CS 570 Graph Theory

Final Exam: Sample Solutions

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This is an in-class examination. You may use your notes and textbook. Clearly show your work, being as formal as possible.

Question 1: *[20pts]* Show that a graph G is connected iff, for any decomposition $V = V_1 \cup V_2$ (and $V_1 \cap V_2 = \emptyset$) of the vertex set of G, there exists an edge $e = vw$ such that $v \in V_1$ and $w \in V_2$.

$\Rightarrow$: Trivial.

$\Leftarrow$: Choose some vertex s in a component V_1 of G. Suppose $V_2 = V \backslash V_1$ is non-empty. Then, there would exist an edge vw with $v \in V_1$ and $w \in V_2$. But then w would be contained in the same component as s, creating a contradiction. Therefore, G must be connected. □

Question 2: *[25pts]* Let $\delta(G) > 0$, and assume G has no induced subgraphs with exactly two edges. Prove that G is complete.

Assume that G is not complete, and let $uv \notin E(G)$. We have easily that $|G| \geq 4$, and thus there are edges uu_1, $vv_1 \in E(G)$, since $\delta(G) > 0$. By the assumption, $u_1 \neq v_1$, and uv_1, $vu_1 \notin E(G)$, and so also $u_1v_1 \notin E(G)$. But now $\{u, u_1, v, v_1\}$ induces a subgraph with exactly two edges. This contradition shows that G is complete.

Question 3: *[25pts]* Show that every cubic 3-edge-connected graph is 3-connected.

By Theorem 3.3.5 of the textbook, every 3-edge-connected graph contains 3 edge-disjoint paths between any pair of vertices. This implies 3 independent paths between arbitrary pair of vertices for cubic graphs (a vertex of degree less than 4 cannot be shared by two edge-disjoint paths). By the same theorem, when every pair of vertices are joined by 3 independent paths the graph is 3-connected. □

Question 4: *[25pts]* A graph G is critically k-chromatic if $\chi(G) = k$ and $\chi(G - x) = k - 1$ for every $x \in V(G)$. Show that if G is a critically k-chromatic graph, then

(a) $\delta(G) \geq k - 1$,

Suppose $\delta(G) < k-1$. Let v be a vertex of degree $\delta(G)$. Since G is critically k-chromatic, $G-v$ is $(k-1)$-colorable. Color the vertices of $G-v$ with $k-1$ colors, and let $V_1, V_2, \ldots, V_{k-1}$ be the corresponding color classes. Since $d_G(v) = \delta(G) < k-1$, there must exist a color class V_i with the property that v is non-adjacent with every vertex in V_i. Thus, v can be assigned color i, producing a $k-1$ coloring of G, and the desired contradiction.

(b) G has no cut-vertices.

Suppose G has a cut-vertex v. $G - v$ will have w components $G_1, \cdots, G_w$ with $w \geq 2$. Let $G'_i := G[V(G_i) \cup v]$. Each G'_i is $k-1$ colorable since G is critically k-chromatic and $|G'_i| < |G|$. By permuting colors of $G'_2, \cdots, G'_w$, we can obtain a coloring of all G'_i, in which v is assigned the same color, resulting in a $k-1$ coloring of G, which contradicts the fact that G is critically k-chromatic graph.

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