CS 570 Graph Theory

Homework 3: Sample Solutions

Lecturer: Uğur Doğrusöz

CS Dept., Bilkent University

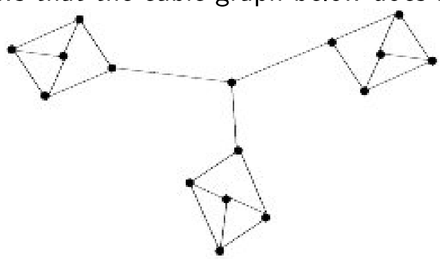
Remember that you are to use no sources other than the textbook, lecture notes provided by the instructor or taken by yourselves during the lectures, and the instructor when solving homework and exam questions unless stated otherwise.

Question 1: Let G be a connected graph that has no induced subgraphs P^3 nor C^4. Show that G has a vertex of degree $|G| - 1$.

Suppose $\Delta(G) < |G| - 1$. Let v be a vertex of degree $\Delta(G)$. Then there exists a vertex $w \notin N_G(v)$, but $wu \in E(G)$ for some $u \in N_G(v)$ (since G is connected). Let $x \in N_G(v)$ be such that $x \neq u$ and $x \notin N_G(u)$. Such a vertex exists, because, otherwise, $d_G(u) > d_G(v)$. The subgraph induced by $\{v, u, w, x\}$ has a path $w \to u \to v \to x$ such that $wv, ux \notin E(G)$. Hence, to avoid a P^3, we have to have $wx \in E(G)$. But then the subgraph is an induced C^4. This contradiction proves the claim.

Question 2: Find a cubic graph of order no more than 16 without a 1-factor.

A simple case argument shows that the cubic graph below does not have a 1-factor.



Question 3: Show that any graph G of order $2n$ with $\delta(G) \geq n$ has a perfect matching.

Let M be a maximum matching of $G = (V, E)$, and suppose that M is not perferct. Hence there are at least two unmatched vertices, say x and y. By maximality, $xy \notin E$.
If there exists an edge $uv \in M$ such that $xu, yv \in E$, then $(M \backslash \{uv\}) \cup \{xu, yv\}$ is a larger matching than M; a contradiction.
Therefore, for all $uv \in M$, at most two of the pairs xu, xv, yu, yv are in E.
Hence the number of edges from the vertices of M to $\{x, y\}$ is less than $2n \leq d_G(x) + d_G(y)$.
Consequently, there exists an edge $xz \in E$ or $yz \in E$ with z unmatched in M. This contradicts our assumption. □

The result may also be proven using Theorem 2.2.1 (show that $q(G-S) \leq |S|$ for all $S \subseteq V(G)$) or Theorem 2.2.3 (show that $|S| = |\mathcal{C}_{G-S}|$ for a vertex set S satisfying the two properties stated in the theorem).

Question 4: Show that any acyclic graph on n vertices has at most $n-1$ edges using induction on n.

- **Base:** For $n = 1$, we have a graph with a single isolated vertex, which has 0 edges.
- **Inductive Hypothesis:** Assume that any acyclic graph G with $n \leq k$ vertices, contains at most $n-1$ edges.
- **Inductive Step:** We need to prove that any acyclic graph G with $n = k+1$ vertices contains at most k edges. Let $G' = G - e$, where e is an arbitrary edge of G. Since G is acyclic, e must be a bridge, and G' will be disconnected with components G'_1 and G'_2. Since each component G'_i, $i = 1, 2$, contains at most k vertices, by the Inductive Hypothesis, $||G'_i|| \leq k_i - 1$. Then, $||G|| - 1 = ||G'|| = ||G'_1|| + ||G'_2|| \leq k + 1 - 2$. Thus, $||G|| \leq k$. □

Question 5: Let G be a 3-connected graph, and let xy be an edge of G. Show that G/xy is 3-connected iff $G - \{x, y\}$ is 2-connected.

$\Rightarrow$: Given a 3-connected graph G with $xy \in E(G)$ and G/xy is 3-connected, assume $G - \{x, y\}$ is not 2-connected. Then a vertex z in $G - \{x, y\}$ separates it. This implies that the set $\{v_{xy}, z\}$ separates G/xy, which is a contradiction. □

$\Leftarrow$: Given a 3-connected graph G with $xy \in E(G)$ and $G - \{x, y\}$ as 2-connected, assume G/xy is not 3-connected. Then at least one vertex pair $\{u, v\}$ separates G/xy. Either u or v must be v_{xy} since otherwise $\{x, y\}$ would separate G, which is not possible (G is 3-connected). Suppose $v = v_{x,y}$. Then u separates $G - \{x, y\}$, which is a contradiction. □