Question 1: [20 pts] Show that a graph $G$ is connected iff, for any decomposition $V = V_1 \cup V_2$ (and $V_1 \cap V_2 = \emptyset$) of the vertex set of $G$, there exists an edge $e = vw$ such that $v \in V_1$ and $w \in V_2$.

$\Rightarrow$: Trivial.

$\Leftarrow$: Choose some vertex $s$ in a component $V_1$ of $G$. Suppose $V_2 = V \setminus V_1$ is non-empty. Then, there would exist an edge $vw$ with $v \in V_1$ and $w \in V_2$. But then $w$ would be contained in the same component as $s$, creating a contradiction. Therefore, $G$ must be connected.

Question 2: [25 pts] Let $\delta(G) > 0$, and assume $G$ has no induced subgraphs with exactly two edges. Prove that $G$ is complete.

Assume that $G$ is not complete, and let $uv \notin E(G)$. We have easily that $|G| \geq 4$, and thus there are edges $uu_1, vv_1 \in E(G)$, since $\delta(G) > 0$. By the assumption, $u_1 \neq v_1$, and $uv_1, vu_1 \notin E(G)$, and so also $u_1v_1 \notin E(G)$. But now $\{u, u_1, v, v_1\}$ induces a subgraph with exactly two edges. This contradiction shows that $G$ is complete.

Question 3: [25 pts] Show that every cubic 3-edge-connected graph is 3-connected.

By Theorem 3.3.5 of the textbook, every 3-edge-connected graph contains 3 edge-disjoint paths between any pair of vertices. This implies 3 independent paths between arbitrary pair of vertices for cubic graphs (a vertex of degree less than 4 cannot be shared by two edge-disjoint paths). By the same theorem, when every pair of vertices are joined by 3 independent paths the graph is 3-connected.

Question 4: [25 pts] A graph $G$ is critically $k$-chromatic if $\chi(G) = k$ and $\chi(G - x) = k - 1$ for every $x \in V(G)$. Show that if $G$ is a critically $k$-chromatic graph, then

(a) $\delta(G) \geq k - 1$, 

(b) $\chi(G) \leq k$.
Suppose $\delta(G) < k - 1$. Let $v$ be a vertex of degree $\delta(G)$. Since $G$ is critically $k$-chromatic, $G - v$ is $(k - 1)$-colorable. Color the vertices of $G - v$ with $k - 1$ colors, and let $V_1, V_2, \ldots, V_{k-1}$ be the corresponding color classes. Since $d_G(v) = \delta(G) < k - 1$, there must exist a color class $V_i$ with the property that $v$ is non-adjacent with every vertex in $V_i$. Thus, $v$ can be assigned color $i$, producing a $k - 1$ coloring of $G$, and the desired contradiction.

(b) $G$ has no cut-vertices.

Suppose $G$ has a cut-vertex $v$. $G - v$ will have $w$ components $G_1, \ldots, G_w$ with $w \geq 2$. Let $G'_i := G[V(G_i) \cup v]$. Each $G'_i$ is $k - 1$ colorable since $G$ is critically $k$-chromatic and $|G'_i| < |G|$. By permuting colors of $G'_2, \ldots, G'_w$, we can obtain a coloring of all $G'_i$, in which $v$ is assigned the same color, resulting in a $k - 1$ coloring of $G$, which contradicts the fact that $G$ is critically $k$-chromatic graph.

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