Algorithms II, CS 502

Elementary Graph Algorithms

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Graphs

- A data structure for maintaining relational information
- A graph G=(V,E)
  - V: discrete set of vertices / nodes
  - E: set of edges linking some pairs of vertices

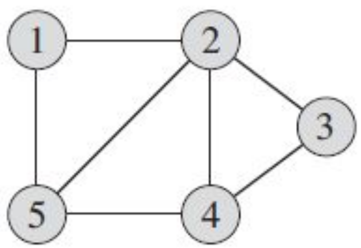
Graphs

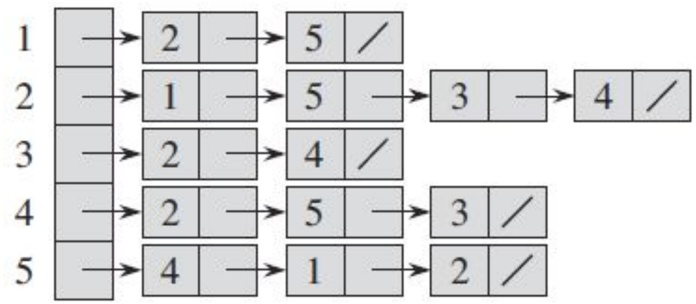
- For a graph G=(V,E),
  - an edge e=(u,v) links / joins vertices u and v
    - edges (hence graphs) may be directed or undirected
  - e is incident upon vertices u and v
    - # of edges incident upon a vertex defines its degree
    - in- and out-degree for directed graphs
  - two edges incident upon a vertex are adjacent
  - u and v are neighboring vertices
  - a path from u to v is an incident sequence of edges without any repetition
    - distance between u and v is the length of a shortest path between u and v

Representation of graphs

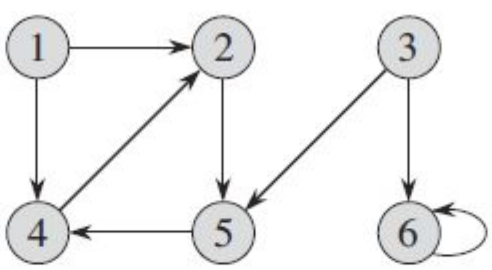
- Adjacency list
  - more popular (will be assumed)
  - much more efficient when $|E| << |V|^2$ (sparse)
  - easy to add weights for edges
  - size is $\theta(|V|+|E|)$
- Adjacency matrix
  - could be preferred when $|E| \approx |V|^2$ (dense)
  - size is $\theta(|V|^2)$
- Access efficiency vs memory requirements
  - to determine whether $(u,v) \in G$ is not $O(1)$ with adjacency lists

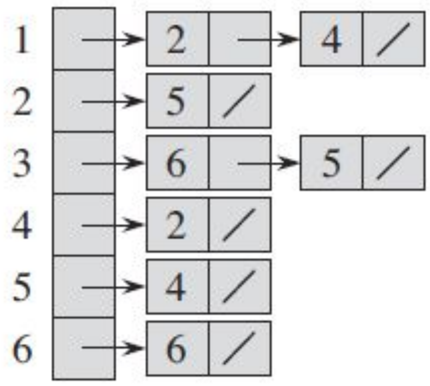
Representation of graphs





	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0





	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

Representing attributes

- Normally need to store per node/edge attributes
  - v.*d* : an attribute *d* of vertex v
  - (u,v).*f* : an attribute *f* of edge (u,v)
  - associating them with graph objects might be tricky
    - use of separate data structures: *d*[1…|V|]
    - instance variables (e.g. of class **`Vertex`**)
    - others?

Breadth-first search

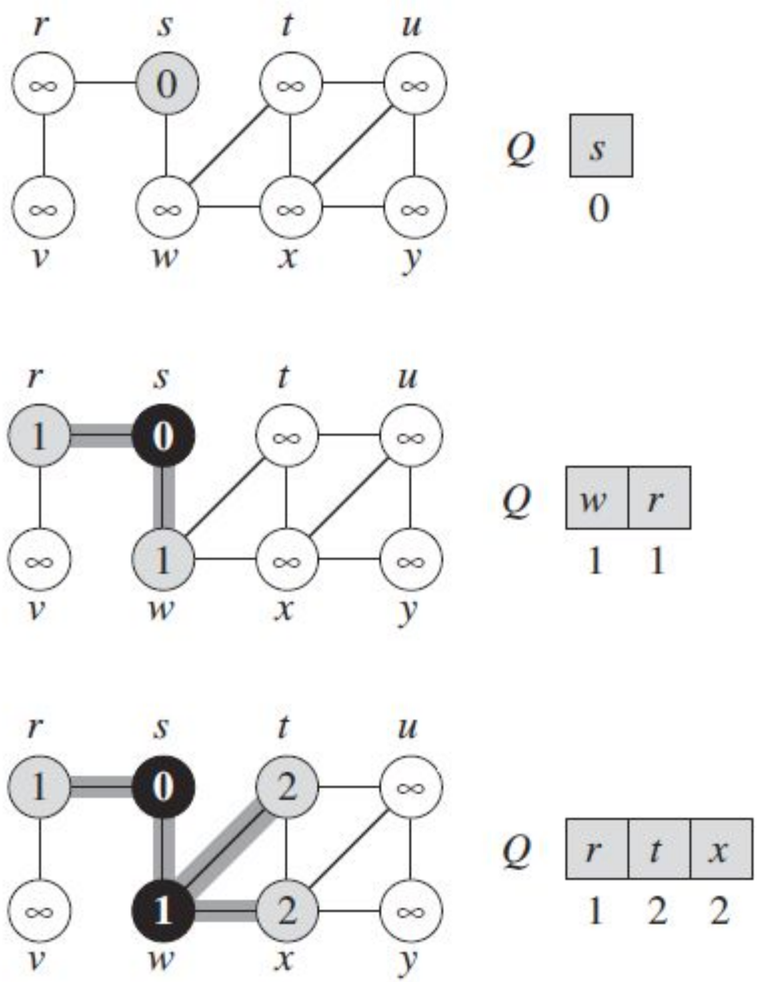
- A simple algorithm to search a graph and basis for many useful graph algorithms
  - Starts from a distinguished source vertex s
  - Systematically explores edges to discover vertices by
    - expanding the frontier between discovered and undiscovered vertices uniformly across breadth of the frontier
    - vertices at distance k from source discovered before those at distance k+1 from source

Breadth-first search

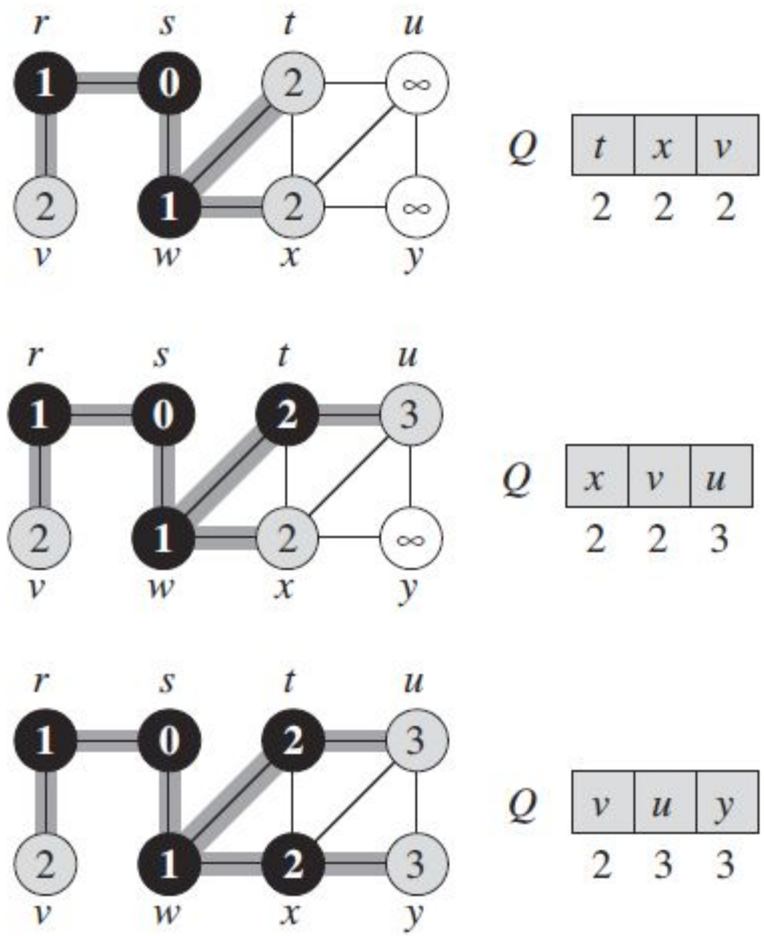
- Assumes adjacency lists
- Has per vertex attributes
  - u.*color* : color of u
    - white, gray, and black
  - u.*π* : predecessor of u
  - u.*d* : distance from source
- Uses a FIFO queue *Q*

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = Ø
ENQUEUE(Q, s)
while Q ≠ Ø
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

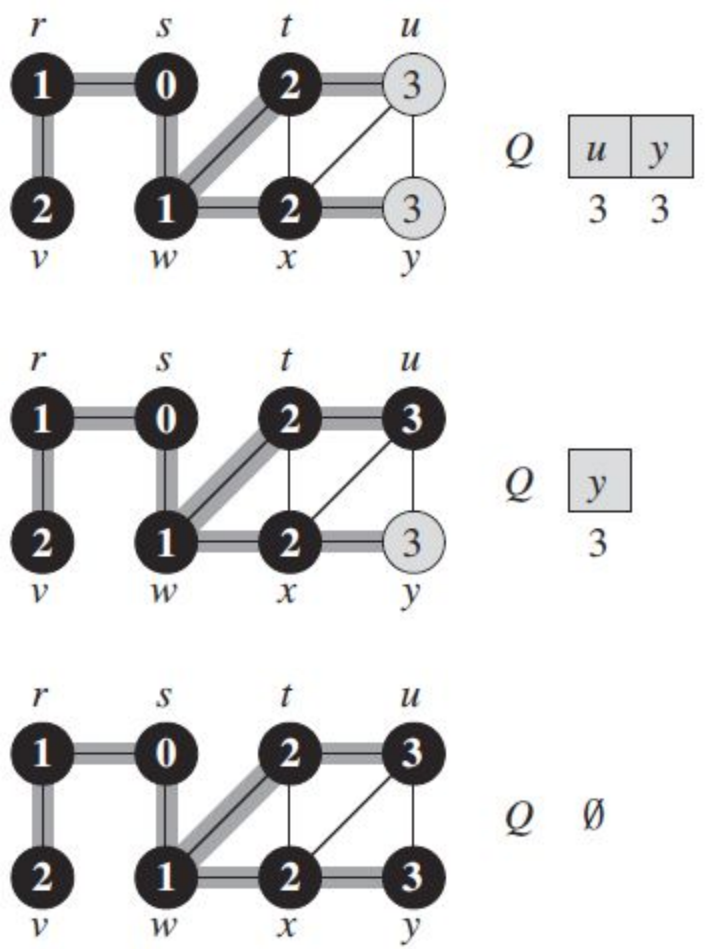
Breadth-first search



Breadth-first search



Breadth-first search



Breadth-first search: analysis

- $O(|V|+|E|)$ since
  - Initialization is $\theta(|V|)$
  - Each vertex enqueued and dequeued only once: $O(|V|)$
  - Each edge visited only once: $\theta(|E|)$

Breadth-first search

- **Lemma 22.1** Let G=(V,E) be directed or undirected graph, and let s ϵ V be an arbitrary vertex. Then, for any edge (u,v) ϵ E,

$$\delta(s,v) \le \delta(s,u) + 1$$

- **Proof** Consider both cases:
  - u is reachable from s,
  - otherwise

Breadth-first search

- **Lemma 22.2** Let G=(V,E) be directed or undirected graph, and suppose that BFS is run on G from a given source vertex s ϵ V. Then, upon termination, for each vertex v ϵ V, the value of v.*d* computed by BFS satisfies

$$v.d \geq \delta(s,v)$$

- **Proof** Use induction on the number of Enqueue operations.
  - **Inductive step**: Consider a white vertex v that is discovered during search from a vertex u
  - v is enqueued only once.

$$\begin{aligned} v.d &= u.d + 1 \\ &\geq \delta(s,u) + 1 \quad \text{(by Inductive Hypotheses)} \\ &\geq \delta(s,v) \quad \text{(by previous Lemma)} \end{aligned}$$

Breadth-first search

- **Lemma 22.3** Suppose that during BFS on a graph G=(V,E), the queue Q contains vertices $<v_1, v_2, \ldots, v_r>$ where v_1 is the head of Q and v_r is the tail. Then

$$v_r.d \le v_1.d + 1 \text{ and } v_i.d \le v_{i+1}.d \text{ for } i = 1,2,\ldots,r-1$$

- **Proof** Use induction on # of queue operations
  - On dequeue

$$v_1.d \le v_2.d \ldots \le v_r.d \quad \text{by the I.H.}$$

$$v_r.d \le v_1.d + 1 \quad \text{by the I.H.}$$

$$\Rightarrow v_r.d \le v_1.d + 1 \le v_2.d + 1$$

$$\Rightarrow v_r.d \le v_2.d + 1 \quad \text{I.S. satisfied for new head}$$

  - Enqueue is similar

Breadth-first search

- **Corollary 22.4** Suppose that vertex v_i is enqueued before vertex v_j during BFS. Then, $v_i.d \le v_j.d$ at the time v_j is enqueued.
- **Proof** Immediate from previous Lemma and the property that each vertex receives a finite *d* value at most once during BFS

Breadth-first search: correctness

- **Theorem 22.5** During execution of BFS on $G=(V,E)$ from source $s \in V$, every vertex $v \in V$ that is reachable from s is discovered, and upon termination, $v.d=\delta(s,v)$ for all $v \in V$. Moreover, for any $v \neq s$ that is reachable from *s*, one of the shortest paths from s to v is a shortest path from s to $v.\pi$ followed by the edge $(v.\pi, v)$.

Breadth-first search: correctness

- **Proof** Let v.$d \neq \delta$(s,v) where δ is minimum
  - v.$d > \delta$(s,v) Lemma 22.2
  - δ(s,v) $\neq \infty$ (v.$d > \infty$ not possible)
  - u is predecessor on a shortest path P from s to v
  - δ(s,u)+1=δ(s,v) $\Rightarrow$ δ(s,u)<δ(s,v) and u.d=δ(s,u) (min)
  - v.$d > \delta$(s,v)= δ(s,u)+1= u.d+1 (Eq. 22.1)
  - At the time u is dequeued from Q, v is:
    - white: line v.d = u.d+1, contradiction
    - black: v already from from Q, v.$d \leq$ u.d (Cor 22.4), contradiction
    - gray: w removed earlier than u from Q:
      - v.d= w.d+1, w.d < u.d (Cor 22.4) $\Rightarrow$ v.$d \leq$ u.d+1, contradiction

Breadth-first search

- **Lemma 22.6** When applied to a directed or undirected graph G=(V,E), procedure BFS constructs π so that predecessor subgraph $G_\pi=(V_\pi,E_\pi)$ is a breadth-first tree.
- **Proof** Apply previous theorem inductively

Breadth-first search

- Print out vertices on a shortest path from s to v (already computed breadth-first tree)

```
PRINT-PATH(G, s, v)
if v == s
    print s
elseif v.π == NIL
    print "no path from" s "to" v "exists"
else PRINT-PATH(G, s, v.π)
    print v
```

- Runs in time linear in the length of the path

Depth-first search

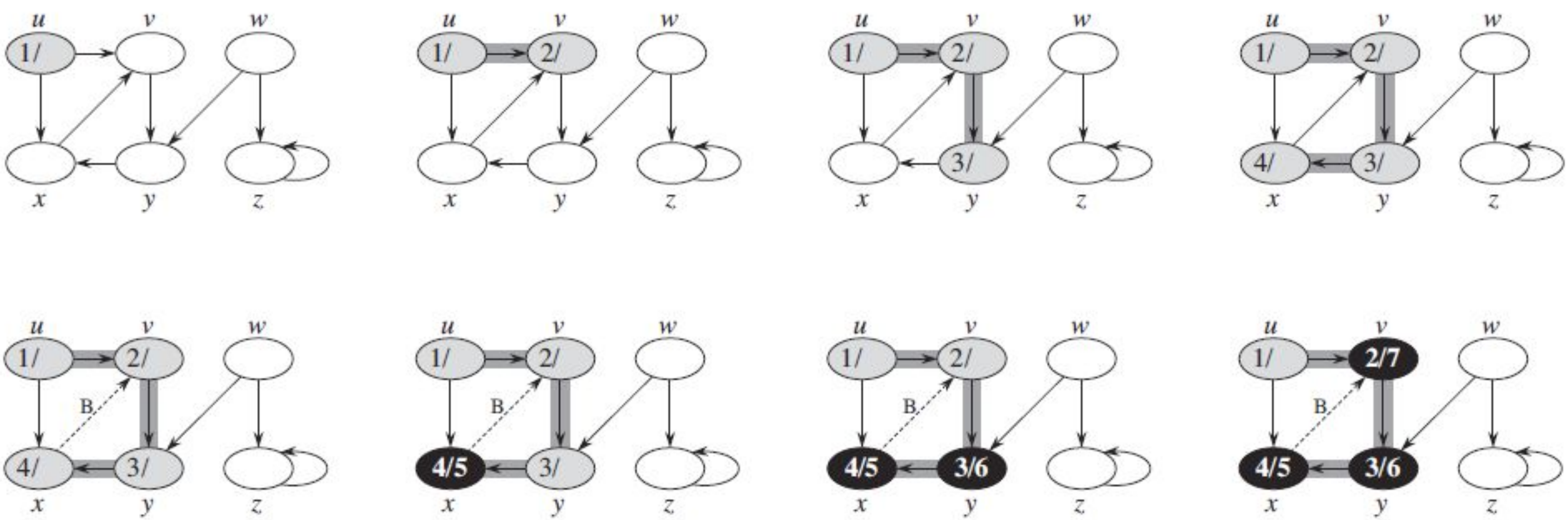
- Search deeper in the graph whenever possible
  - Explore edges out of the most recently discovered vertex v that still has unexplored edges leaving it
  - Once all of v's edges have been explored, backtrack to explore edges leaving the vertex from which v was discovered
  - Predecessor subgraph of DFS forms a depth-first forest
  - Records when it discovers and finishes a vertex u in attributes u.*d* and u.*f*
    - u: white before u.*d*, gray between u.*d* & u.*f*, and black thereafter
    - u.*d* < u.*f* for each vertex u

Depth-first search

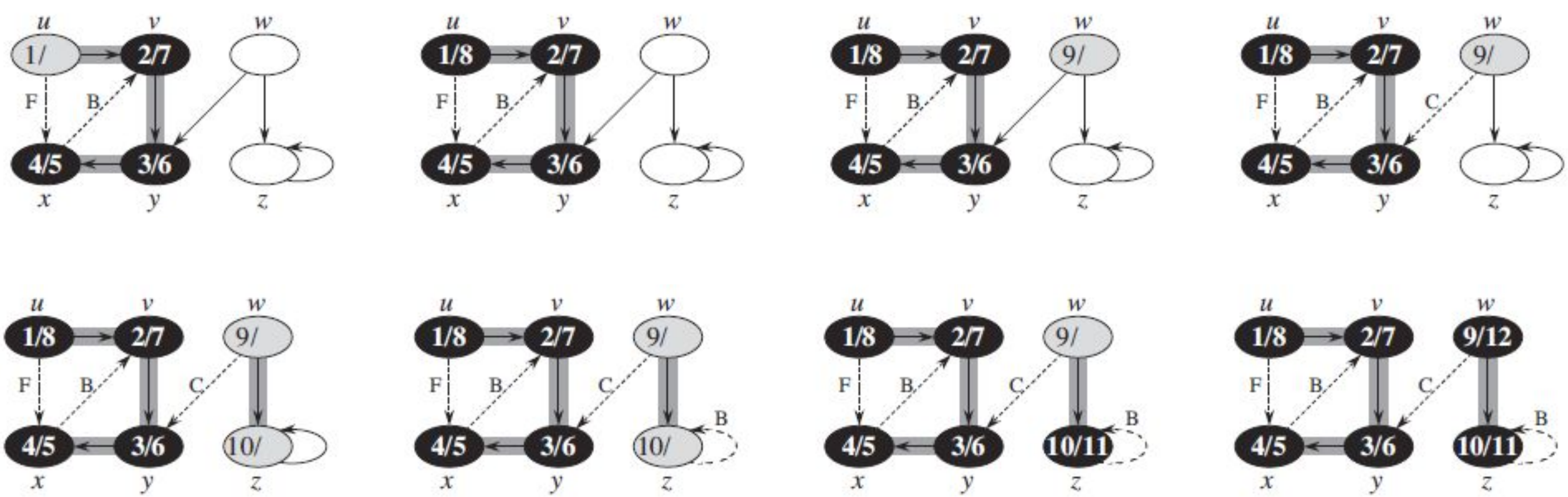
```
DFS(G)
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
time = time + 1              // white vertex u has just been discovered
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]        // explore edge (u, v)
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK              // blacken u; it is finished
time = time + 1
u.f = time
```

Depth-first search



Depth-first search



Depth-first search: analysis

- Depth-first forest mirrors the structure of recursive calls of Dfs-Visit
- O(|V|+|E|) since
  - Dfs-Visit is called exactly once per vertex
  - lines 4-7 executes |Adj[v]| times and $\sum_{v \in V} |Adj[v]| = \Theta(|E|)$

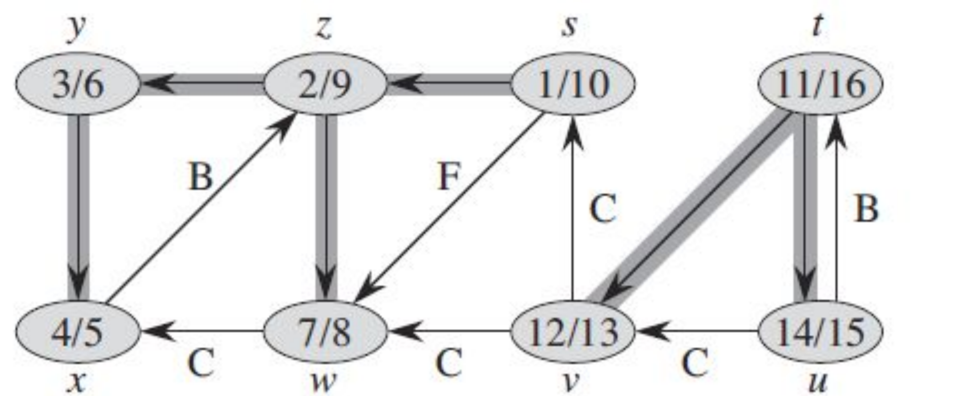
Depth-first search: analysis

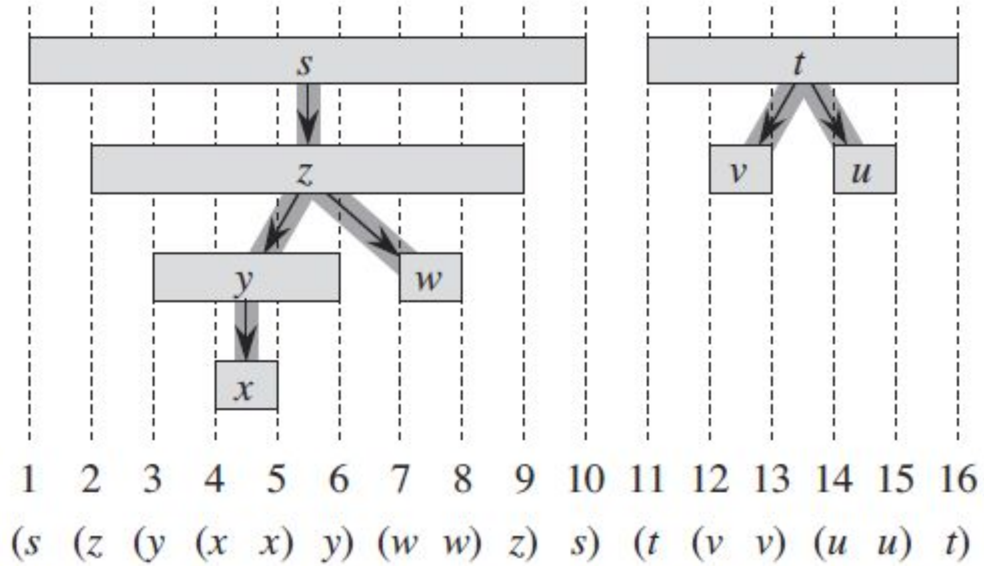
- **Theorem 22.7 (Parenthesis theorem)** In any DFS of a graph G=(V,E), for any two vertices u and v, exactly one of following holds:
  - intervals [u.*d*,u.*f*] and [v.*d*,v.*f*] are entirely disjoint, and neither u nor v is a descendant of the other in the depth-first forest,
  - interval [u.*d*,u.*f*] is contained entirely within interval [v.*d*,v.*f*], and u is a descendant of v in a depth-first tree, or vice versa.

Depth-first search: analysis

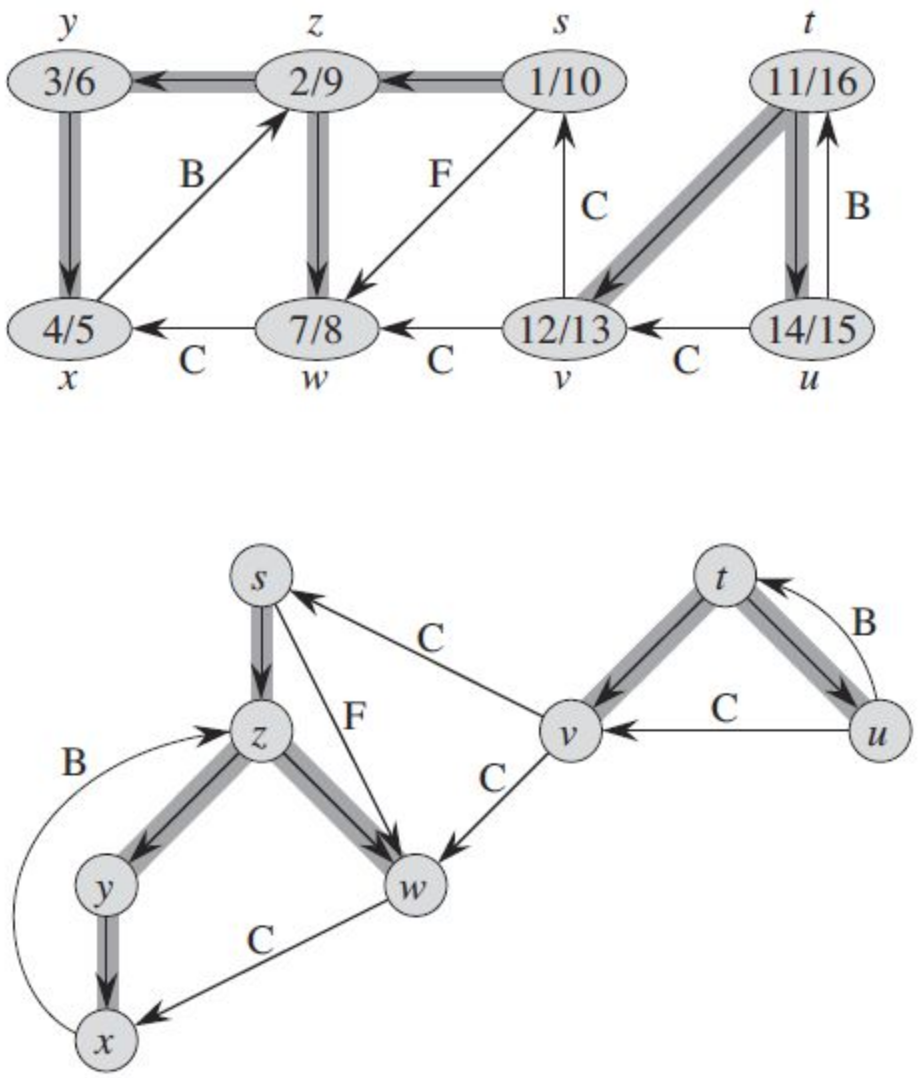
- **Theorem 22.7 (Parenthesis theorem)**
- **Proof**
  - W.l.o.g. suppose u.*d* < v.*d* (< v.*f*). Then we have two cases:
    - v.*d* < u.*f* : v was discovered while u was gray, thus v is a descendant of u, thus v's interval entirely contained within u's
    - u.*f* < v.*d* : means u.*d* < u.*f* < v.*d* < v.*f*, making two intervals disjoint

Depth-first search





Depth-first search



Depth-first search: analysis

- **Corollary 22.8 (Nesting of descendants' intervals)** Vertex v is a proper descendant of vertex u in the depth-first forest for a graph G if and only if u.d < v.d < v.f < u.f.
- **Proof** Follows from Parenthesis theorem

Depth-first search: analysis

- **Theorem 22.9 (White path theorem)** In a depth-first forest of a graph G=(V,E), vertex v is a descendant of vertex u if and only if at the time u.*d* that the search discovers u, there is a path from u to v consisting entirely of white vertices.
- Proof
  - □: if v is a proper descendant of u, then u.*d* < v.*d* and v is white at time u.*d* (by previous Corollary)

Depth-first search: analysis

- **Theorem 22.9 (White path theorem)**
- Proof
  - : Suppose on the white path from u to v, w is a descendant of u but not v. Then, $u.d < v.d$. Also, $w.f \le u.f$ (by Cor. 22.8) and $v.d < w.f$. Hence: $u.d < v.d < w.f \le u.f$.

    By Th. 22.7 then, [v.d,v.f] is completely contained within [u.d,u.f]. Hence, by Cor. 22.8 v is a descendant of u in DFS forest, which is not possible (would form a cycle).

Depth-first search: analysis

1. **Tree edges**: edges (u,v) in depth-first forest; v was first discovered by exploring edge (u,v).
2. **Back edges**: edges (u,v) connecting a vertex u to an ancestor v in a depth-first tree. Self-loops of directed graphs are back edges.
3. **Forward edges**: non-tree edges (u,v) connecting a vertex u to a descendant v in a depth-first tree.
4. **Cross edges**: all other edges; they go between vertices in the same depth-first tree, as long as one vertex is not an ancestor of the other, or they can go between vertices in different depth-first trees.

Depth-first search: analysis

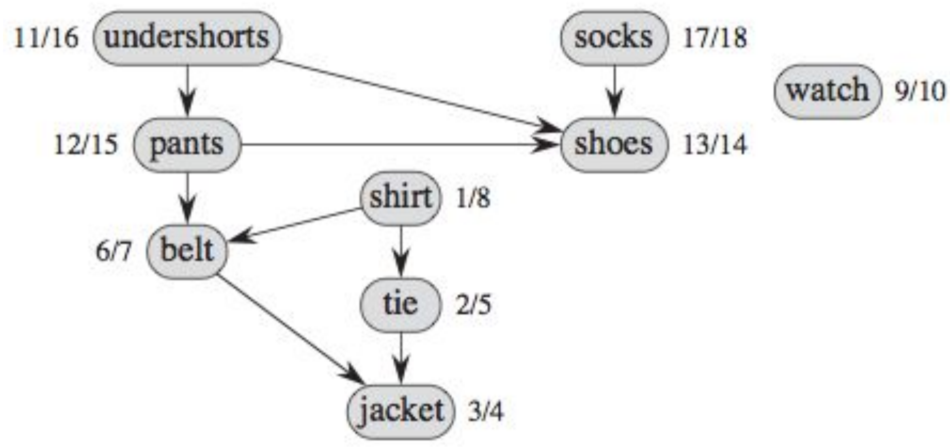
- When we first explore an edge (u,v), the color of vertex v tells us something about the edge:
  - WHITE indicates a tree edge,
  - GRAY indicates a back edge, and
  - BLACK indicates a forward or cross edge. For an edge (u,v):
    - u.d < v.d: forward edge (v's lifetime contained within u's)
    - u.d > v.d: cross edge (u & v's lifetimes are disjoint)

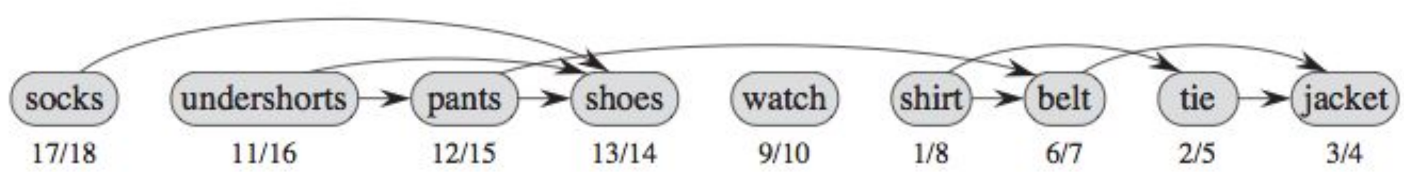
Depth-first search: analysis

- **Theorem 22.10** In a depth-first search of an undirected graph G, every edge of G is either a tree edge or a back edge.
- **Proof** Suppose w.l.o.g. u.*d* < v.*d* for an edge (u,v). Search must discover and finish v before it finishes u (since v is on u's adjacency list)
  - First time (u,v) is explored from u to v: v is undiscovered (white), hence a tree edge
  - First time (u,v) is explored from v to u: u is gray, hence a back edge

Topological sort

- A linear ordering of all vertices of a directed acyclic graph (dag) G=(V,E) such that if (u,v) in V, then u appears before v in the ordering
- Not unique
(partial vs. total order)





Topological sort

- Takes O(V+E) since a straightforward DFS with O(V) (O(1) per vertex) extra processing performed

```
TOPOLOGICAL-SORT(G)
    call DFS(G) to compute finishing times v.f for each vertex v
    as each vertex is finished, insert it onto the front of a linked list
    return the linked list of vertices
```

Topological sort

- **Lemma 22.11** A directed graph G is acyclic if and only if a depth-first search of G yields no back edges
- **Proof**
  - : A back edge (u,v) produced by a DFS implies v is an ancestor of vertex u in the depth-first forest, resulting in a path from v to u, and the back edge (u,v) completes a cycle, contradiction
  - : Suppose G contains a cycle c and let v be the first vertex discovered in c. Let (u,v) be the preceding edge in c. At time v.*d*, the vertices of c form a path of white vertices from v to u. By the white-path theorem, vertex u becomes a descendant of v in the depth-first forest; hence (u,v) is a back edge.

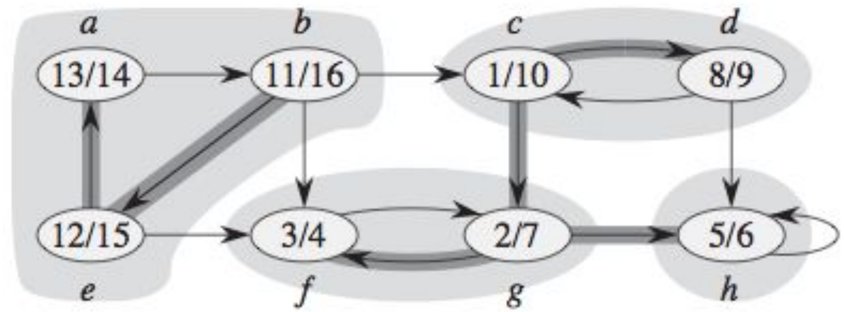
Topological sort

- **Theorem 22.12** Topological-Sort produces a topological sort of the directed acyclic graph provided as its input.
- **Proof** Need to show v.*f* < u.*f* for any edge (u,v) discovered by DFS. v cannot be gray since (u,v) cannot be a back edge (by previous Lemma):
  - v is white: v is a descendant of u, so v.*f* < u.*f*
  - v is black: v has been finished and v.f has been set; still exploring from u, yet to assign a timestamp to u, thus we will have v.*f* < u.*f*

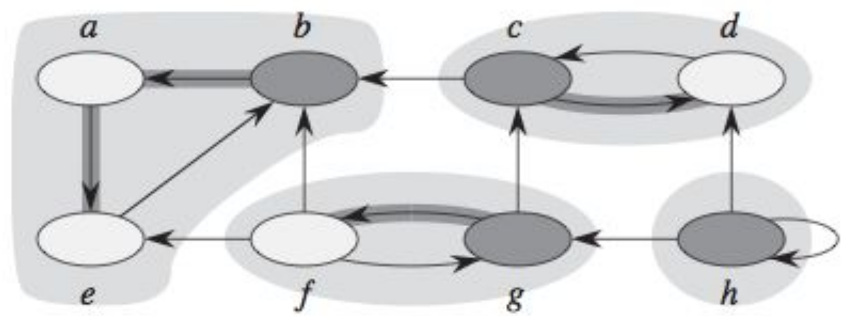
Strongly connected components

- Another application of DFS to decompose a directed graph into strongly connected components, a maximal set of vertices C in V such that for every vertex pair u and v are reachable from each other in C.

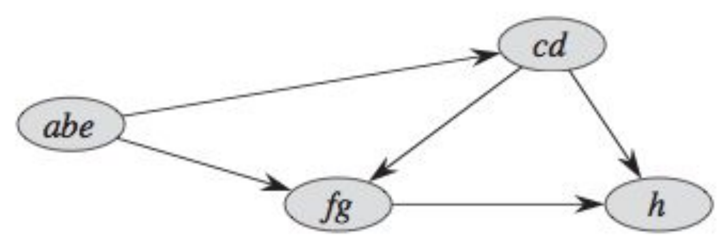
Strongly connected components



The transpose of a graph G is $G^T=(V,E^T)$, where $E^T=\{(u,v) \mid (v,u) \text{ in } E\}$, edges of G with their directions reversed.



Acyclic component graph G^{SCC} obtained by contracting all edges within each strongly connected component of G so that only a single vertex remains in each component.



Strongly connected components

STRONGLY-CONNECTED-COMPONENTS(G)

1 call DFS(G) to compute finishing times $u.f$ for each vertex u
2 compute G^{T}
3 call DFS(G^{T}), but in the main loop of DFS, consider the vertices
  in order of decreasing $u.f$ (as computed in line 1)
4 output the vertices of each tree in the depth-first forest formed in line 3 as a
  separate strongly connected component

Strongly connected components

- **Lemma 22.13** Let C and C' be distinct strongly connected components in directed graph G=(V,E), with u and v in C and u' and v' in C'. Suppose G contains a path u -> u'. Then G cannot also contain a path v' -> v.
- **Proof** If G contains a path v' -> v, then it contains paths u -> u' -> v' and v' -> v -> u. Thus, u and v' are reachable from each other, thereby contradicting the assumption that C and C' are distinct strongly connected components.

Strongly connected components

- **Lemma 22.14** Let C and C' be distinct strongly connected components in directed graph G=(V,E). Suppose that there is an edge (u,v) in E, where u in C and v in C'. Then f(C) > f(C').
- **Proof**
  - d(C) < d(C'): Let x be the first vertex discovered in C. At time x.*d*, all vertices in C and C' are white. At that time, G contains a path from x to each vertex in C consisting only of white vertices. Because (u,v) in E, for any vertex w in C', there is also a path in G at time x.*d* from x to w consisting only of white vertices: x -> u -> v -> w. By the white-path theorem, all vertices in C and C' become descendants of x in the depth-first tree. By previous corollary, x has the latest finishing time of any of its descendants, and so x.f = f(C) > f(C') .

Strongly connected components

- **Proof cntd**
  - d(C) > d(C'): Let y be the first vertex discovered in C'. At time y.*d*, all vertices in C' are white and G contains a path from y to each vertex in C' consisting only of white vertices. By the white-path theorem, all vertices in C' become descendants of y in the depth-first tree, and by previous corollary (nesting of descendants' intervals), y,*f* = f(C'). At time y.*d*, all vertices in C are white. Since there is an edge (u,v) from C to C', Lemma 22.13 implies that there cannot be a path from C' to C. Hence, no vertex in C is reachable from y. At time y.*f* , therefore, all vertices in C are still white. Thus, for any vertex w in C, we have w.*f* > y.*f*, which implies that f(C) > f(C').

Strongly connected components

- **Corollary 22.15** Let C and C' be distinct strongly connected components in directed graph G=(V,E). Suppose that there is an edge (u,v) in E^T, where u in C and v in C'. Then f(C) < f(C').
- **Proof** Since (u,v) in E^T, we have (v,u) in E (the strongly connected components of G and G^T are the same), Lemma 22.14 implies that f (C) < f(C').

Strongly connected components

- **Theorem 22.16** The Strongly-Connected-Components procedure correctly computes the strongly connected components of the directed graph G provided as its input.
- **Proof** Use induction on the number of depth-first trees found in the depth-first search of G^T in line 3:
  - **I.H.**: First k trees produced in line 3 are strongly connected components
  - **Basis**: k=0 is trivial
  - **I.S.**: Consider the $(k+1)^{st}$ tree produced