Question 1: In the Petersen graph, find:

(a) a trail of length 5,
(b) a path of length 9,
(c) cycles of lengths 5, 6, 8, and 9,
(d) separating sets with 3, 4, and 5 edges.

(a) $a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow j$
(b) $a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow j \rightarrow h \rightarrow f \rightarrow i \rightarrow g$
(c) $a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow a$,
   $a \rightarrow b \rightarrow c \rightarrow d \rightarrow i \rightarrow f \rightarrow a$,
   $a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow j \rightarrow h \rightarrow f \rightarrow a$,
   $a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow j \rightarrow g \rightarrow i \rightarrow f \rightarrow a$
(d) $\{ab, ae, af\}$, $\{ab, af, de, ej\}$, $\{ab, af, de, ej\}$, $\{ab, af, cd, di, ej\}$.

Question 2: Show for a simple graph $G$ that both $G$ and its complement $\overline{G}$ cannot be disconnected.
Suppose one of $G = (V, E)$ and $\overline{G} = (\overline{V}, \overline{E})$ is disconnected; say $G$ with components $G_1, \ldots, G_k$, $k > 1$, w.l.o.g. since $G = \overline{\overline{G}}$. Any two vertices $v \in G_i$ and $w \in G_j$ will be connected in $\overline{G}$ since

(a) if $i \neq j$ then $vw \notin E$, so $vw \in \overline{E}$.

(b) if $i = j$ then there must be a third vertex $u$ in another component such that $vu \notin E$ and $wu \notin E$. In this case, $v$ and $w$ would be connected in $\overline{G}$ through edges $vu, wu \in \overline{E}$.

Since any pair of vertices in $\overline{G}$ are connected, $\overline{G}$ is connected.

Question 3: Show that $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$ for every graph $G$.

Let us prove it in two parts:

(i) There must exist a vertex $u$ whose distance is $\text{rad}(G)$ from a central vertex $v_c$ by definition. Also by definition, the distance between no pair of vertices is greater than $\text{diam}(G)$, yielding $\text{rad}(G) \leq \text{diam}(G)$.

(ii) Let $v_c$ be a central vertex in $G$ and $u$ and $v$ be two vertices with $d_G(u, v) = \text{diam}(G) = |P|$ where $P$ is a shortest path between $u$ and $v$. Also let $P_1$ and $P_2$ be shortest paths from $u$ to $v_c$ and $v_c$ to $v$, respectively; and by definition $|P_1|, |P_2| \leq \text{rad}(G)$.

Clearly $|P| \leq |P_1| + |P_2|$ regardless of $v_c$ being on $P$ or not. Combining this with the statements above, we have $\text{diam}(G) \leq 2\text{rad}(G)$.

Question 4: Show that in any simple graph there are always two vertices with exactly the same degree.

The degrees of vertices in a simple graph $G$ are in the range $[0 \ldots n - 1]$. Suppose all vertices have distinct degrees. Then, the $n$ vertices in the graph must have degrees 0 through $n - 1$. However, a graph can not have two vertices, one not connected to any other and another connected to all others at the same time. By contradiction at least two vertices must have the same degree.

Question 5: Find an example of a graph $G$ where $\kappa(G) = 2$, $\lambda(G) = 3$, and $\delta(G) = 4$, if possible.