Question 1: Let $G$ be a 2-connected graph but not a triangle, and let $e$ be an edge of $G$. Show that either $G - e$ or $G/e$ is again 2-connected.

Let $e = xy$ be an arbitrary edge of a 2-connected graph $G$ which is not a triangle. If $G - e$ is 2-connected, we have no problem. Otherwise, $x$ and $y$ must be connected by only one path $P_{xy}$ in $G - e$, hence by exactly two paths, $P_{xy}$ and the trivial $\{xy\}$ path in $G$. Now we need to show that $G/e$ is 2-connected when $G - e$ is not 2-connected. Now take any two vertices $u$ and $v$ in $G/e$.

If $v_{xy} = u \neq v$ (or $v_{xy} = v \neq u$), then since $G$ is 2-connected, there must be at least two paths, say $P_1$ and $P_2$, between $x$ and $v$ in $G$, which remain intact and independent in $G/e$, forming two independent paths between $u$ and $v$.

If $v_{xy} \neq u, v$, then there are three cases for the two independent paths $P_1$ and $P_2$ of $G$ in regards to $xy$:

(i) $xy$ is edge and vertex-disjoint with both $P_1$ and $P_2$. In this case, these paths remain intact and independent in $G/e$.

(ii) $xy$ is edge-disjoint but not vertex-disjoint. We have two subcases here:

(a) Exactly one of $x$ and $y$ is on $P_1$ or $P_2$. Again, these two paths remain intact and independent in $G/e$.

(b) Both $x$ and $y$ are on $P_1$ and $P_2$, respectively. This is not possible since then there would be more than two paths between $x$ and $y$ in $G$.

(iii) $xy$ is on (exactly) one of $P_1$ or $P_2$. The two paths remain intact and independent in $G/e$.

Since there are at least two independent paths in between arbitrary vertices $u$ and $v$ in $G/e$, $G/e$ must be 2-connected.

Question 2: Prove that any graph $G$ with $\delta(G) \geq 1$ (i.e., without an isolated vertex) and with an Euler tour/circuit is connected.
Suppose $G$ is disconnected with components $G_1$ through $G_k$, $k \geq 2$. Note that each such component will have at least one edge since $G$ doesn’t contain any isolated vertices. Let $G_i$, $1 \leq i \leq k$, be the component where an Euler tour starts. Since an Euler tour must visit all the edges in $G$ and $G$ has at least one component $G_j$, $j \neq i$, with at least one edge, the edges in $G_j$ will never be reached by this tour; this creates a contradiction to the definition of an Euler tour. Thus $G$ must be connected.

Question 3: Let $G$ be a 3-regular graph. Show that $G$ has an even cycle.

Consider a maximal path $P = u_0 u_1 \ldots u_l$. By maximality, all three neighbours of $u_0$ are on this path. Say, $u_0 u_i, u_0 u_j \in E(G)$ for $1 < i < j$. If the cycles $u_0 \ldots u_i \ldots u_0$ and $u_0 \ldots u_j \ldots u_0$ have odd length, then the cycle $u_0 \ldots u_i \ldots u_j \ldots u_0$ is even, since the paths $u_0 \ldots u_i$ and $u_0 \ldots u_j$ have even lengths (and hence so does $u_i \ldots u_j$).

Question 4: A graph $G$ is called self-complementary, if it is isomorphic with its complement.

(a) Show that if $G$ is self-complementary, then $|G| \equiv 0$ or $1 \mod 4$.

(b) Show that the paths $P^0$ and $P^3$ are the only self-complementary paths.

(c) Determine the values of $k$ for which $C^k$ is self-complementary.

(a) Let $n = |G|$. When $G$ and $\overline{G}$ are superimposed, we obtain $K^n$; that is, $E(K^n) = E(G) \cup E(\overline{G})$. Therefore $||G|| + ||\overline{G}|| = n(n - 1)/2$. Suppose that $G$ and $\overline{G}$ are isomorphic. Then $||G|| = ||\overline{G}||$ and so $n(n - 1) = 4 \cdot ||G|| \equiv 0 \mod 4$. It follows that $n \equiv 0$ or $1 \mod 4$.

(b) $P^3$ is the only self-complementary path other than the trivial path $P^0$, since $P_1$, $P_2$ are not, and for $n \geq 4$, $P^n$ has a leaf $v$, and hence $d_{\overline{G}}(v) = n - 1 \geq 3$.

(c) $C^5$ is self-complementary. It is the only one, since $C^5$ and $C^4$ are not self-complementary, and for $n \geq 6$, $d_{\overline{G}}(v) > 2$, but $d_{C^n}(v) = 2$ for all $v$.

Question 5: Find an infinite counterexample to the statement of the marriage theorem.

Let $G = (V, E)$ be an infinite graph with bipartitions $A$ and $B$ where $A = \{a_0, a_1, a_2, \ldots\}$ and $B = \{b_0, b_1, b_2, \ldots\}$. Also suppose $E = \{a_ib_j \mid i = 0 \text{ or } i = j + 1, j \geq 0\}$. This bipartite graph satisfies the marriage condition as the number of neighbors of any vertex set $A'$ in $A$ will be at least $|A'|$. However, it does not contain a matching of $A$ as either $a_0$ will remain unmatched or the only neighbor in $A$ of the vertex matched to $a_0$ will remain unmatched.