Question 1: [20pts]

(a) Show that the line graph of a simple Eulerian graph is Eulerian.

Let’s show this constructively:
For any given simple Eulerian graph \( G \), we know that the degree of each vertex is even. Take an arbitrary vertex \( v_{ij} \) of \( L(G) \) corresponding to an edge \( v_i v_j \) in \( G \). Since both \( \deg_G(v_i) = 2d_i \) and \( \deg_G(v_j) = 2d_j \) (for integer \( d_i \) and \( d_j \)) are even, so is \( \deg_{L(G)}(v_{ij}) = (\deg_G(v_i) - 1) + (\deg_G(v_j) - 1) = 2(d_i + d_j - 1) \).
In addition, \( L(G) \) is connected (as a matter of fact, there exists a cycle that goes through every vertex of \( L(G) \) corresponding to an Euler tour in \( G \). Thus \( L(G) \) is connected and every vertex in \( L(G) \) is of even degree, so \( L(G) \) must be Eulerian. \( \square \)

(b) If the line graph of a simple graph \( G \) is Eulerian, must \( G \) be Eulerian?

Not necessarily since the degree sums mentioned above is even when both vertices are of odd degree as well.

Question 2: [25pts] Show that a tree \( T \) of order at least three has at least \( \Delta(T) \) leaves using induction.
Let us use induction on $|T|$: 

- **Base**: There is only one distinct tree on three vertices, and it has $\Delta(T) = 2$ leaves.

- **Inductive Hypothesis**: Assume in any tree $T$ of order $|T| < n$, there are at least $\Delta(T)$ leaves.

- **Inductive Step**: We need to show that in any tree $T$ of order $|T| = n$, there are at least $\Delta(T)$ leaves. Let $T' := T - v$, where $v$ is an arbitrary leaf of $T$ and $w$ is the only neighbor of $v$ in $T$ (and thus in $T'$). We have two cases:

  (i) if $d_T(w) = \Delta(T)$, then $\Delta(T') = \Delta(T)$ or $\Delta(T) - 1$ and number of leaves of $T'$ is one less than number of leaves of $T$ (since $d_{T'}(w) \geq 2$ as there is at least one non-leaf vertex in a tree of order at least three). Since $T'$ has at least $\Delta(T')$ leaves by the I.H., $T$ must have at least $\Delta(T') + 1 \geq \Delta(T)$ leaves.

  (ii) if $d_T(w) < \Delta(T)$, then $\Delta(T') = \Delta(T)$ and $T$ has at least as many leaves as $T'$. Since $T'$ has at least $\Delta(T') (= \Delta(T))$ leaves by the I.H., $T$ has at least $\Delta(T)$ leaves.

**Question 3**: [25pts] A graph $G$ is called self-complementary, if it is isomorphic with its complement. Find a self-complementary graph of order $4n$ and of order $4n + 1$ for all $n \geq 1$.

For the order $4n$, expand the self-complementary path $P^3$ as follows: replace the two leaves of $P^3$ by $K^n$ and the two other vertices by $K^n$. The result is a self-complementary graph. For the order $4n + 1$, expand $C^5$ as follows: replace the vertices in order around $C^5$ by $K^n$, $K^n$, $K^n$, $x$, and $K^n$. The result is a self-complementary graph. (Note: There are also other solutions to the problem.)

**Question 4**: [25pts] Show that every $k$-connected graph of order at least $2k$ contains a cycle of length at least $2k$, for $k \geq 2$.

Let $G$ be a $k$-connected graph of order at least $2k$ with $k \geq 2$. Such a graph contains a cycle since $\delta(G) \geq 2$. Let $C$ be a longest cycle in $G$ ($|C| \geq k + 1$ by Proposition 1.3.1). Assume $|C| < 2k$. Then there must be at least one vertex $v$ outside $C$. Consider the paths between $v$ and $V(C)$ forming a $v_k - C$ fan by Corollary 3.3.3. There are at least $k$ such paths since $G$ is $k$-connected. By the pigeon-hole principle, at least two of these paths, say $P_1$ and $P_2$ must end in adjacent vertices, say $v_1$ and $v_2$, of $C$, respectively. Removing the edge $v_1v_2$ from $C$ and adding $P_1$ and $P_2$ results in a longer cycle than $C$, creating a contradiction.

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