CS 570 Graph Theory

Homework 1: Sample Solutions

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Remember that you are to use no sources other than the textbook, lecture notes provided by the instructor or taken by yourselves during the lectures, and the instructor when solving homework and exam questions unless stated otherwise.

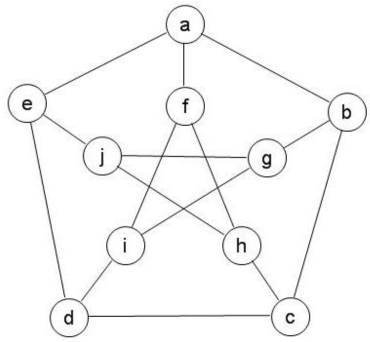
Question 1: In the Petersen graph, find:

(a) a trail of length 5,

(b) a path of length 9,

(c) cycles of lengths 5, 6, 8, and 9,

(d) separating sets with 3, 4, and 5 edges.



(a) $a \to b \to c \to d \to e \to j$;

(b) $a \to b \to c \to d \to e \to j \to h \to f \to i \to g$;

(c) $a \to b \to c \to d \to e \to a$,
$a \to b \to c \to d \to i \to f \to a$,
$a \to b \to c \to d \to e \to j \to h \to f \to a$,
$a \to b \to c \to d \to e \to j \to g \to i \to f \to a$;

(d) $\{ab, ae, af\}$, $\{ab, af, de, ej\}$, $\{ab, af, de, ej\}$, $\{ab, af, cd, di, ej\}$.

Question 2: Show for a simple graph G that both G and its complement $\overline{G}$ cannot be disconnected.

Suppose one of $G = (V, E)$ and $\overline{G} = (\overline{V}, \overline{E})$ is disconnected; say G with components $G_1, \ldots, G_k$, $k > 1$, w.l.o.g since $G = \overline{\overline{G}}$. Any two vertices $v \in G_i$ and $w \in G_j$ will be connected in $\overline{G}$ since

(a) if $i \neq j$ then $vw \notin E$, so $vw \in \overline{E}$.

(b) if $i = j$ then there must be a third vertex u in another component such that $vu \notin E$ and $wu \notin E$. In this case, v and w would be connected in $\overline{G}$ through edges $vu, wu \in \overline{E}$.

Since any pair of vertices in $\overline{G}$ are connected, $\overline{G}$ is connected. □

Question 3: Show that $rad(G) \leq diam(G) \leq 2rad(G)$ for every graph G.

Let us prove it in two parts:

(i) There must exist a vertex u whose distance is $rad(G)$ from a central vertex v_c by definition. Also by definition, the distance between no pair of vertices is greater than $diam(G)$, yielding $rad(G) \leq diam(G)$.

(ii) Let v_c be a central vertex in G and u and v be two vertices with

$$d_G(u, v) = diam(G) = |P|$$

where P is a shortest path between u and v. Also let P_1 and P_2 be shortest paths from u to v_c and v_c to v, respectively; and by definition

$$|P_1|, |P_2| \leq rad(G).$$

Clearly $|P| \leq |P_1| + |P_2|$ regardless of v_c being on P or not. Combining this with the statements above, we have

$$diam(G) \leq 2rad(G).$$

□

Question 4: Show that in any simple graph there are always two vertices with exactly the same degree.

The degrees of vertices in a simple graph G are in the range $[0 \ldots n-1]$. Suppose all vertices have distinct degrees. Then, the n vertices in the graph must have degrees 0 through $n-1$. However, a graph can not have two vertices, one not connected to any other and another connected to all others at the same time. By contradiction at least two vertices must have the same degree. □

Question 5: Find an example of a graph G where $\kappa(G) = 2$, $\lambda(G) = 3$, and $\delta(G) = 4$, if possible.