We follow the approach given in our textbook (Huth & Ryan, 2000). Since proofs for $\land i$ and $\lor e$ are already given there (page 60), we must supply a proof for each of the remaining natural deduction rules.\(^1\)

*Caveat:* Although the proof in the book is for sequents with a non-empty set of premises, we can convert any given theorem (i.e. $\vdash \phi$) to a regular sequent by supplying $\eta \lor \neg \eta$ (that is, $\top$) as a premise, thus obtaining $\eta \lor \neg \eta \vdash \phi$.

N.B. Unless noted otherwise, the $k^{th}$ line of the proof contains $\psi$. Also remember that we assume $\phi_1, \ldots, \phi_n$ to be all $T$.)

$\land e_1$: Suppose that the rule used in line $k$ is $\land e_1$. Then, there is a previous line (say, line $k_1$) which contains $\psi \land \gamma$. Since $k_1 < k$, we see that there exists a proof of the sequent $\phi_1, \ldots, \phi_n \vdash \psi \land \gamma$ with length less than $k$, viz. take the first $k_1$ lines of the original proof. By the induction hypothesis, $\psi \land \gamma$ evaluates to $T$. According to the truth table of $\land$, $\psi$ must be $T$ in order this to be true.

$\land e_2$: Suppose that the rule used in line $k$ is $\land e_2$. Then, there is a previous line (say, line $k_1$) which contains $\gamma \land \psi$. Since $k_1 < k$, we see that there exists a proof of the sequent $\phi_1, \ldots, \phi_n \vdash \gamma \land \psi$ with length less than $k$, viz. take the first $k_1$ lines of the original proof. By the induction hypothesis, $\gamma \land \psi$ evaluates to $T$. According to the truth table of $\land$, $\psi$ must be $T$ in order this to be true.

\(^1\)We’ll omit “copy” since it constitutes a trivial case.
\(\forall i_1\): Suppose that the last rule is \(\forall i_1\). Then \(\psi\) is of the form \(\psi_1 \lor \psi_2\) and the justification in line \(k\) refers to a previous line \(k_1\) which has \(\psi_1\) as its conclusion. Since \(k_1 < k\), we see that there exists a proof of the sequent \(\phi_1, \ldots, \phi_n \vdash \psi_1\) with length less than \(k\). By the induction hypothesis, \(\psi_1\) evaluates to \(T\). According to the truth table of \(\lor\), \(\psi_1 \lor \psi_2\) evaluates to \(T\) as well.

\(\forall i_2\): Suppose that the last rule is \(\forall i_2\). Then \(\psi\) is of the form \(\psi_1 \lor \psi_2\) and the justification in line \(k\) refers to a previous line \(k_2\) which has \(\psi_2\) as its conclusion. Since \(k_2 < k\), we see that there exists a proof of the sequent \(\phi_1, \ldots, \phi_n \vdash \psi_2\) with length less than \(k\). By the induction hypothesis, \(\psi_2\) evaluates to \(T\). According to the truth table of \(\lor\), \(\psi_1 \lor \psi_2\) evaluates to \(T\) as well.

→ \(i\): In this case, \(\psi\) is of the form \(\psi_1 \rightarrow \psi_2\) and the justification in line \(k\) refers to the preceding block of lines (a box starting at \(k_1\) and ending at \(k-1\)). Line \(k_1\) has the assumption \(\psi_1\) and line \(k-1\) has the conclusion \(\psi_2\). Since \(k-1 < k\), we see that there exists a proof of the sequent \(\phi_1, \ldots, \phi_n, \psi_1 \vdash \psi_2\) with length less than \(k\). By the induction hypothesis, \(\psi_2\) must evaluate to \(T\) when \(\psi_1\) is \(T\). This means that \(\psi_1 \rightarrow \psi_2\) evaluates to \(T\).

→ \(e\): In this case, the justification in line \(k\) refers to previous lines \(k_1\) and \(k_2\) which have \(\gamma\) and \(\gamma \rightarrow \psi\) as their conclusions. Since \(k_1 < k\) and \(k_2 < k\), there exist proofs of \(\phi_1, \ldots, \phi_n \vdash \gamma\) and \(\phi_1, \ldots, \phi_n \vdash \gamma \rightarrow \psi\) with length less than \(k\). By the induction hypothesis, \(\gamma\) and \(\gamma \rightarrow \psi\) both evaluate to \(T\). According to the truth table of \(\rightarrow\), \(\psi\) must evaluate to \(T\) as well.

→ \(i\): In this case, the justification in line \(k\) refers to the preceding block of lines (a box starting at \(k_1\) and ending at \(k-1\)). Line \(k_1\) has the assumption \(\psi\) and line \(k-1\) has the conclusion \(\bot\). Since \(k-1 < k\), we see that there exists a proof of the sequent \(\phi_1, \ldots, \phi_n, \psi \vdash \bot\) with length less than \(k\). By the induction hypothesis, \(\bot\) must evaluate to \(T\) when \(\psi\) is \(T\). Since this is impossible, \(\psi\) evaluates to \(F\) and hence \(\neg \psi\) evaluates to \(T\).

→ \(e\): In this case, the justification in line \(k\) refers to previous lines \(k_1\) and \(k_2\) which have \(\gamma\) and \(\neg \gamma\) as their conclusions. Since \(k_1 < k\) and \(k_2 < k\), there exist proofs of \(\phi_1, \ldots, \phi_n \vdash \gamma\) and \(\phi_1, \ldots, \phi_n \vdash \neg \gamma\) with length less than \(k\). By the induction hypothesis, \(\gamma\) and \(\neg \gamma\) both evaluate to \(T\). But this is a contradiction and hence \(\psi\) is \(\bot\), as required.
\(\bot e\): In this case, the justification in line \(k\) refers to a previous line \(k_1\) which has \(\bot\) as its conclusion. Since \(k_1 < k\), there exists a proof of \(\phi_1, \ldots, \phi_n \vdash \bot\) with length less than \(k\). Now replace \(\bot\) with \(\psi \land \neg \psi\). By the induction hypothesis, this last formula must evaluate to \(T\) but it never can. But then the given premises cannot all be satisfied (compute to \(T\)). This means that we are free to select the truth value of \(\psi\) to be \(T\).

\(\neg\neg e\): In this case the justification in line \(k\) refers to a previous line \(k_1\) which has \(\neg \neg \psi\) in it. Since \(k_1 < k\), there exists a proof of \(\phi_1, \ldots, \phi_n \vdash \neg \neg \psi\) with length less than \(k\). By the induction hypothesis \(\neg \neg \psi\) evaluates to \(T\). According to the truth table of \(\neg\), \(\psi\) is also \(T\) then.

P.S. See (Bergmann et al., 1998), pages 229-234, for a much longer and hence detailed proof.

References