CS 570 GRAPH THEORY

Homework 4: Sample Solutions

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Remember that you are to use no sources other than the textbook, lecture notes provided by the instructor or taken by yourselves during the lectures, and the instructor when solving homework and exam questions unless stated otherwise.

Question 1: Let t_1, t_2, t_3 be vertices of a tree T. Prove that there is a unique vertex t of T such that for every $i, j = 1, 2, 3$ with $i \neq j$ the vertex t lies on the unique path between t_i and t_j in T.

Use P_{12} to denote the path from t_1 to t_2. Then let $P_{12} \cap P_{13} = P_{1a}$, $P_{12} \cap P_{23} = P_{2b}$. If $a \neq b$, then $a \to t_3 \to b \to a$ must be a closed trail, which contains a cycle, contradicting with the fact that T is a tree. So $a = b = t$.

Question 2: Show that a k-connected graph with at least $2k$ vertices contains a matching of size k. Is this best possible? *Hint:* Use Theorem 2.2.3 of the textbook.

Let G be a k-connected graph with at least $2k$ vertices. G contains a vertex set S satisfying the two properties given in Theorem 2.2.3; thus, S is matchable to $G - S$. Then we have the following possible cases:

- $|S| = 1$: Then $G - S$ will be connected. Suppose the single vertex $v \in S$ is matched to w of $G - S$ (using Theorem 2.2.3). Also since $G - S$ is factor critical by Theorem 2.2.3, $G - S - w$ contains a 1-factor of size at least $\frac{2k-2}{2} = k - 1$ ($|G - S - w| \geq 2k - 2$), combined with edge vw, we have a matching of size at least k in G.
- $2 \leq |S| \leq k - 1$: This case is not possible since separation of less than k vertices will leave $G - S$ connected, making it impossible to satisfy one of the properties of Theorem 2.2.3: S is matchable to $G - S$.
- $|S| \geq k$: This gives us a matching of size of at least k by Theorem 2.2.3. □

This is the most we can guarentee but there sure are some graphs with a larger matching.

Question 3: Prove that any two edges of a 2-connected graph lie on a common cycle.

Let $e_1 = x_1 y_1$ and $e_2 = x_2 y_2$ be two arbitrary edges of a 2-connected graph G. By Menger's theorem (Theorem 3.3.1 of the textbook with $A = \{x_1, y_1\}$ and $B = \{x_2, y_2\}$), we have 2 independent $A - B$ paths, which, together with edges e_1 and e_2, form a cycle. □

Question 4: Assume that both G and $\overline{G}$ are connected. Show that G contains an induced subgraph P^3.

We use induction on $|G| = n$. For $|G| \leq 4$, the claim holds. Let then $|G| > 4$, and let $v \in G$ be a chosen vertex.
There are vertices u, u' such that $vu', uu' \in E(G)$, but $vu \notin E(G)$. Indeed, since $\overline{G}$ is connected, there is u_0 such that $vu_0 \notin E(G)$. Let $u_0\ u_1 \ldots u_k\ v$ be a shortest path from u_0 to v in G. Then $u = u_{k-1}$ and $u' = u_k$ will do.
If $G - v$ is disconnected, then G contains an induced P^3. For, otherwise, let x be in a different component of $G - v$ than u and u', and let $x \ldots w\ v$ be a shortest path in G from x to v. Then u, u', v, w form an induced P^3.
Similarly, when we replace G by $\overline{G}$ in the above, we have that $\overline{G - v} = \overline{G} - v$ is connected, or $\overline{G}$ (and thus G) has an induced P^3.
On the other hand, if both $G - v$ and $\overline{G - v}$ are connected, then the induction hypothesis gives that $G - v$ and thus G has an induced P^3. $\square$