CS 570 Graph Theory

Homework 5: Sample Solutions

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Remember that you are to use no sources other than the textbook, lecture notes provided by the instructor or taken by yourselves during the lectures, and the instructor when solving homework and exam questions unless stated otherwise.

Question 1: Prove that every connected k-regular bipartite graph on at least three vertices is 2-connected for $k \geq 2$.

Suppose for a contradiction that G is a k-regular graph with bipartition (A, B) so that G is not 2-connected. G must have at least three vertices, so it follows that there exists a cut vertex v of G. Let $G_1, ..., G_j$ be the components of $G - v$. We may assume (without loss) that $v \in A$. Since $j \geq 2$, there must be at least one component G_i so that $A_i = A \cap V(G_i)$ and $B_i = B \cap V(G_i)$ satisfy $|A_i| \geq |B_i|$. Let S be the set of edges with exactly one end in $V(G_i)$. Then, since every edge in S has one end in B_i, we have

$$k|B_i| = \sum_{v \in B_i} deg(v) = |S| + e(G_i) > e(G_i) = \sum_{v \in A_i} deg(v) = k|A_i|$$

which is contradictory. It follows that G must be connected.

Question 2:

(a) Given a 4-regular connected graph G, prove or disprove that the edges of G can be colored red and blue with exactly two red and two blue edges incident to each vertex.

Since every vertex in G is of even degree, G contains an Eulerian circuit, say C. Suppose we color the edges of C alternatingly, red and blue. Since the length of C ($|C| = m = 4n/2$) is even, we won't have to color any subsequent edges on C the same. Let v be an arbitrary vertex of G. Since C will enter and leave v exactly twice and each time v is visited one of its incident edges will be colored with red and another with blue, in the end, exactly two of its incident edges will be red and exactly two will be blue. □

(b) Given a 6-regular connected graph G, is it possible to color edges of G with red, green, and blue such that exactly two red, two green, and two blue edges are incident to each vertex.

Corollary 2.1.5 can be used here as well to find a 2-factor of G, say G'. Since $G - G'$ is also a regular graph of positive even degree (4-regular), it contains a 2-factor, say G''. Notice that $||G'|| = ||G''|| = ||G - G' - G''|| = ||G||/3$. If we color all vertices in G', G'', and $G - G' - G''$ red, green, and blue, respectively, each such vertex will have exactly two red, two green, and two blue edges incident to itself. □

Question 3: A graph G is *critically k-chromatic*, if $\chi(G) = k$ and $\chi(G - v) = k - 1$ for every $v \in V(G)$.

Show that every critical k-chromatic graph ($k \geq 2$) is $(k-1)$-edge connected.

Suppose $G = (V, E)$ is a critically k-chromatic graph where $k \geq 2$. If $k = 2$, then G is K^2, while if $k = 3$, then G is an odd cycle. Thus, G is 1- or 2-edge connected, respectively.
Now, consider the case where $k \geq 4$. Assume G is not $(k-1)$-edge connected. Thus there must exist a partition of V into nonempty subsets V_1 and V_2 such that there are fewer than $k-1$ edges joining V_1 and V_2. Call the set of these edges E_{12}. Since G is critically k-chromatic, the subgraphs G_1 and G_2, induced on vertices V_1 and V_2, respectively, must both have $k-1$ coloring. If the edges in E_{12} are all incident to vertices assigned different colors, then we have a $k-1$ coloring of G, a contradiction.
Let $V_{11}, V_{12}, \ldots, V_{1l}$ be the set of vertices of G_1 assigned color $1, 2, \ldots, l$, respectively, with at least one edge to G_2. Suppose there are q_i edges from V_{1i} to G_2. Thus we have $\sum_{i=1}^{l} q_i \leq k-2$.
We now try to permute colors to obtain the desired coloring. If all vertices in $V_1 1$ are adjacent with vertices of G_2 with different colors, then we do nothing. If, however, there is some v_1 that is adjacent to some vertex of G_2 of the same color, then in G_1 we permute the colors so that no vertex of V_{11} is adjacent to a vertex of G_2 having the same color. This is possible since vertices of V_{11} may be assigned any one of at least $k - 1 - q_1 (> 0)$ colors.
We repeat the same process for V_{1i} where $2 \leq i \leq l$, leaving the colors assigned to V_{11} through $V_{1,i-1}$ fixed. This is possible since the vertices of V_i can be assigned any one of $k - 1 - (q_i - (i-1))$ colors, and this value is greater than zero. Thus we arrive at a $k-1$ coloring, the desired contradiction. □

Question 4: Let u and v be two non-adjacent vertices in G such that $deg(u) + deg(v) \geq n$. Then, show that G is Hamiltonian iff $G + uv$ is Hamiltonian.

Obviously if G is Hamiltonian then so is $G + uv$.
Now assume that $G + uv$ is Hamiltonian. Obviously, if G has a Hamilton cycle, we are done. Thus, we may assume that G has no Hamilton cycle and, therefore, G has a Hamilton path $P = w_0 w_1 \cdots w_n$, with $w_0 = u$, $w_n = v$ and $n = |V(G)|$. Let S be the set of neighbours of u, which are necessarily on P, and let T be the set of vertices on P which follow neighbours of v on P. That is, w_{i+1} is in T if and only if w_i is a neighbour of v. But, $|S| + |T|$ is at least $|V(G)|$, and therefore there is a vertex w_{i+1} in both S and T. But then $C[w_0, w_i] \cup C[w_{i+1}, w_n] \cup \{w_0 w_{i+1}, w_i w_n\}$ is a Hamilton cycle of G.