CS 570 Graph Theory

Lecture 3: Connectivity

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We have already defined k-connectedness as "A graph is **k-connected** if we need at least k vertices to disconnect it." Alternative definition is "A graph is k-connected if any two of its vertices can be joined by k independent paths." These two definitions are in fact equivalent as we will see.

1 2-connected graphs and subgraphs

A maximal connected subgraph without a cutvertex is called a **block** (2-connected anologues of components). Thus, every block of a graph G is either a maximal 2-connected subgraph, or a bridge (with its end vertices), or an isolated vertex. Conversely, every such subgraph is a block.

Unlike components of a graph, the structure of a graph G is not completely captured by its blocks since the blocks need not be disjoint. Let A denote the set of cutvertices and $\mathcal{B}$ the set of its blocks. The **block graph** of a graph G is a natural bipartite graph on $A \cup \mathcal{B}$ formed by the edges aB with $a \in B$.

Proposition 1.1 The block graph of a connected graph is a tree. □

Proposition 1.2 [3.1.1] ♠ A graph is 2-connected iff it can be constructed from a cycle by successively adding H-paths to graphs H already constructed. □

2 The structure of 3-connected graphs

Lemma 2.1 [3.2.4] ♠ If G is 3-connected and $|G| > 4$, then G has an edge e such that G/e is again 3-connected. □

Theorem 2.2 [3.2.5][Tutte 1961] A graph G is 3-connected iff there exists a sequence $G_0 \ldots G_n$ of graphs with the following properties:

(i) $G_0 = K^4$ and $G_n = G$

(ii) G_{i+1} has an edge xy with $d(x), d(y) \geq 3$ and $G_i = G_{i+1}/xy$, for every $i < n$.

□

The theorem above is the essential core of a result of Tutte known as his *wheel theorem*. Similar to the construction of 2-connected graphs by successively adding H-paths, it enables us to construct all 3-connected graphs by a simple inductive process depending only on local information: starting with K^4, we pick a vertex v in a graph constructed already, split it into two adjacent vertices v', v'', and join these to the former neighbors of v as we please - provided only that v' and v'' each acquire at least 3 incident edges, and that every former neighbor of v becomes adjacent to at least one of v' and v''.

Theorem 2.3 [3.2.6][Tutte 1963] The cycle space of a 3-connected graph is generated by its non-separating induced cycles. □

3 Menger's theorem

Theorem 3.1 [3.3.1][Menger 1927] ♠ Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Then the minimum number of vertices separating A from B in G is equal to the maximum number of disjoint $A - B$ paths in G. □

A set of $a - B$ paths is called an $a - B$ **fan** if any two of the paths have only a in common.

Corollary 3.2 [3.3.4] ♠ For $B \subseteq V$ and $a \in V \backslash B$, the minimum number of vertices, different from a, separating a from B is equal to the maximum number of paths forming an $a - B$ fan in G. □

Corollary 3.3 [3.3.5] ♠ Let a and b be two distinct vertices of G.

(i) If $ab \notin E$, then the minimum number of vertices $\neq a, b$, separating a from b in G is equal to the maximum number of independent $a - b$ paths in G.

(ii) The minimum number of edges separating a from b in G is equal to the maximum number of edge-disjoint $a - b$ paths in G.

□

Theorem 3.4 [3.3.6][Global version of Menger's theorem]

(i) A graph is k-connected iff it contains k independent paths between any two vertices.

(ii) A graph is k-edge-connected iff it contains k edge-disjoint paths between any two vertices.

□

4 Mader's theorem

In analogy to Menger's theorem we may consider the following question: given a graph G with an induced subgraph H, up to how many independent H-paths can we find in G?

Let $X \subseteq V(G-H)$ and $F \subseteq E(G-H)$ such that every H-path in G has a vertex or an edge in $X \cup F$. Then G cannot contain more than $|X \cup F|$ independent H-paths. Hence the least cardinality of such a set is a natural upper bound for the maximum number of independent H-paths.

In contrast to Menger's theorem, this bound can still be improved. Clearly, we may assume that no edge in F has an end in X; otherwise, this edge would not be needed in the separator. Let $Y := V(G-H)\backslash X$, and $\mathcal{C}_F$ denote the set of components of the graph (Y, F). Since every H-path avoiding X contains an edge from F, it has at least two vertices in ∂C for some $C \in \mathcal{C}_F$, where ∂C denotes the set of vertices in C with a neighbor in $G - X - C$. The number of independent H-paths in G is therefore bounded above by

$$M_G(H) = min(|X| + \sum_{C \in \mathcal{C}_F} \lfloor \frac{1}{2} |\partial C| \rfloor),$$

where the minimum is taken over all X and F as described above: $X \subseteq V(G-H)$ and $F \subseteq E(G-H-X)$ such that every H-path in G has a vertex or an edge in $X \cup F$.

Mader's theorem says this upper bound is always attained by some set of independent H-paths:

Theorem 4.1 [3.4.1][Mader 1978] Given a graph G with an induced subgraph H, there are always $M_G(H)$ independent H-paths in G. □

5 Edge-disjoint spanning trees

The edge version of Menger's theorem tells us when a graph G contains k **edge-disjoint paths**.

In a situation where a quick access to a set of k edge-disjoint paths between any two vertices is desirable, it may be a good idea to ask for more: k **edge-disjoint spanning trees**. If such trees exist, then the graph is clearly k-edge-connected.

Let's look into some obvious necessary conditions for the existence of k edge-disjoint spanning trees. With respect to any partition of $V(G)$ into r sets, every spanning tree of G has at least $r-1$ **cross-edges**, edges whose ends lie in different sets. Thus if G has k edge-disjoint spanning trees, it has at least $k(r-1)$ cross-edges.

This obvious necessary condition is also sufficient:

Theorem 5.1 [Tutte 1961; Nash-Williams 1961] A multigraph contains k edge-disjoint spanning trees iff for every partition P of its vertex set it has at least $k(|P|-1)$ cross-edges. □

Corollary 5.2 ♠ Every $2k$-edge-connected multigraph G has k edge-disjoint spanning trees. □

6 Paths between given pairs of vertices

A graph with at least $2k$ vertices is said to be k-linked if for every $2k$ distinct vertices $s_1, s_2, \ldots, s_k$ and $t_1, t_2, \ldots, t_k$ it contains k disjoint paths $P_1, P_2, \ldots, P_k$ with $P_i = s_i \ldots t_i$ for all i. Unlike in Menger's theorem, we are not merely asking for k disjoint paths between two *sets* of vertices; we insist that each of these paths link a specified distinct pair of vertices.

Theorem 6.1 [3.5.1][Mader 1967] There is a function $h : N \to N$ such that every graph with average degree at least $h(r)$ contains K^r as a topological minor, for every $r \in N$. □

The function used in the proof of the above theorem is $h(r) = 2^{\binom{r}{2}}$.

Theorem 6.2 [3.5.2][Jung 1970; Larman & Mani 1970] There is a function $f : N \to N$ such that every $f(k)$-connected graph is k-linked, for all $k \in N$. □

The function f used in the proof of the above theorem (i.e. $f(k) = h(3k) + 2k$ where $h(r) = 2^{\binom{r}{2}}$) is exponential and far from being best possible. It is still remarkable, though, that f can be chosen linear: as Bollobas & Thomason (1996) have shown, every $22k$-connected graph is k-linked.