CS 570 Graph Theory

Lecture 1: The Basics

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1 The Basics: Definitions and Terminology

1.1 Graphs

A *graph* is a pair $G = (V, E)$ of sets such that $E \subseteq [V]^2$; thus, the elements of E are 2-element subsets of V. To avoid notational ambiguities we assume $V \cap E = \emptyset$. The elements of V are the *vertices* (or *nodes* or *points*) of the graph G, the elements of E are its *edges* (or *lines*).

The number of vertices of a graph G is its **order** denoted by $|G|$. Its number of edges is denoted by $||G||$. Graphs can be *finite*, *infinite*, *countable* and so on according to their order. We assume graphs to be finite unless otherwise stated. A graph of order 0 or 1 is called *trivial* and disregarded in most cases.

A vertex is *incident* with an edge e if $v \in e$; then e is an edge *at* v. The two vertices incident with an edge are its *endvertices* or *ends*, and an edge *joins* its ends. An edge $\{x, y\}$ is usually written as xy (or yx). If $x \in X$ and $y \in Y$, then xy is an $X - Y$ edge. The set of all $X - Y$ edges in a set E is denoted by $E(X, Y)$; instead of $E(\{x\}, Y)$ and $E(X, \{y\})$ we simply write $E(x, Y)$ and $E(X, y)$. The set of all the edges in E at a vertex v is denoted by $E(v)$.

Two vertices x, y of G are *adjacent*, or *neighbors*, if xy is an edge of G. Two edges $e \neq f$ are *adjacent* if they have an end in common. If all the vertices of G are pairwise adjacent, then G is *complete*. A complete graph on n vertices is a K^n.

Pairwise non-adjacent vertices or edges are called *independent*.

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We call G and G' isomorphic

If $G = (V, E)$ is a simple graph, its **complement** $\overline{G}$ is the simple graph with vertex set V in which two vertices are adjacent if and only if they are *not* adjacent in G.

Two graphs G_1 and G_2 are **isomorphic** if there is a one-one correspondence between the vertices of G_1 and those of G_2 such that the number of edges joining any two vertices of G_1 is equal to the number of edges joining the corresponding vertices of G_2.

1.2 Representation

One way of storing a graph in a computer is by listing the vertices adjacent to each vertex of the graph, called the **adjacency list representation**. Other useful representations involve matrices. If G is a graph with vertices labelled $1, 2, \ldots, n$, its **adjacency matrix** A, is the $n \times n$ matrix whose ij-th entry is the number of edges joining vertex i and vertex

j. If, in addition, the edges are labelled $1, 2, \ldots, m$, its **incident matrix** M is the $n \times m$ matrix whose ij-th entry is 1 if vertex i is incident with edge j, and 0 otherwise.

1.3 Subgraphs

A **subgraph** of a graph $G = (V, E)$ is a graph, each of whose vertices belongs to V and each of whose edges belongs to E.

If $G' \subseteq G$ and G' contains all the edges $xy \in E$ with $x, y \in V'$, then G' is an **induced subgraph** of G; we say V' induces G' in G, and we write $G' = G[V']$.

If U is a set of vertices (usually of G), we write $G - U$ for $G[V \backslash U]$.

We call G **edge-maximal** with a given graph property if G itself has the property but no graph $G + xy$ does, for non-adjacent vertices $x, y \in G$.

1.4 Adjacency

We say two vertices v and w of a graph G are **adjacent** if there is an edge vw joining them, and the vertices v and w are then **incident** with such an edge. Similarly, two distinct edges e and f are **adjacent** if they have a vertex in common.

The **degree** of a vertex v of G is the number of edges incident with v. A vertex of degree 0 is an **isolated**, or disconnected, vertex and a vertex of degree 1 is an **end-vertex**.

$L(G) = (V_L, E_L)$ is the **line graph** of $G = (V, E)$ where $V_L := E$, vertices xy and uv are in V_L and $\{xy, uv\} \in E_L$ if and only if edges xy and uv are adjacent in G.

The number $\delta(G) = min\{d(v)|v \in V\}$ is the **minimum degree** of G. Similarly, the number $\Delta(G) = max\{d(v)|v \in V\}$ is the **maximum degree** of G. The **average degree** of a graph G is

$$d(G) = \frac{1}{|V|} \sum_{v \in V} d(v)$$

Clearly,

$$\delta(G) \leq d(G) \leq \Delta(G)$$

The number of edges of G per vertex is expressed by

$$\epsilon(G) = \frac{|E|}{|V|}$$

Average degree of a vertex, d, and the number of edges per vertex, ϵ, are of course related:

$$|E| = \frac{1}{2} \sum_{v \in V} d(v) = \frac{1}{2} d(G)|V|$$

and therefore

$$\epsilon(G) = \frac{1}{2} d(G)$$

Proposition 1.1 [1.2.1] ♠ The number of vertices of odd degree in a graph is always even. □

1.5 Paths and Cycles

A **path** is a non-empty graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, \ldots, x_k\} \qquad E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$$

where x_i are all *distinct*. The number of edges of a path is its length.

Given a graph H, we call P an *H-path* if P is non-trivial and meets H exactly in its ends.

If $P = x_0 \ldots x_{k-1}$ is a path and $k \geq 3$, then the graph $C := P + x_{k-1}x_0$ is called a **cycle** denoted by $x_0x_1 \ldots x_{k-1}x_0$. An edge which joins two vertices of a cycle but is *not* itself an edge of the cycle is a **chord** of that cycle.

Proposition 1.2 [1.3.1] ♠ Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$ provided that $\delta(G) \geq 2$. □

The minimum length of a cycle contained in a graph G is the **girth**, $g(G)$, of G; the maximum length of a cycle in G is its **circumference**.

The **distance** $d_G(x, y)$ in G of two vertices x and y is the length of a shortest $x - y$ path in G; if no such path exists we set $d_G(x, y) = \infty$. The greatest distance between any two vertices in G is the **diameter** of G, denoted by $\mathrm{diam}(G)$.

Proposition 1.3 [1.3.2] ♠ Every graph G containing a cycle satisfies $g(G) \leq 2diam(G) + 1$. □

A vertex is **central** in G if its greatest distance from any other vertex is as small as possible. This distance is the **radius** of G, denoted by $\mathrm{rad}(G)$. So,

$$rad(G) \leq diam(G) \leq 2rad(G)$$

A **walk** of length k in G is a non-empty alternating sequence $v_0e_0v_1e_1 \ldots e_{k-1}v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If $v_0 = v_k$, the walk is **closed**. If the vertices in a walk are all distinct, it obviously defines a **path**. If the edges in a walk are all distinct, it defines a **trail**. A **circuit** is a closed trail.

1.6 Connectivity

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where V_1 and V_2 are disjoint, their **union** $G_1 \cup G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge family $E_1 \cup E_2$. A graph is **connected** if it cannot be expressed as the union of two graphs, and **disconnected** otherwise. Clearly any disconnected graph can be expressed as the union of connected graphs, each of which is a **component** of G.

If $A, B \subseteq V$ and $X \subseteq V \cup E$ are such that every $A - B$ path in G contains a vertex or an edge from X, we say that X **separates** the sets A and B in G. This implies in particular that $A \cap B \subseteq X$. More generally, we say that X **separates** G, and call X a **separating set** in G.

A vertex that separates two other vertices of the same component is a **cutvertex**, and an edge separating its ends is a **bridge**. Thus, bridges in a graph are precisely those edges that do not lie on any cycle.

G is called k**-connected** if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. In other words, no two vertices are separated by fewer than k other vertices. The greatest integer k such that G is k-connected is the **connectivity** $\kappa(G)$ of G.

If $|G| > 1$ and $G - F$ is connected for every set $F \subseteq E$ of fewer than l edges, then G is called l**-edge-connected**. The greatest integer l such that G is l-edge-connected is the **edge-connectivity** $\lambda(G)$ of G.

1.7 Trees and Forests

An **acyclic** graph, one not containing any cycles, is called a **forest**. A connected forest is called a **tree**. The vertices of degree 1, end-vertices, in a tree are its **leaves**.

Theorem 1.4 [1.5.1] ♠ The following assertions are equivalent for a graph T:

(i) T is a tree.

(ii) Any two vertices of T are linked by a unique path in T.

(iii) T is minimally connected.

(iv) T is maximally acyclic.

□

1.8 Contraction and Minors

Let $e = xy$ be an edge of a graph $G = (V, E)$. By G/e we denote the graph obtained from G by **contracting** the edge e into a new vertex v_e, which becomes adjacent to all the former neighbors of x and y.

More generally, if X is another graph and $\{V_x | x \in V(X)\}$ is a partition of V into connected subsets such that, for any two vertices $x, y \in X$, there is a $V_x - V_y$ edge in G if and only if $xy \in E(X)$, we call G an MX and write $G = MX$. The sets V_x are the **branch sets** of this MX. Intuitively, we obtain X from G by contracting every branch set into a single vertex and deleting any parallel edges or loops that may arise.

If $G = MX$ is a subgraph of another graph Y, we call X a **minor** of Y and write $X \preceq Y$. Note that every subgraph of a graph is also its minor.

If we replace the edges of X with independent paths between their ends (so that none of these paths has an inner vertex on another path or in X), we call the graph G obtained a **subdivision** of X and write $G = TX$. If $G = TX$ is the subgraph of another graph Y, then X is a **topological minor** of Y.

If $G = TX$, we view $V(X)$ as a subset of $V(G)$ and call these vertices the **branch vertices** of G; the other vertices of G are its **subdividing vertices**. Thus, all subdividing vertices have degree 2, while the branch vertices retain their degree from X.

Notice that every topological minor of a graph is also its (ordinary) minor.

1.9 Euler Tours

A closed walk in a graph is called an **Euler tour** if it traverses every edge of the graph exactly once. A graph is **Eulerian** if and only if it admits an Euler tour.

Theorem 1.5 [1.8.1] ♠ A connected graph is Eulerian iff every vertex has even degree. □

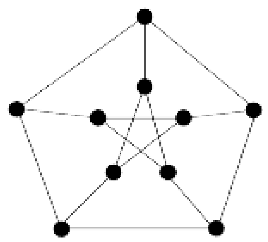
1.10 Types of Graphs

There are certain types or classes of graphs that are quite popular.

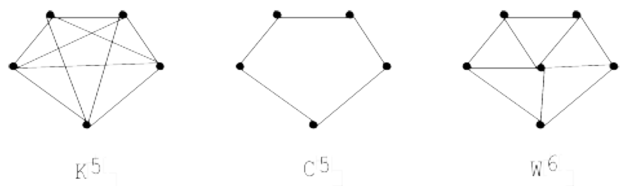
A graph whose edge set is empty is a **null graph**.

A simple graph in which each pair of distinct vertices are adjacent is a **complete graph**. We denote the complete graph on n vertices by K^n.

A graph in which each vertex has the same degree is a **regular graph**. If each vertex has degree r, the graph is **regular of degree-r** or simply r-regular. Of special importance are the **cubic graphs**, which are regular of degree 3; an example of a cubic graph is the **Petersen graph**:



A connected graph that is regular of degree 2 is a **cycle graph**, denoted by C^n. The graph obtained from C^n by removing an edge is the **path graph** on n vertices, denoted by P^n. The graph obtained from C^{n-1} by joining each vertex to a new vertex v is the **wheel** on n vertices, denoted by W^n.

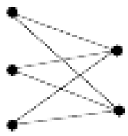


If the vertex set of a graph G can be split into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B, then G is a **bipartite graph**. A **complete bipartite graph** is a bipartite graph in which each vertex in A is joined to each vertex in B by a single edge, denoted by $K_{r,s}$ assuming the two partitions have r and s vertices, respectively.

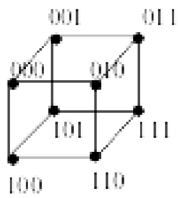
In general, a graph $G = (V, E)$ is called r-**partite** if V admits a partition into r classes such that every edge has its ends in different classes; that is, vertices in the same partition must not be adjacent. An r-partite graph in which every two vertices from different partition classes are adjacent is called **complete**.

Proposition 1.6 [1.6.1] ♠ A graph is bipartite iff it contains no odd cycle. □

Of special interest among the regular bipartite graphs are the cubes. The **k-cube** Q_k is the graph whose vertices correspond to the sequences $(a_1, a_2, \ldots, a_n)$, where each $a_i = 0$ or 1, and whose edges join those sequences that differ in just one place. Note that the graph of the cube is the graph Q_3. Check that Q_k has 2^k vertices and $k2^{k-1}$ edges, and is regular of degree k.



$K_{3,2}$



Q_3

1.11 Other Notions of Graphs

A **hypergraph** is a pair (V, E) of disjoint sets, where the elements of E are non-empty subsets (of any cardinality) of V. Thus, graphs are special hypergraphs.

A **directed graph** or digraph is a pair (V, E) of disjoint sets of vertices and edges together with two maps init:$E \to V$ and ter:$E \to V$ assigning to every edge e an initial vertex init(e) and a terminal vertex ter(e). The edge is said to be directed from init(e) to ter(e). Note that a directed graph may have several edges between the same two vertices. Such edges are called **multiple edges**; if they have the same direction, they are **parallel**.

A **multigraph** is a graph which can have multiple edges or loops.