Question 1: Let \( t_1, t_2, t_3 \) be vertices of a tree \( T \). Prove that there is a unique vertex \( t \) of \( T \) such that for every \( i, j = 1, 2, 3 \) with \( i \neq j \) the vertex \( t \) lies on the unique path between \( t_i \) and \( t_j \) in \( T \).

Use \( P_{12} \) to denote the path from \( t_1 \) to \( t_2 \). Then let \( P_{12} \cap P_{13} = P_a \), \( P_{12} \cap P_{23} = P_b \). If \( a \neq b \), then \( a \to t_3 \to b \to a \) must be a closed trail, which contains a cycle, contradicting with the fact that \( T \) is a tree. So \( a = b = t \).

Question 2: Show that a \( k \)-connected graph with at least \( 2k \) vertices contains a matching of size \( k \). Is this best possible? Hint: Use Theorem 2.2.3 of the textbook.

Let \( G \) be a \( k \)-connected graph with at least \( 2k \) vertices. \( G \) contains a vertex set \( S \) satisfying the two properties given in Theorem 2.2.3; thus, \( S \) is matchable to \( G - S \). Then we have the following possible cases:

- \( |S| = 1 \): Then \( G - S \) will be connected. Suppose the single vertex \( v \in S \) is matched to \( w \) of \( G - S \) (using Theorem 2.2.3). Also since \( G - S \) is factor critical by Theorem 2.2.3, \( G - S - w \) contains a 1-factor of size at least \( \frac{2k-2}{2} = k - 1 \) ((\( G - S - w \) \geq 2k - 2)), combined with edge \( vw \), we have a matching of size at least \( k \) in \( G \).

- \( 2 \leq |S| \leq k - 1 \): This case is not possible since separation of less than \( k \) vertices will leave \( G - S \) connected, making it impossible to satisfy one of the properties of Theorem 2.2.3: \( S \) is matchable to \( G - S \).

- \( |S| \geq k \): This gives us a matching of size of at least \( k \) by Theorem 2.2.3.

This is the most we can guarantee but there sure are some graphs with a larger matching.

Question 3: Prove that any two edges of a 2-connected graph lie on a common cycle.

Let \( e_1 = x_1y_1 \) and \( e_2 = x_2y_2 \) be two arbitrary edges of a 2-connected graph \( G \). By Menger’s theorem (Theorem 3.3.1 of the textbook with \( A = \{x_1, y_1\} \) and \( B = \{x_2, y_2\} \)), we have 2 independent \( A - B \) paths, which, together with edges \( e_1 \) and \( e_2 \), form a cycle.

Question 4: Assume that both \( G \) and \( \overline{G} \) are connected. Show that \( G \) contains an induced
We use induction on $|G| = n$. For $|G| \leq 4$, the claim holds. Let then $|G| > 4$, and let $v \in G$ be a chosen vertex.

There are vertices $u, u'$ such that $vu', uu' \in E(G)$, but $vu \notin E(G)$. Indeed, since $\overline{G}$ is connected, there is $u_0$ such that $vu_0 \notin E(G)$. Let $u_0 u_1 \ldots u_k v$ be a shortest path from $u_0$ to $v$ in $G$. Then $u = u_{k-1}$ and $u' = u_k$ will do.

If $G - v$ is disconnected, then $G$ contains an induced $P^3$. For, otherwise, let $x$ be in a different component of $G - v$ than $u$ and $u'$, and let $x \ldots w v$ be a shortest path in $G$ from $x$ to $v$. Then $u, u', v, w$ form an induced $P^3$.

Similarly, when we replace $G$ by $\overline{G}$ in the above, we have that $\overline{G} - v = \overline{G} - v$ is connected, or $\overline{G}$ (and thus $G$) has an induced $P^3$.

On the other hand, if both $G - v$ and $\overline{G} - v$ are connected, then the induction hypothesis gives that $G - v$ and thus $G$ has an induced $P^3.$