CS 570 Graph Theory

Homework 2: Sample Solutions

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Remember that you are to use no sources other than the textbook, lecture notes provided by the instructor or taken by yourselves during the lectures, and the instructor when solving homework and exam questions unless stated otherwise.

Question 1: Let G be a 2-connected graph but not a triangle, and let e be an edge of G. Show that either $G - e$ or G/e is again 2-connected.

Let $e = xy$ be an arbitrary edge of a 2-connected graph G which is not a triangle. If $G - e$ is 2-connected, we have no problem. Otherwise, x and y must be connected by only one path P_{xy} in $G - e$, hence by exactly two paths, P_{xy} and the trivial $\{xy\}$ path in G. Now we need to show that G/e is 2-connected when $G - e$ is not 2-connected. Now take any two vertices u and v in G/e.
If $v_{xy} = u \neq v$ (or $v_{xy} = v \neq u$), then since G is 2-connected, there must be at least two paths, say P_1 and P_2, between x and v in G, which remain intact and independent in G/e, forming two independent paths between u and v.
If $v_{xy} \neq u, v$, then there are three cases for the two independent paths P_1 and P_2 of G in regards to xy:

(i) xy is edge and vertex-disjoint with both P_1 and P_2. In this case, these paths remain intact and independent in G/e.

(ii) xy is edge-disjoint but not vertex-disjoint. We have two subcases here:

  (a) Exactly one of x and y is on P_1 or P_2. Again, these two paths remain intact and independent in G/e.

  (b) Both x and y are on P_1 and P_2, respectively. This is not possible since then there would be more than two paths between x and y in G.

(iii) xy is on (exactly) one of P_1 or P_2. The two paths remain intact and independent in G/e.

Since there are at least two independent paths in between arbitrary vertices u and v in G/e, G/e must be 2-connected. □

Question 2: Prove that any graph G with $\delta(G) \geq 1$ (i.e., without an isolated vertex) and with an Euler tour/circuit is connected.

Suppose G is disconnected with components G_1 through G_k, $k \geq 2$. Note that each such component will have at least one edge since G doesn't contain any isolated vertices. Let G_i, $1 \leq i \leq k$, be the component where an Euler tour starts. Since an Euler tour must visit all the edges in G and G has at least one component G_j, $j \neq i$, with at least one edge, the edges in G_j will never be reached by this tour; this creates a contradiction to the definition of an Euler tour. Thus G must be connected. $\square$

Question 3: Let G be 3-regular graph. Show that G has an even cycle.

Consider a maximal path $P = u_0\, u_1 \dots u_l$. By maximality, all three neighbours of u_0 are on this path. Say, $u_0u_i, u_0u_j \in E(G)$ for $1 < i < j$. If the cycles $u_0 \dots u_i \dots u_0$ and $u_0 \dots u_j \dots u_0$ have odd length, then the cycle $u_0\, u_i \dots u_j\, u_0$ is even, since the paths $u_0 \dots u_i$ and $u_0 \dots u_j$ have even lengths (and hence so does $u_i \dots u_j$). $\square$

Question 4: A graph G is called *self-complementary*, if it is isomorphic with its complement.

(a) Show that if G is self-complementary, then $|G| \equiv 0$ or $1 \bmod 4$.

(b) Show that the paths P^0 and P^3 are the only self-complementary paths.

(c) Determine the values of k for which C^k is self-complementary.

(a) Let $n = |G|$. When G and $\overline{G}$ are superimposed, we obtain K^n; that is, $E(K^n) = E(G) \cup E(\overline{G})$. Therefore $||G|| + ||\overline{G}|| = n(n-1)/2$. Suppose that G and $\overline{G}$ are isomorphic. Then $||G|| = ||\overline{G}||$ and so $n(n-1) = 4 \cdot ||G|| \equiv 0 \bmod 4$. It follows that $n \equiv 0$ or $1 \bmod 4$.

(b) P^3 is the only self-complementary path other than the trivial path P^0, since P_1, P_2 are not, and for $n \geq 4$, P^n has a leaf v, and hence $d_{\overline{G}}(v) = n - 1 \geq 3$.

(c) C^5 is self-complementary. It is the only one, since C^3 and C^4 are not self-complementary, and for $n \geq 6$, $d_{\overline{C^n}}(v) > 2$, but $d_{C^n}(v) = 2$ for all v.

Question 5: Find an infinite counterexample to the statement of the marriage theorem.

Let $G = (V, E)$ be an infinite graph with bipartitions A and B where $A = \{a_0, a_1, a_2, \dots\}$ and $B = \{b_0, b_1, b_2, \dots\}$. Also suppose $E = \{a_ib_j \mid i = 0$ or $i = j+1, j \geq 0\}$. This bipartite graph satisfies the marriage condition as the number of neighbors of any vertex set A' in A will be at least $|A'|$. However, it does not contain a matching of A as either a_0 will remain unmatched or the only neighbor in A of the vertex matched to a_0 will remain unmatched.