(S) = \pi_C(S)\lambda p_{0r} + \sum_{r'\in E} \left\{ \left[\sum_{S'\in\sigma} \pi_C(S')\nu_{r'}(S') \right] p_{r',r} \right\}, \tag{10}$$

where E is the subchain to which class r belongs and σ is the set of states S' such that $S'_{-r'} = S$. It is perhaps worth emphasizing again that if E is closed, then (a) the first term in (10) is zero, and (b) the state S that is seen is of network configuration C_{-r}, not C.

Now we are going to invoke the product form of the solution of BCMP networks (so far, only general properties of Markov and Markov renewal processes have been used) in order to simplify the right-hand side of (10). In particular, the following result holds.

LEMMA 1. *In the notation of eq.* (10), *if the subchain E is open, then*

$$\sum_{S'\in\sigma} \pi_C(S')\nu_{r'}(S') = \lambda e_{r'}\pi_C(S), \qquad r' \in E, \tag{11}$$

where λ is the external arrival rate and $\{e_r\}$ is the solution of (1). *If the subchain E is closed, then*

$$\sum_{S'\in\sigma} \pi_C(S')\nu_{r'}(S') = ce_{r'}\pi_{C-r'}(S), \qquad r' \in E, \tag{12}$$

where c is a constant independent of S.

The proof of Lemma 1 is in the appendix. Intuitively, (11) equates the service completion rate for class r' in states such that state S is left behind with the arrival rate for class r' in state S. This should not be confused with the local balance equations, since it ignores the destination of the departing job and the origin of the arriving job. Equation (11) coincides with the local balance equations only in the case of a one-node network without feedback. Equation (12) has a similar character, except that the rates involved are in different network configurations.

Lemma 1 and the flow balance eqs. (1) reduce (10) to

$$\eta_r^{\mathrm{i}}(S) = \begin{cases} \lambda e_r \pi_C(S) & \text{if } E \text{ is open,} \\ c e_r \pi_{C-r}(S) & \text{if } E \text{ is closed} \end{cases} \tag{13}$$

(note that the configurations C_{-r} and $C_{-r'}$ are identical when r and r' belong to the same subchain).

Now, substituting (13) into (8), we obtain our result.

THEOREM 1. *In nonsaturated BCMP queuing networks with Poisson external arrival streams,*

$$\xi_r^{\mathrm{i}}(S) = \pi_C(S) \tag{14a}$$

if r belongs to an open subchain, and

$$\xi_r^{\mathrm{i}}(S) = \pi_{C-r}(S) \tag{14b}$$

if r belongs to a closed subchain.

Thus, input jobs in open subchains see the time-average equilibrium distribution of the network state, regardless of which class they join or whether they come from inside or outside the network. Inputs in closed subchains see the equilibrium distribution of a different network configuration with one less job in the relevant subchain. Again that is regardless of which class they leave or which class they join.

Going back to the original model formulation, with nodes $(1, 2, \ldots, M)$ and job classes $(1, 2, \ldots, K)$, the network state distribution seen by a class-r job coming into node m depends on whether the pair (m, r) belongs to an open or a closed subchain. Different jobs coming into the same node may see states of different configurations depending on their class. Note also that if (14) is true in terms of a detailed network state S, it will obviously be true in terms of various aggregated states; in particular, it applies to the marginal distribution of an individual node state seen by jobs arriving there. The distribution of sojourn times at a node in an open Jackson network can, therefore, be found by isolating that node and treating it as an independent M/M/1 or M/M/c queue with the appropriate parameters.

For closed subchains, an expression for $\xi_r^{\mathrm{i}}(S)$ in terms of network configuration C can be obtained by substituting (12) into (14b):

$$\xi_r^{\mathrm{i}}(S) = \left(\frac{1}{c e_{r'}}\right)\left[\sum_{S' \in \sigma} \pi_C(S')\nu_{r'}(S')\right], \qquad r' \in E. \tag{15}$$

If r' should happen to be associated with a type-1 node (with average service time $1/\mu_{r'}$), then σ consists of a single state; denote that state by S^*. Bearing in mind that according to (13), $c e_{r'} = \eta_{r'}^{\mathrm{i}}$ is the throughput of class r', we can rewrite (15) as

$$\xi_r^{\mathrm{i}}(S) = \frac{\pi_C(S^*)}{\rho_{r'}}, \tag{16}$$

where $\rho_{r'} = c e_{r'}/\mu_{r'}$ is the utilization of class r'. The right-hand side of (16) is the conditional probability of state S^*, given that a class r' job is in service. Equation (16) is a generalization of the results mentioned in the introduction.

The derivation of the distribution $\xi_r^{\mathrm{o}}(S)$ of the network state left behind at output instants proceeds along similar lines. We write (see eq. (8))

$$\xi_r^{\mathrm{o}}(S) = \frac{\eta_r^{\mathrm{o}}(S)}{\eta_r^{\mathrm{o}}},$$

where $1/\eta_r^o(S)$ is the expected time between two class-r output instants at which state S is left behind and $\eta_r^o = \sum_s \eta_r^o(S)$. As for eq. (10), we obtain

$$\eta_r^o(S) = \sum_{S' \in \sigma} \pi_C(S')\nu_r(S'),$$

where σ is the set of states S' such that $S'_{-r} = S$ and $\nu_r(S')$ is the instantaneous service completion rate for class r in state S'. Application of Lemma 1 yields

$$\eta_r^o(S) = \eta_r^i(S)$$

(this last equation follows also directly from the fact that the network is in equilibrium). Therefore,

THEOREM 2. *In nonsaturated BCMP queuing networks with Poisson external arrival streams,*

$$\xi_r^o(S) = \pi_C(S)$$

if r belongs to an open subchain, and

$$\xi_r^o(S) = \pi_{C-r}(S)$$

if r belongs to a closed subchain.

All remarks that were made following Theorem 1, including eq. (16), apply to output instants also.

4. *Generalizations*

We saw that the key to Theorems 1 and 2 was Lemma 1. If the model is generalized so as to preserve the validity of Lemma 1 (without destroying the Markov properties of the continuous parameter process), those theorems, or similar ones, will hold. We describe here several such generalizations.

The external arrival process in BCMP networks may be state-dependent, in a restricted way. Let $N_o(S)$ be the total number of jobs in the open subchains and $N_l(S)$ be the number of jobs in open subchain E_l, when the network is in state S. The external arrivals may be generated according to either one, but not both, of the following two rules.

(a) By a single nonstationary Poisson process whose instantaneous rate $\lambda(N_o(S))$ depends on the network state via $N_o(S)$. As before, a new arrival joins class r with probability p_{0r} ($\sum_{r=1}^{R} p_{0r} = 1$).

(b) By separate nonstationary Poisson processes, one for each open subchain. The instantaneous rate of the lth process, $\lambda_l(N_l(S))$, depends on the network state via $N_l(S)$. A new arrival in the lth stream joins class r with probability p_{0r} ($\sum_{r \in E_l} p_{0r} = 1$ if E_l is open).

The proof of Lemma 1 under these assumptions is almost identical to the previous proof. The only change is to replace the state-independent arrival rate λ with

$$\Lambda_r(S) = \lambda(N_o(S))$$

if the external arrivals are of type (a), and with

$$\Lambda_r(S) = \lambda_j(N_j(S))$$

if arrivals are of type (b) and r belongs to the open subchain E_j.

Equation (13) for open subchains now reads

$$\eta_r^i(S) = \Lambda_r(S)e_r\pi_C(S),$$

and Theorems 1 and 2 for open subchains become (for closed subchains they are unchanged)

$$\xi_r^i(S) = \xi_r^o(S) = \frac{\Lambda_r(S)\pi_C(S)}{\sum_{S' \in C} \Lambda_r(S')\pi_C(S')]}. \tag{17}$$

If we consider the Markov renewal process embedded at consecutive moments of *external* arrivals to (departures from) class r, we easily discover that the network state distribution seen (left behind) at those moments is also given by (17). Thus all inputs (outputs) in open subchains see (leave behind) the same network state distribution as the external arrivals (departures).

The definition of BCMP networks also includes the possibility of state-dependent service rates at either individual nodes or subsets of nodes. The rate of service for class-r jobs may depend on the number k_r of class-r jobs (except at type-1 nodes), and/or on the total number n_m of jobs at the node m associated with class r, and/or on $\sum n_m$ over a certain subset of nodes (but in the last case the same dependency applies to all other classes at that node or nodes). These dependencies are reflected as factor terms in the product-form solution. It is a tedious but routine task to show that Lemma 1 and hence Theorems 1 and 2 continue to hold in their present form.

The class of networks known to have product form has been expanded recently with respect to permissible service disciplines and service requirement distributions. The mathematical (but not practical) limitation that service-time distributions must have rational Laplace transforms can be relaxed to allow general absolutely continuous distributions with finite means [1, 6]. A parameterized family of service disciplines which includes the four disciplines listed in the last section as special cases [6, 7, 16] can be defined.

Both Chandy et al. [6] and Kelly [16] view each queue as having "stations." In general, customers arrive into, receive service in, and depart from any station in the queue. A queuing discipline is defined in terms of how an arriving customer selects a station (possibly forcing other customers to change station) and what proportion of the server's attention is devoted to each station in each state of the queue. A queue discipline is said to be "station balancing" if it is exponential local balancing and in each possible state of the queue the probability that an arriving customer picks a particular station is proportional to the share of the server's attention devoted to that station. (The disciplines LCFSPR, PS and IS are station balancing, whereas FCFS is not.)

The system state is defined by the class and remaining service time of the customer at each station of each queue. Whenever all service disciplines are exponential local balancing and, at all nodes, either the discipline is station balancing or the service requirements for all classes are distributed exponentially with the same mean, the solution for the equilibrium state probability density function has a product form.

Lemma 1 holds for these more general networks. However, Theorems 1 and 2 cannot be claimed directly, because the state space is not denumerable. Nevertheless, we conjecture that the theorems continue to hold, interpreting $\xi_r^i(S)$ and $\xi_r^o(S)$ as probability density functions rather than probabilities.

The product form of the equilibrium state probabilities is retained in certain cases where the routing of jobs is not represented as a simple transition matrix. Kelly [17] considers the case in which each job has a specific route that it follows through the network. Kobayashi and Reiser [18] allow the transitions among service centers to be governed by a Markov process of arbitrary (finite) order. The technique used in both cases is to introduce additional "artificial" classes (typically a lot of them) in order to retain more information about the past or future of the job. Then, in terms of the enlarged set of classes, all transitions among service centers and nonartificial classes can be represented by a simple transition matrix (or first-order Markov process). Moreover, the artificial job classes resulting from one class belong to the same type of subchain (open or closed) as the original class. Theorems 1 and 2 apply to the detailed states involving the artificial job classes. Eliminating the distinctions between the artificial classes—that is, returning to the original ones—corresponds to

(a) merging the embedded Markov renewal process, and
(b) aggregating the detailed networks states.

Both actions preserve the validity of our theorems. We conclude, therefore, that Theorems 1 and 2 apply to models with these more general routing patterns.

The product form of the equilibrium probability distribution is also retained when constraints on the populations in various subchains are more general than having each chain contain either a constant (closed) or an unlimited (open) number of jobs. Lam [19] studies "loss" and "trigger" functions for each job class based on the number of customers currently in each chain. His work is a generalization of the ideas used by Jackson to unify open and closed networks [15], and by Reiser and Kobayashi to create "semiclosed" subchains in which the number of customers is allowed to vary between minimum and maximum values, but independently of the populations in other subchains [24].

Lam proves a sufficient condition for the equilibrium state probabilities to retain the same product form as for the networks described by Baskett et al. [3]. Let $T_r(S)$ be the "trigger" function whose value is 0 if the departure of a job from the same subchain as r in state S triggers the arrival of a new job in that subchain and is 1 otherwise. Similarly, let $L_r(S)$ be the "loss" function whose value is 0 if arrivals in the subchain of class r in state S are lost and is 1 otherwise. Then Lam's sufficient condition is

$$T_r(S') = L_r(S)$$

for all r and all S' such that $S'_{-r} = S$.

Thus an arrival is triggered to stop a change in subchain population if and only if arrivals that would cause the reverse change are lost. Because systems with loss and trigger functions satisfying the above equation have a product-form solution, Lemma 1 holds for them. However, the expressions for $\eta_r^1(S)$ and $\eta_r^0(S)$ must be modified to include the loss and trigger functions. For example, (10) becomes

$$\eta_r^1(S) = \lambda L_r(S)\pi_C(S)p_{0r} + \sum_{r' \in E} \left\{ \sum_{S' \in \sigma} \pi_C(S')\nu_{r'}(S')[p_{r'r} + p_{r0}(1 - T_{r'}(S'))p_{0r}] \right\}.$$

When $L_r(S) = 1$, $T_{r'}(S') = 1$ because r and r' are in the same subchain; the above equation is then identical to (10). When $L_r(S) = T_{r'}(S') = 0$, it again reduces to (13) after some algebraic manipulation and the use of the identity

$$\sum_{r \in E} \left(1 - \sum_{r' \in E} p_{rr'}\right) e_r = 1.$$

Thus if the loss and trigger functions satisfy Lam's sufficient condition, Theorems 1 and 2 remain applicable.

5. *Conclusions*

We have considered queuing networks with multiple customer classes, Markov routing chains of arbitrary order, state-dependent arrival and service rates, general service-time distributions at service centers with station balancing disciplines, and general constraints on the populations of the routing chains. The presence of even a single feedback loop in such networks causes the input processes at nodes to be non-Poisson and, in general, nonrenewal. Yet we have found that these networks with product-form equilibrium state distributions have input and output instant distributions related simply to the equilibrium distribution of the same network but with the population constraints (if any) on one chain decreased by one customer. Thus, with negligible additional computation, existing algorithms for calculating the equilibrium state probabilities can also calculate the distribution at instants just before inputs or arrivals or just after outputs or departures.

Among networks with equilibrium distributions that do not have product form, the simple relationship between input instant and equilibrium distributions seem to be retained only in very special cases. The M/Er-2/1 FCFS queue with feedback is perhaps the simplest network without product-form equilibrium probabilities. In this case the input-instant and the steady-state distributions are not the same because fed-back customers never see the second service stage busy, whereas the equilibrium utilizations of the two stages are equal. However, if we consider aggregate states n, defined as the number of customers at the service center, then for an M/G/1 FCFS queue with feedback, the distributions of n at input instants and in equilibrium are the same. This can be shown by modifying the approach in [11] so that fed-back customers do not see themselves.

However, in a two-node closed network where the service requirement distribution is highly skewed, the distribution of n input instants is not the same as the equilibrium distribution of n in the same closed network with one less customer. We conjecture that Theorems 1 and 2 fail to hold in any stochastic network where a FCFS nonexponential node contributes to the input of another node.

Appendix

PROOF OF EQUATION (9). Let $\{X(t),\ t > 0\}$ be an arbitrary irreducible and recurrent Markov process with a denumerable state space $\{1, 2, \ldots\}$. Denote by $\pi(i)$ the steady-state probability of state i, by $a(i, j)$ the instantaneous transition rate from state i to state j, and by $\gamma(i, j, t)$ the expected number of transitions from i to j on the interval $(0, t)$, $i, j = 1, 2, \ldots; t > 0$.

The probability $p(i, j)$ that the next transition is to state j given that the last one was to state i is

$$p(i, j) = \frac{a(i, j)}{a(i)},$$

where

$$a(i) = \sum_{j \neq i} a(i, j)$$

is the instantaneous transition rate out of state i (we assume that $a(i) \neq 0$ for all i). Moreover, given that the last transition was to state i, the time to the next transition and the state of the next transition are mutually independent (see [8, Sec. 8.3.3.]). Therefore, given that there are k transitions out of state i on the interval $(0, t)$, the expected number of transitions from i to j is $p(i, j)k$. Hence

$$\gamma(i, j, t) = \gamma(i, t)p(i, j),$$

where $\gamma(i, t)$ is the expected number of transitions out of state i in $(0, t)$. That number satisfies

$$\lim_{t \to \infty} \frac{\gamma(i, t)}{t} = \eta(i),$$

where $1/\eta(i)$ is the expected time between two returns to state i. On the other hand,

$$\eta(i) = \pi(i)a(i)$$

(see [8, Sec. 10.4.5 and 10.5.22]). Combining the above equations yields

$$\lim_{t\to\infty} \frac{\gamma(i, j, t)}{t} = \pi(i)a(i, j). \qquad \square$$

PROOF OF LEMMA 1. Baskett et al. [3] have shown that the stationary distribution $\pi_C(S)$ in network configuration C is given by (we have reorganized the expression to conform with present notation)

$$\pi_C(S) = \frac{1}{G_C} \lambda^{N_0(S)} \prod_{m=1}^{M} q_m(n_m) \prod_{r=1}^{R} [e_r^{k_r} f_r(S_r)], \tag{A1}$$

where

G_C = a normalization constant chosen so that (A1) is a distribution;

$N_0(S)$ = the total number of jobs in the open subchains (this power term changes to a product term when arrivals are state-dependent);

n_m = the number of jobs at node m;

$$q_m(i) = \begin{cases} i! & \text{if node } m \text{ is of type 2,} \\ 1 & \text{otherwise;} \end{cases}$$

$$f_r(S_r) = \begin{cases} (1/\mu_r)^{k_r} & \text{if } r \text{ is associated with a type} = 1 \text{ node;} \\ \prod_{j=1}^{Q_r} [(A_{rj}/\mu_{rj})^{k_{rj}}/k_{rj}!] & \text{if } r \text{ is associated with a type-2 or -3 node;} \\ \prod_{i=1}^{k_r} (A_{rj_i}/\mu_{rj_i}) & \text{if } r \text{ is associated with a type-4 node;} \end{cases}$$

where A_{rj} and μ_{rj} are parameters of the Coxian distribution of class-r service times, the probability of reaching stage j and the rate of completing stage j, respectively (at type-1 nodes there is only one stage); j_i is the stage reached by the ith job in the LCFS ordering.

The network configuration is reflected in (A1) only through the set of feasible states and hence through the normalization constant G_C.

It follows from (A1) that if a state S' differs from the state S by the presence of an additional class-r job (at the head of the queue for type-1 or type-4 nodes), then, depending on the nature of the subchain E to which class r belongs,

$$\begin{aligned} \pi_C(S') &= \pi_C(S)\lambda e_r g(r) && \text{if } E \text{ is open,} \\ \pi_C(S') &= \left(\frac{G_{C-r}}{G_C}\right) \pi_{C-r} e_r g(r) && \text{if } E \text{ is closed,} \end{aligned} \tag{A2}$$

where the factor $g(r)$ depends on the type of node m associated with class r and on the stage j reached by the extra job:

$$g(r) = \begin{cases} \dfrac{1}{\mu_r} & \text{if type 1,} \\ \left(\dfrac{n_m + 1}{k_{rj} + 1}\right)\left(\dfrac{A_{rj}}{\mu_{rj}}\right) & \text{if type 2,} \\ \dfrac{A_{rj}/\mu_{rj}}{k_{rj} + 1} & \text{if type 3,} \\ \dfrac{A_{rj}}{\mu_{rj}} & \text{if type 4.} \end{cases} \tag{A3}$$

The service completion rate for the extra class-r job is given by

$$\nu_r(S') = \begin{cases} \mu_r & \text{if type 1,} \\ \dfrac{\mu_{rj} b_{rj}(k_{rj}+1)}{n_m+1} & \text{if type 2,} \\ \mu_{rj} b_{rj}(k_{rj}+1) & \text{if type 3,} \\ \mu_{rj} b_{rj} & \text{if type 4,} \end{cases} \tag{A4}$$

where b_{rj} is the probability that the job exists after stage j.

Combining (A2) and (A4), and bearing in mind that for all r,

$$\sum_{j=1}^{Q_r} A_{rj} b_{rj} = 1,$$

we find

$$\sum_{S' \in \sigma} \pi_C(S')\nu_r(S') = \begin{cases} \lambda e_r\, \pi_C(S) & \text{if } E \text{ is open,} \\ c e_r \pi_{C-r}(S) & \text{if } E \text{ is closed,} \end{cases}$$

where $c = G_{C-r}/G_C$. This completes the proof of Lemma 1. □

ACKNOWLEDGMENTS. We appreciate the helpful suggestions we have received from P. J. Denning, G. S. Graham, J. Zahorjan, and anonymous referees.

REFERENCES

(Note Reference [13] is not cited in the text.)

1 BARBOUR, A D Networks of queues and the method of stages *Adv Appl. Prob 8*, 3 (Sept 1976), 584–591

2 BARD, Y Some extensions to multiclass queueing network analysis Proc 4th Int Symp Modelling and Performance Evaluation, Vienna, Austria, 1979.

3 BASKETT, F , CHANDY, K M , MUNTZ, R R , AND PALACIOS, F G Open, closed, and mixed networks of queues with different classes of customers *J ACM 22*, 2 (April 1975), 248–260.

4 BURKE, P.J Proof of a conjecture on the interarrival time distribution in an M/M/1 queue with feedback *IEEE Trans. Comput TC 24*, 5 (May 1976), 575–576

5. BUZEN, J P , AND DENNING, P J Operational treatment of queueing distributions and mean value analysis Tech Rep , Computer Science Dep , Purdue Univ., Lafayette, Ind , Aug , 1979

6 CHANDY, K M , HOWARD, J H JR , AND TOWSLEY, D F Product form and local balance in queueing networks *J ACM 24*, 2 (April 1977), 250–263

7. CHANDY, K M In preparation.

8 CINLAR, E. *Introduction to Stochastic Processes* Prentice-Hall, Englewood Cliffs, N.J., 1975

9. COOPER, R B *Introduction to Queueing Theory.* MacMillan Publishing Co , New York, 1972

10 COX, D R A use of complex probabilities in the theory of stochastic processes *Proc. Cambridge Phil Soc. 51* (1955), 313–319

11 DISNEY, R L Random flow in queueing networks· A review and critique *AIIE Trans. 7*, 3 (Sept 1975) 268–288

12 DISNEY, R L , AND MCNICKLE, D C The M/G/1 queue with instantaneous Bernoulli feedback Tech Rep 77-4, Dep of Industrial and Operational Engineering, Univ of Michigan, Ann Arbor, Mich , April 1977.

13 GORDON, W J , AND NEWELL, G F. Closed queueing systems with exponential servers *Oper Res 15* (1967), 254–265

14 JACKSON, J R Networks of waiting lines *Oper Res 5* (1957), 518–521

15 JACKSON, J.R Jobshop-like queueing systems *Manage Sci. 10*, 1 (Oct 1963), 131–142

16 KELLY, F.P Networks of queues with customers of different classes *J Appl Prob 12* (1975), 542–554.

17 KELLY, F.P Networks of queues *Adv Appl. Prob 8*, 2 (June 1976), 416–432

18. KOBAYASHI, H , AND REISER, M On generalization of job routing behavior in a queueing network model Res Rep RC5252, IBM Thomas J. Watson Research Center, Yorktown Heights, N Y , Feb 1975

19 LAM, S.S Queueing networks with population size constraints *IBM J Res Devel 21*, 4 (July 1977), 370–378

20. Lavenberg, S S , and Reiser, M. Stationary state probabilities at arrival instants for closed queueing networks with multiple types of customers. Res Rep. RC7592, IBM Thomas J Watson Research Center, Yorktown Heights, N Y , April, 1979
21 Lemoine, A J Networks of queues—a survey of equilibrium results *Manage Sci 24*, 4 (Dec 1977), 464–481
22 Melamed, B On Markov jump processes imbedded at jump epochs and their queueing-theoretic applications, Dep. of Industrial Engineering and Management Sciences, Northwestern Univ , Nov. 1978
23 Mitrani, I Nonpriority multiprogramming systems under heavy demand conditions—customer's viewpoint *J ACM 19*, 3 (July 1972), 445–452
24 Reiser, M , and Kobayashi, H. Numerical solution of semi-closed exponential server queueing networks. Proc 7th Asilomar Conf , Asilomar, Calif , 1973, pp 308–312.
25 Reiser, M , and Lavenberg, S S Mean-value analysis of closed multichain queuing networks. *J ACM 27*, 2 (April 1980), 313–322
26 Reiser, M Mean-value analysis of queueing networks. In *Proc 4th Int Symp Modelling and Performance Evaluation*, M Arato, E Gelenbe, and A Butrimenko, Eds., North-Holland Publishing, Amsterdam, 1979

RECEIVED MAY 1978, REVISED FEBRUARY 1980, ACCEPTED MARCH 1980