CS 570 Graph Theory

Lecture 4: Planar Graphs

Lecturer: Uğur Doğrusöz

CS Dept., Bilkent University

When we draw a graph on a piece of paper, we naturally try to do this as transparent as possible. One obvious way to limit the mess created by all lines is to avoid intersections.

Graphs drawn without any intersections are called **plane graphs**. Abstract graphs that *can* be drawn without any intersections are called **planar**.

1 Topological prerequisites

If P is an arc between x and y, we denote the point set $P \backslash \{x, y\}$, the **interior** of P, by $\mathring{P}$.

Let $O \subseteq \Re^2$ be an open set. Being linked by an arc in O defines an equivalence relation on O. The corresponding equivalence classes are again open; they are the **regions** of O. A closed set $X \subseteq \Re^2$ is said to **separate** O if $O \backslash X$ has more than one region. The **frontier** of a set $X \subseteq \Re^2$ is the set Y of all points $y \in \Re^2$ such that every neighborhood of y meets both X and $\Re^2 \backslash X$.

2 Plane Graphs

A **plane graph** is a pair (V, E) of finite sets with the following:

(i) $V \subseteq \Re^2$;

(ii) every edge is an arc between two vertices;

(iii) different edges have different sets of endpoints;

(iv) the interior of an edge contains no vertex and no point of any other edge.

For every plane graph G, the set $\Re^2 \backslash G$ is open; its regions are the **faces** of G. Since G is bounded, exactly one of its faces is unbounded called the **outer face** of G. We denote the set of faces of G by $F(G)$.

Corollary 2.1 [4.2.3] The frontier of a face is always the point set of a subgraph. □

The subgraph of G whose point set is the frontier of a face f is said to **bound** f and is called its **boundary**, denoted by $G[f]$. A face is said to be incident with the vertices and edges of its boundary. Note that every face of a plane graph is also a face of its own boundary.

Proposition 2.2 [4.2.4] A plane forest has exactly one face. □

Lemma 2.3 [4.2.5] If a plane graph has different faces with the same boundary, then the graph is a cycle. □

Proposition 2.4 [4.2.6] In a 2-connected plane graph, every face is bounded by a cycle. □

A plane graph G is called **maximally plane**, or just **maximal**, if we cannot add a new edge to form a plane graph $G' \supset G$ with $V(G') = V(G)$. We call G a **plane triangulation** if every face of G, including the outer face, is bounded by a triangle.

Proposition 2.5 [4.2.8] A plane graph of order at least 3 is maximally plane iff it is a plane triangulation. □

Theorem 2.6 [4.2.9][Euler's Formula] ♠ Let G be a connected plane graph with n vertices, m edges, and l faces. Then $n - m + l = 2$. □

We have $n - m + l = w + 1$ for a graph with w components.

The general form of Euler's theorem asserts the sum obtained is always a fixed number depending only on the surface, not on the graph, and this number differs for distinct surfaces.

Corollary 2.7 [4.2.10] ♠ A plane graph with $n \geq 3$ vertices has at most $3n - 6$ edges. Every plane triangulation with n vertices has $3n - 6$ edges. □

Euler's formula is useful for showing that certain graphs cannot occur as plane graphs (e.g. K^5 and $K_{3,3}$; **why?**)

Corollary 2.8 [4.2.9] ♠ A plane graph contains neither K^5 nor $K_{3,3}$ as a topological minor. □

Proposition 2.9 [4.2.7] The face boundaries in a 3-connected plane graph are precisely its non-separating induced cycles. □

3 Drawings

An embedding in the plane, or **planar embedding**, of an (abstract) graph G is an isomorphism between G and a plane graph $\tilde{G}$, the latter being called a **drawing** of G.

Two planar embeddings of a graph may obviously differ. Two planar embeddings σ_1 and σ_2 of a graph G are called **topologically equivalent** if $\sigma_2 \circ \sigma_1^{-1}$ is a topological isomorphism between $\sigma_1(G)$ and $\sigma_2(G)$.

Theorem 3.1 [4.3.2][Whitney 1932] Any two planar embeddings of a 3-connected graph are equivalent. □

4 Planar graphs: Kuratowski's theorem

A graph is called **planar** if it can be embedded in the plane; that is, if it is isomorphic to a plane graph. A planar graph is **maximal**, or maximally planar, if it cannot be extended to a larger planar graph by adding an edge.

Drawings of maximal planar graphs are obviously maximally plane. The converse, however, is not obvious but true.

Proposition 4.1 [4.4.1] ♠

(i) Every maximal plane graph is maximally planar.

(ii) A planar graph with $n \geq 3$ vertices is maximally planar iff it has $3n - 6$ edges.

□

Proposition 4.2 [4.4.2] A graph contains K^5 or $K_{3,3}$ as a minor iff it contains K^5 or $K_{3,3}$ as a topological minor. □

Theorem 4.3 [4.4.6][Kuratowski 1930; Wagner 1937] The following assertions are equivalent for graphs G:

(i) G is planar;

(ii) G contains neither K^5 nor $K_{3,3}$ as a minor;

(iii) G contains neither K^5 nor $K_{3,3}$ as a topological minor.

□

5 Algebraic planarity criteria

Planarity can be characterized by purely algebraic terms, by a certain abstract property of its cycle space.

Let $G = (V, E)$. We call a subset $\mathcal{F}$ of its edge space $\varepsilon(G)$ **simple** if every edge of G lies in at most two sets of $\mathcal{F}$.

Theorem 5.1 [4.5.1][MacLane 1937] A graph is planar iff its cycle space has a simple basis. □

Theorem 5.2 [4.5.2][Tutte 1963] A 3-connected graph is planar iff every edge lies on at most (equivalently: exactly) two non-separating induced cycles. □

6 Plane duality

Let G be a plane multigraph. The plane multigraph G^* formed by placing a new vertex inside each face of G and linking these vertices in G^* whenever the corresponding two faces are adjacent (share an edge) in G gives the **plane dual** of G.

Proposition 6.1 [4.6.1] For any connected plane multigraph G, an edge set $E \subseteq E(G)$ is the edge set of a cycle in G iff $E^* := \{e^* | \; e \in E\}$ is a minimal cut in G^*. $\square$

Let us call a multigraph G^* an **abstract dual** of a multigraph G if $E(G^*) = E(G)$ and the minimal cuts in G^* are precisely the edge sets of cycles in G.

Proposition 6.2 [4.6.2] If G^* is an abstract dual of G, then the cut space of G^* is the cycle space of G, i.e.

$$\mathcal{C}^*(G^*) = \mathcal{C}(G).$$

$\square$

Theorem 6.3 [4.6.3][Whitney 1933] A graph is planar iff it has an abstract dual. $\square$