CS 570 Graph Theory

# Midterm Exam: Sample Solutions

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This is an in-class examination. You may use your notes and textbook. Clearly show your work, being as formal as possible.

**Question 1:** *[20pts]*

(a) Show that the line graph of a simple Eulerian graph is Eulerian.

Let's show this *constructively*:
For any given simple Eulerian graph $G$, we know that the degree of each vertex is even. Take an arbitrary vertex $v_{ij}$ of $L(G)$ corresponding to an edge $v_iv_j$ in $G$. Since both $deg_G(v_i) = 2d_i$ and $deg_G(v_j) = 2d_j$ (for integer $d_i$ and $d_j$) are even, so is $deg_{L(G)}(v_{ij}) = (deg_G(v_i) - 1) + (deg_G(v_j) - 1) = 2(d_i + d_j - 1)$.
In addition, $L(G)$ is connected (as a matter of fact, there exists a cycle that goes through every vertex of $L(G)$ corresponding to an Euler tour in $G$. Thus $L(G)$ is connected and every vertex in $L(G)$ is of even degree, so $L(G)$ must be Eulerian. □

(b) If the line graph of a simple graph $G$ is Eulerian, must $G$ be Eulerian?

Not necessarily since the degree sums mentioned above is even when both vertices are of odd degree as well.

**Question 2:** *[25pts]* Show that a tree $T$ of order at least three has at least $\Delta(T)$ leaves **using induction**.

Let us use induction on $|T|$:

- **Base:** There is only one distinct tree on three vertices, and it has $\Delta(T) = 2$ leaves.
- **Inductive Hypothesis:** Assume in any tree $T$ of order $|T| < n$, there are at least $\Delta(T)$ leaves.
- **Inductive Step:** We need to show that in any tree $T$ of order $|T| = n$, there are at least $\Delta(T)$ leaves. Let $T' := T - v$, where $v$ is an arbitrary leaf of $T$ and $w$ is the only neighbor of $v$ in $T$ (and thus in $T'$). We have two cases:
  (i) if $d_T(w) = \Delta(T)$, then $\Delta(T') = \Delta(T)$ or $\Delta(T) - 1$ and number of leaves of $T'$ is one less than number of leaves of $T$ (since $d_{T'}(w) \geq 2$ as there is at least one non-leaf vertex in a tree of order at least three). Since $T'$ has at least $\Delta(T')$ leaves by the I.H., $T$ must have at least $\Delta(T') + 1 \geq \Delta(T)$ leaves.
  (ii) if $d_T(w) < \Delta(T)$, then $\Delta(T') = \Delta(T)$ and $T$ has at least as many leaves as $T'$. Since $T'$ has at least $\Delta(T')$ $(= \Delta(T))$ leaves by the I.H., $T$ has at least $\Delta(T)$ leaves.

**Question 3:** *[25pts]* A graph $G$ is called *self-complementary*, if it is isomorphic with its complement. Find a self-complementary graph of order $4n$ and of order $4n+1$ for **all** $n \geq 1$.

For the order $4n$, expand the self-complementary path $P^3$ as follows: replace the two leaves of $P^3$ by $\overline{K^n}$ and the two other vertices by $K^n$. The result is a self-complementary graph.
For the order $4n+1$, expand $C^5$ as follows: replace the vertices in order around $C^5$ by $K^n$, $K^n$, $\overline{K^n}$, $x$, and $\overline{K^n}$. The result is a self-complementary graph. (*Note:* There are also other solutions to the problem.)

**Question 4:** *[25pts]* Show that every $k$-connected graph of order at least $2k$ contains a cycle of length at least $2k$, for $k \geq 2$.

Let $G$ be a $k$-connected graph of order at least $2k$ with $k \geq 2$. Such a graph contains a cycle since $\delta(G) \geq 2$. Let $C$ be a longest cycle in $G$ ($|C| \geq k+1$ by Proposition 1.3.1). Assume $|C| < 2k$. Then there must be at least one vertex $v$ outside $C$. Consider the paths between $v$ and $V(C)$ forming a $v_k - C$ fan by Corollary 3.3.3. There are at least $k$ such paths since $G$ is $k$-connected. By the piegon-hole principle, at least two of these paths, say $P_1$ and $P_2$ must end in adjacent vertices, say $v_1$ and $v_2$, of $C$, respectively. Removing the edge $v_1v_2$ from $C$ and adding $P_1$ and $P_2$ results in a longer cycle than $C$, creating a contradiction.

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