CS473-Algorithms I

Lecture 13

Disjoint Set Operations

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Disjoint Set Operations

A disjoint-set data structure

- Maintains a collection $S=\{S_1,\ldots,S_k\}$ of disjoint dynamic sets
- Each set is identified by a representative which is some member of the set

In some applications,

- It does not matter which member is used as the representative
- We only care that if we ask for the representative of a set twice without modifying the set between the requests,

  ✓ we get the same answer both times

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Disjoint Set Operations

In other applications, there may be a prescribed rule for choosing the representative (e.g., choose the smallest member in the set)

Each element of a set is represented by an object "x"

MAKE-SET(x) creates a new set whose only member is x

- Object x is the representative of the set
- x is not already a member of any other set

UNION(x, y) unites dynamic sets S_x & S_y that contain x & y

- S_x & S_y are assumed to be disjoint prior to the operation
- The new representative is some member of $S_x \cup S_y$

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Disjoint Set Operations

Usually, the representative of either S_x or S_y is chosen as the new representative

We destroy sets S_x and S_y, removing them from the collection S since we require the sets in the collection to be disjoint

FIND-SET(x) returns a pointer to the representative of the unique set containing x

We will analyze the running times in terms of two parameters

n : The number of MAKE-SET operations

m : The total number of MAKE-SET, UNION, and FIND-SET operations

Disjoint Set Operations

Each union operation reduces the number of sets by one since the sets are disjoint

- Therefore, only one set remains after $n - 1$ union operations
- Thus, the number of union operations is $\leq n - 1$

Also note that, $m \geq n$ always hold since MAKE-SET operations are included in the total number of operations

An Application of Disjoint-Set Data Structures

Finding the connected components of an undirected graph G=(V, E)

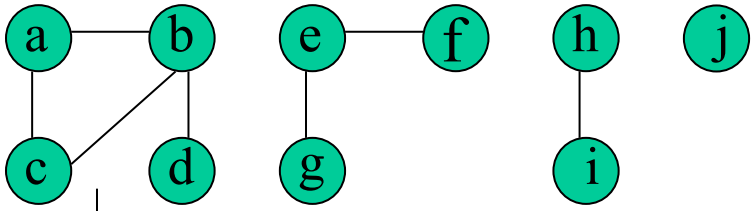
```
CONNECTED-COMPONENTS (G)
  for each vertex v ∈ V[G] do
        MAKE-SET(v)

  for each edge (u, v) ∈ E[G] do
      if   FIND-SET(u) ≠ FIND-SET(v) then
          UNION(u, v)

SAME-COMPONENT(u,v)
      if FIND-SET(u) = FIND-SET(v) then
          return TRUE
      else
          return FALSE
```

An Application of Disjoint-Set Data Structures

Finding the connected components of an undirected graph G=(V,E)



Initial	{a}	{b}	{c}	{d}	{e}	{f}	{g}	{h}	{i}	{j}

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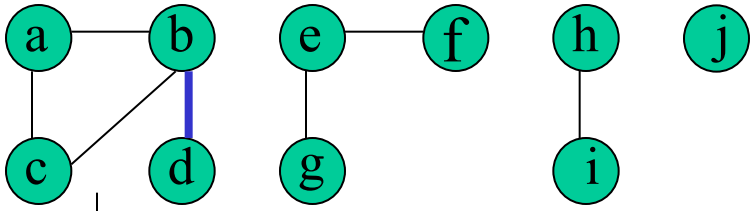
An Application of Disjoint-Set Data Structures

Finding the connected components of an undirected graph G=(V,E)



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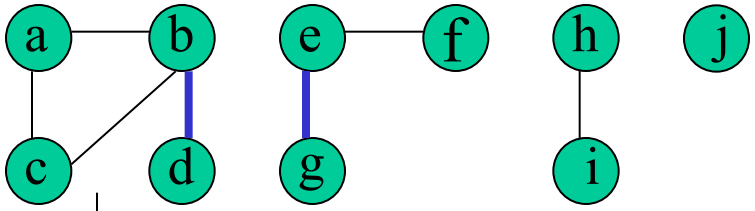
(b, d)	{a}	{b, d}	{c}		{e}	{f}	{g}	{h}	{i}	{j}

An Application of Disjoint-Set Data Structures

Finding the connected components of an undirected graph G=(V,E)



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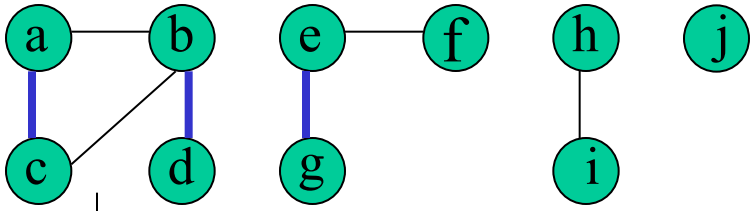
An Application of Disjoint-Set Data Structures

Finding the connected components of an undirected graph G=(V,E)



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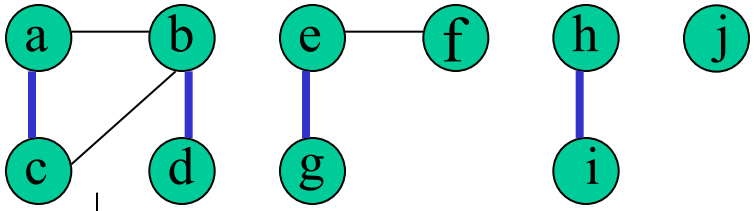
An Application of Disjoint-Set Data Structures

Finding the connected components of an undirected graph G=(V,E)



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(a, c)	{a, c}	{b, d}			{e, g}	{f}		{h}	{i}	{j}
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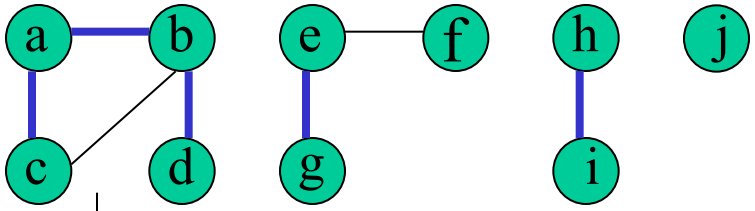
An Application of Disjoint-Set Data Structures

Finding the connected components of an undirected graph G=(V,E)



Initial	{a}	{b}	{c}	{d}	{e}	{f}	{g}	{h}	{i}	{j}
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(e, g)	{a}	{b, d}	{c}		{e, g}	{f}		{h}	{i}	{j}
(a, c)	{a, c}	{b, d}			{e, g}	{f}		{h}	{i}	{j}
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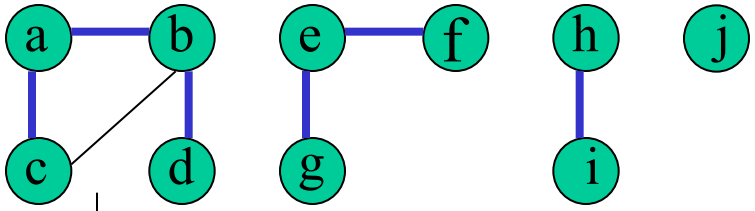
An Application of Disjoint-Set Data Structures

Finding the connected components of an undirected graph G=(V,E)



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(a, b)	{a, b, c, d}				{e, g}	{f}		{h, i}		{j}
(e, f)	{a, b, c, d}				{e, f, g}			{h, i}		{j}

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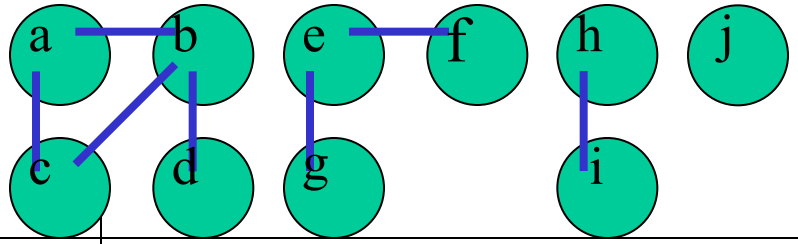
An Application of Disjoint-Set Data Structures



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(e, f)	{a, b, c, d}				{e, f, g}			{h, i}		
(b, c)	{a, b, c, d}				{e, f, g}			{h, i}		

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Linked-List Representation of Disjoint Sets

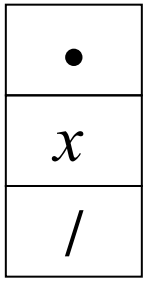
Represent each set by a linked-list

The first object in the linked-list serves as its set representative

Each object in the linked-list contains

- A set member
- A pointer to the object containing the next set member
- A pointer back to the representative

MAKE-SET(x) : $O(1)$



Set/representative pointer

Next object pointer

FIND-SET(x) : We return the representative pointer of x

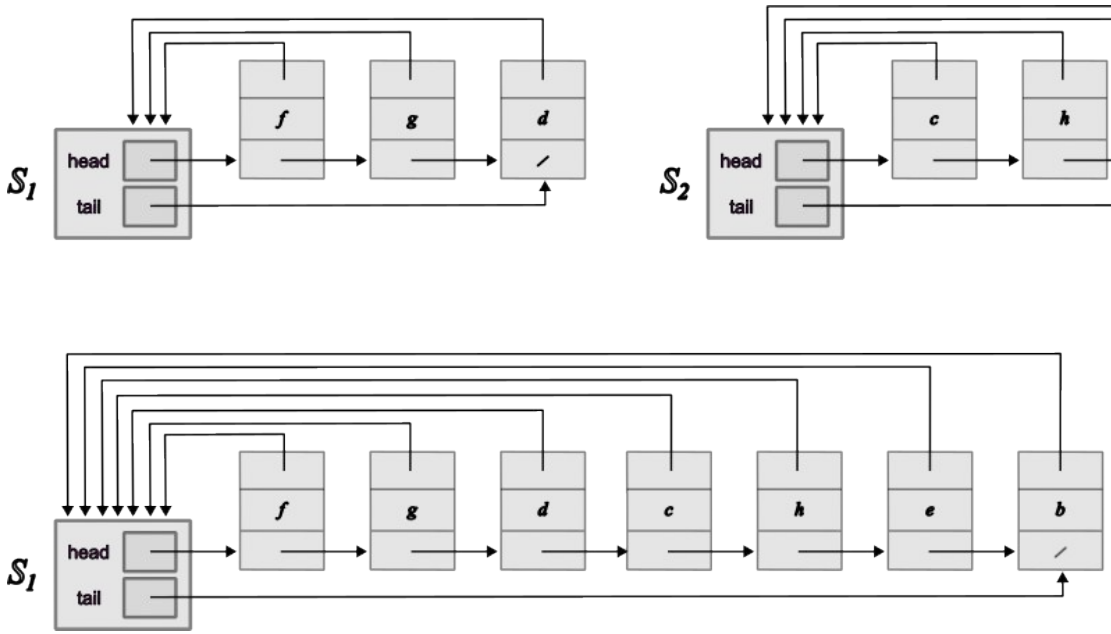
Linked-List Representation of Disjoint Sets

A Simple Implementation of Union : UNION(x,y)

APPEND x's list to the end of y 's list

The representative of y 's list becomes the new representative

UPDATE the representative pointer of each object originally on x's list which takes time linear in the length of x's list



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Analysis of Simple Union Implementation

- A sequence of m operations that requires $O(m^2)$ time
- Suppose that we have n objects $\{x_1, x_2, \ldots, x_n\}$ and let $m = 2n - 1$

Analysis of Simple Union Implementation

Operation	Number of objects updated	Updated objects (denoted in **bold green**)
MAKE-SET(x_1)	1	$\{\mathbf{x_1}\}$
MAKE-SET(x_2)	1	$\{\mathbf{x_2}\}$
…	…	…
MAKE-SET(x_n)	1	$\{\mathbf{x_n}\}$
UNION(x_1,x_2)	1	$\{x_1\} \cup \{x_2\} = \{x_1,\mathbf{x_2}\}$
UNION(x_2,x_3)	2	$\{x_1,x_2\} \cup \{x_3\} = \{x_1,\mathbf{x_2},\mathbf{x_3}\}$
UNION(x_3,x_4)	3	$\{x_1,x_2,x_3\} \cup \{x_4\} = \{x_1,\mathbf{x_2},\mathbf{x_3},\mathbf{x_4}\}$
…	…	…
UNION(x_{n-1},x_n)	n - 1	$\{x_1,x_2,\ldots,x_{n-1}\} \cup \{x_n\} = \{x_1,\mathbf{x_2},\ldots,\mathbf{x_n}\}$

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Analysis of Simple Union Implementation

The total number of representative pointer updates

$$= n + \sum_{i=1}^{n-1} i = n + \frac{1}{2}(n-1)n = \frac{1}{2}n^2 + \frac{1}{2}n = \Theta(n^2)$$

MAKE-SET operations

UNION operations

$= \Theta(m^2)$ since $n = \lceil m/2 \rceil$

Thus, on the average, each operation requires $\Theta(m)$ time
That is, the amortized time of an operation is $\Theta(m)$

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A Weighted-Union Heuristic

The simple implementation is inefficient because

- We may be appending a longer list to a shorter list during a UNION operation
- so that we must update the representative pointer of each member of the longer list

The weighted-union heuristic

- Maintains the length of each list
- Always appends the smaller list to the longer list (with ties broken arbitrarily)

!! A single UNION can still take $\Omega(m)$ time if both sets have $\Omega(m)$ members

Weighted Union Heuristic

Theorem: A sequence of m MAKE-SET, UNION & FIND-SET operations, n of which are MAKE-SET operations, takes $O(m+n \lg n)$ time

Proof: Try to compute an upper bound on the number of representative pointer updates for each object in a set of size n

Consider a fixed object x. Each time x's R-PTR was updated, x was a member of the smaller set:

$\{x\} \cup \{v\} = \{/, v\}$ 1st update $|S_x| \geq 2$

$\{x, v\} \cup \{w_1, w_2\} = \{/, v, w_1, w_2\}$ 2nd update $|S_x| \geq 4$

$\{x, v, w_1, w_2\} \cup \{z_1, z_2, z_3, z_4\} = \{/, v, w_1, w_2, z_1, z_2, z_3, z_4\}$; $|S_x| \geq 4$

3rd update $|S| \geq 8$

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Weighted Union Heuristic - Analysis

- For any k ≤ n, after x's R-PTR has been updated $\lceil \lg k \rceil$ times, the resulting set must have at least k members
- R-PTR of each object can be updated at most $\lceil \lg n \rceil$ time over all UNION operations
- The figure below illustrates a worst case sequence for a set with $n = 16$ objects
- The total number of R-PTR updates

$$= \frac{16}{2}\times 1 + \frac{16}{4}\times 2 + \frac{16}{8}\times 4 + \frac{16}{16}\times 8 = 8\times 1 + 4\times 2 + 2\times 4 + 1\times 8 = 8\times 4 = 32$$

$$= \underbrace{\frac{n}{2} + \frac{n}{2} + \ldots\ldots + \frac{n}{2}}_{\lg n} = \frac{n}{2}\lg n = O(n \lg n)$$

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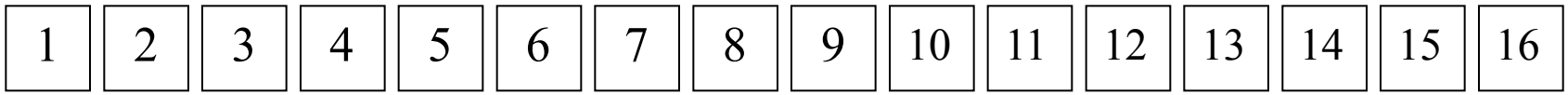
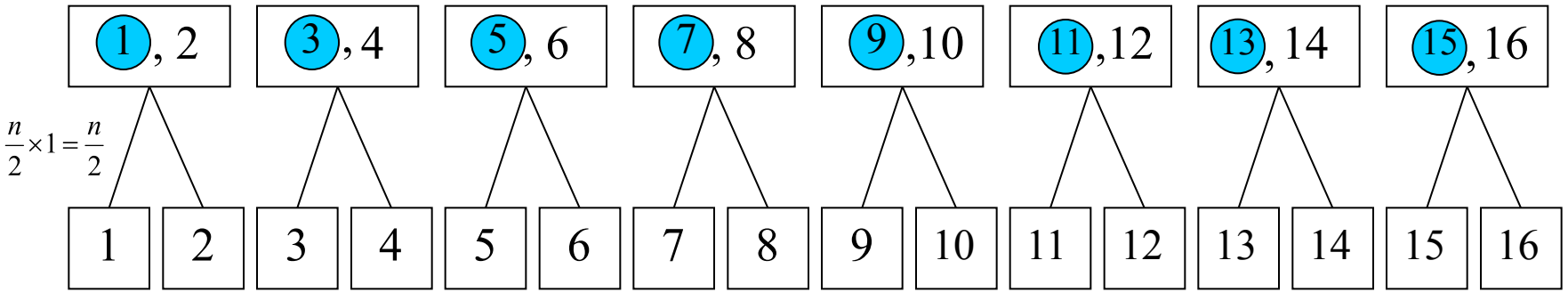
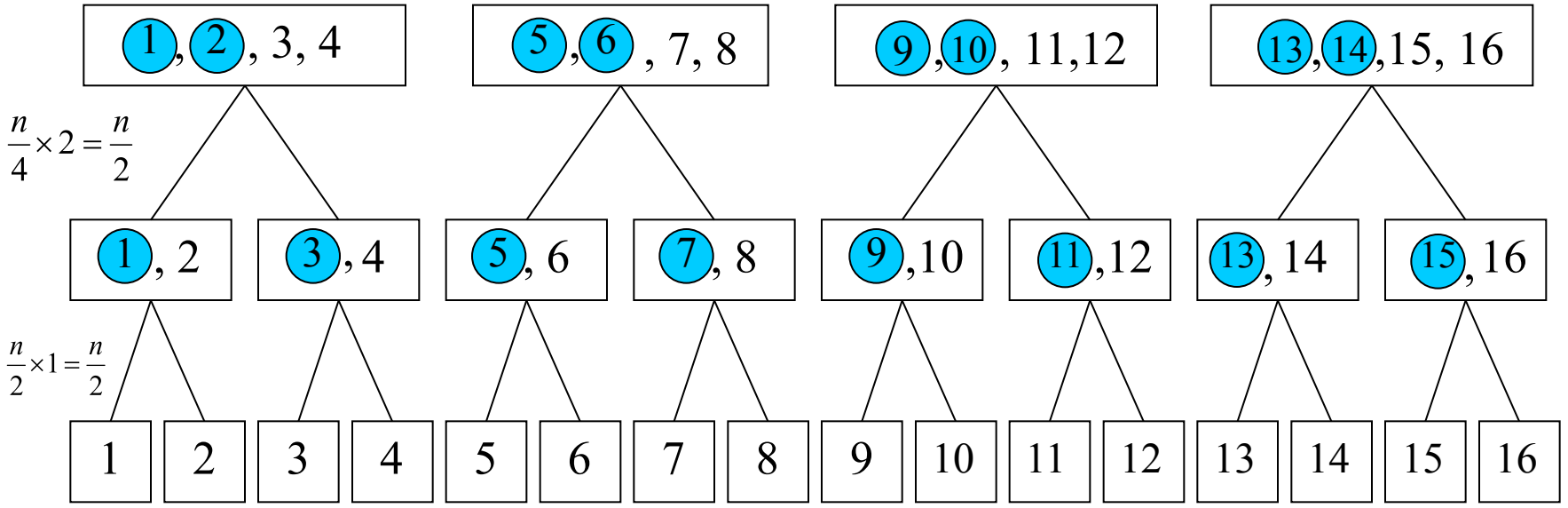
Weighted Union Heuristic - Analysis



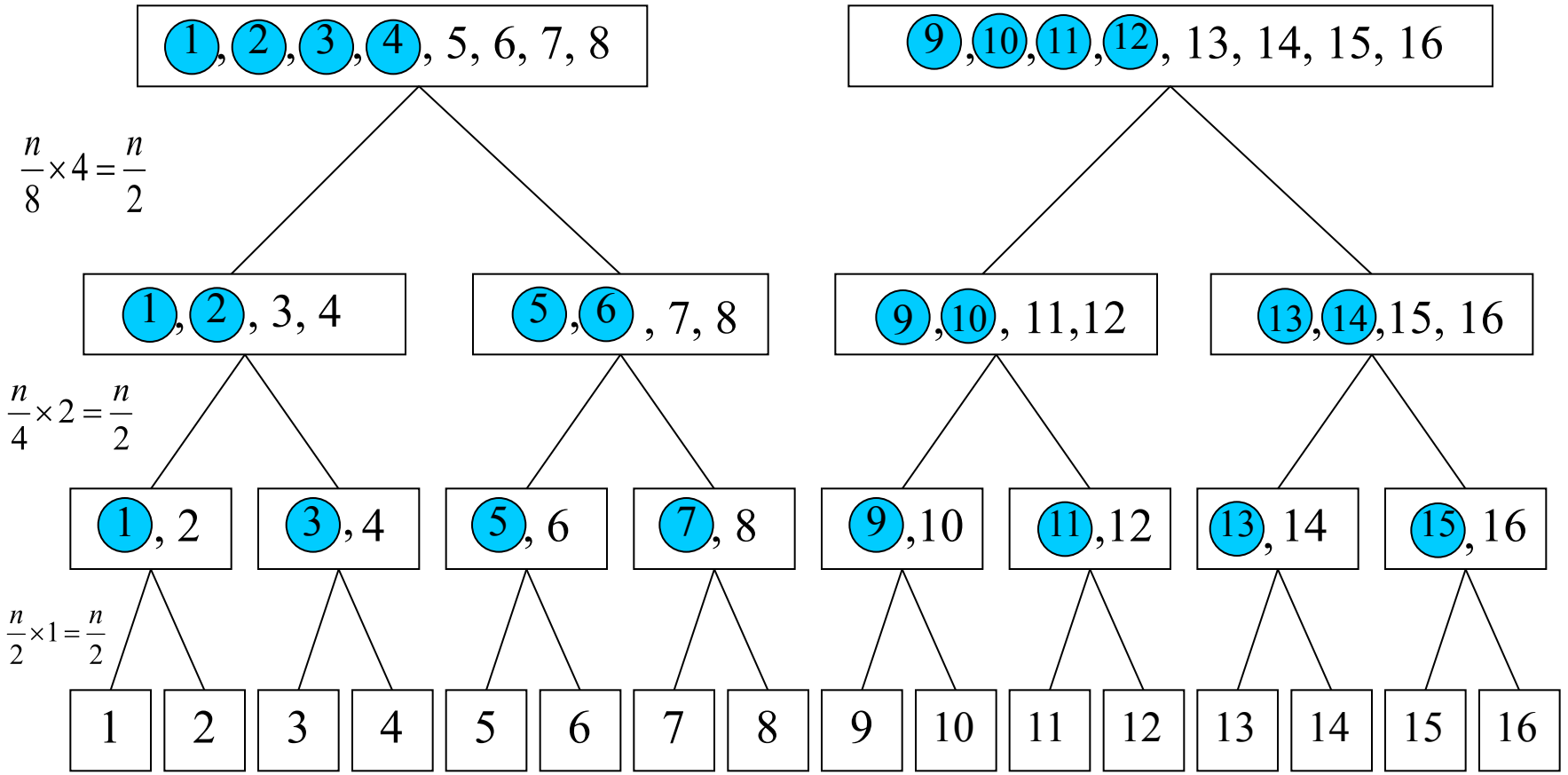
Weighted Union Heuristic - Analysis



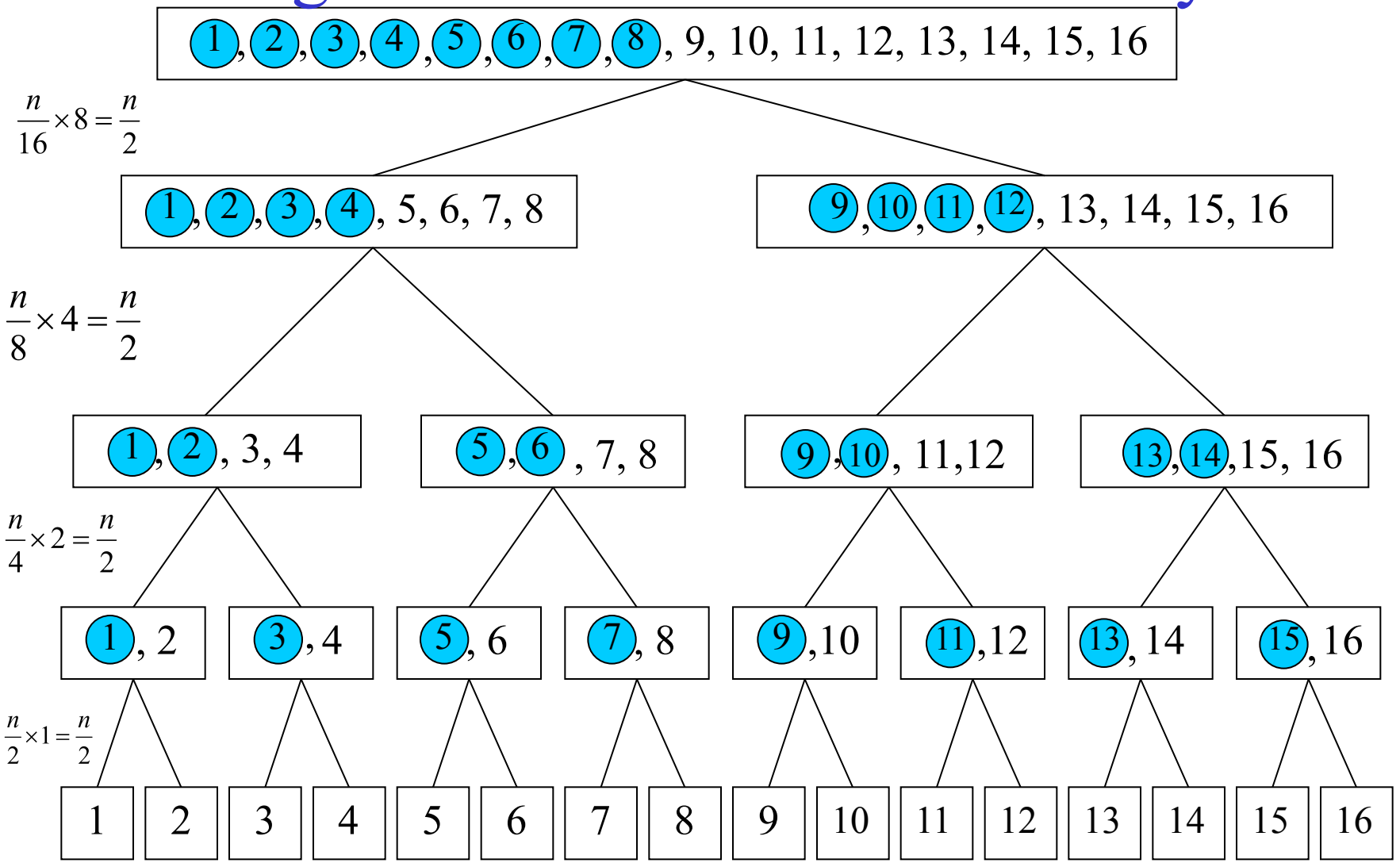
Weighted Union Heuristic - Analysis



Weighted Union Heuristic - Analysis



Weighted Union Heuristic - Analysis



Weighted Union Heuristic - Analysis

Each MAKE-SET & FIND-SET operation takes $O(1)$ time, and there are $O(m)$ of them

The total time for the entire sequence

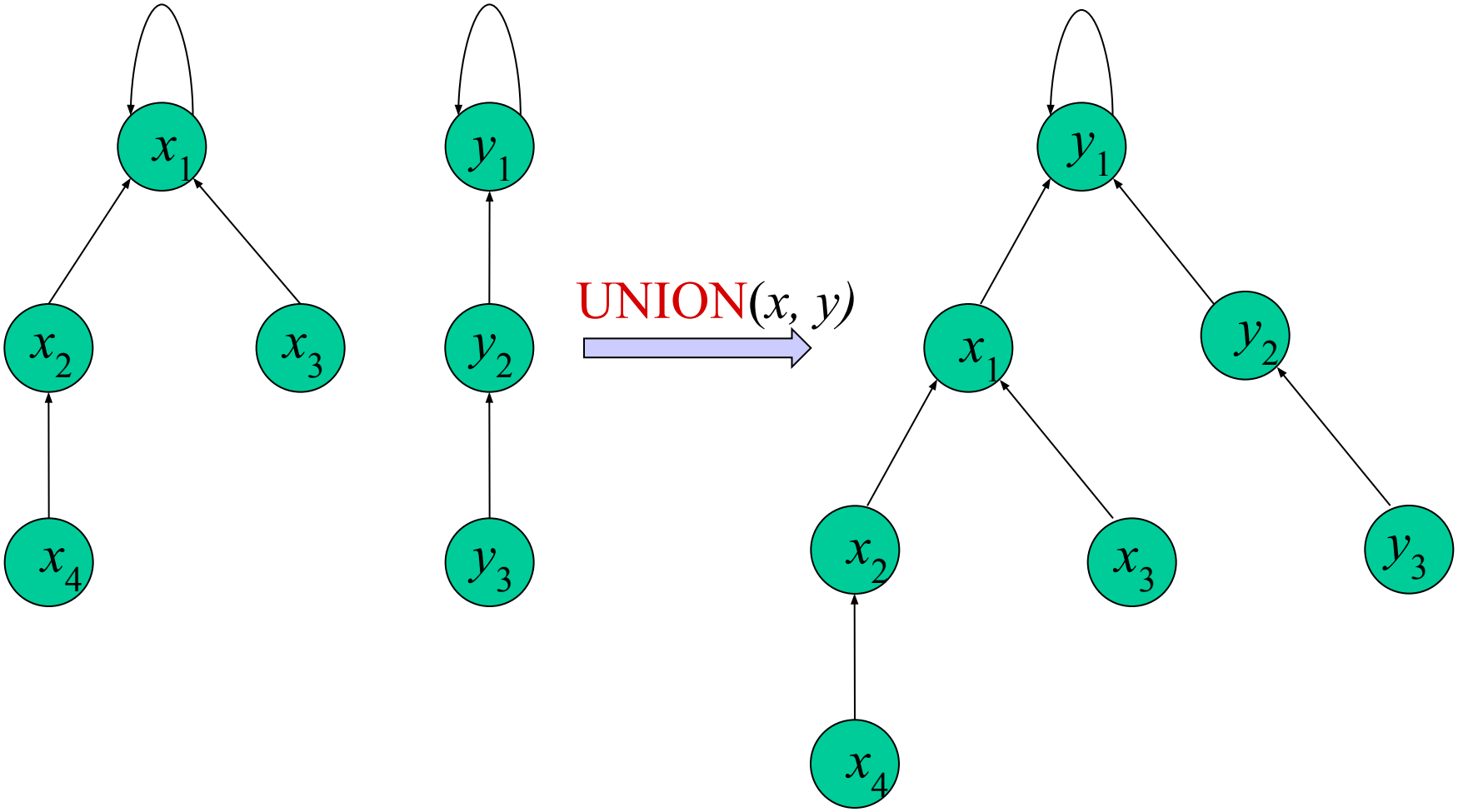
$= O(m+n \lg n)$

Disjoint Set Forests

In a faster implementation, we represent sets by rooted trees

- Each node contains one member
- Each tree represents one set
- Each member points only to its parent
- The root of each tree contains the representative
- Each root is its own parent

Disjoint Set Forests



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Disjoint Set Forests

Straightforward Implementation

MAKE-SET : Simply creates a tree with just one node : $O(1)$

FIND-SET : Follows parent pointers until the root node is found
The nodes visited on this path toward the root constitute the FIND-PATH

UNION : Makes the root of one tree to point to the other one

Heuristics To Improve the Running Time

- Straightforward implementation is no faster than ones that use the linked-list representation
- A sequence of $n - 1$ UNIONs, following a sequence of n MAKE-SETs, may create a tree, which is just a linear chain of n nodes

Heuristics To Improve the Running Time

First Heuristic : UNION by Rank

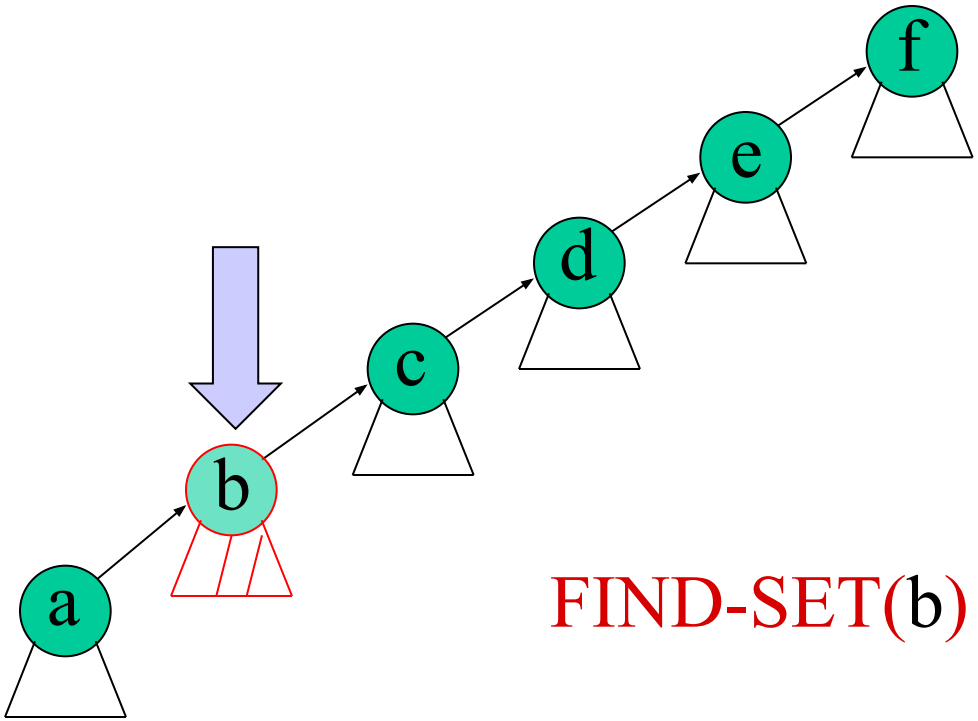
- Similar to the weighted-union used for the linked-list representation
- The idea is to make the root of the tree with fewer nodes point to the root of the tree with more nodes
- Rather than explicitly keeping the size of the subtree rooted at each node

  We maintain a rank
  - that approximates the logarithm of the subtree size
  - and is also an upper bound on the height of the node
- During a UNION operation
  - make the root with smaller rank to point to the root with larger rank

Heuristics To Improve the Running Time

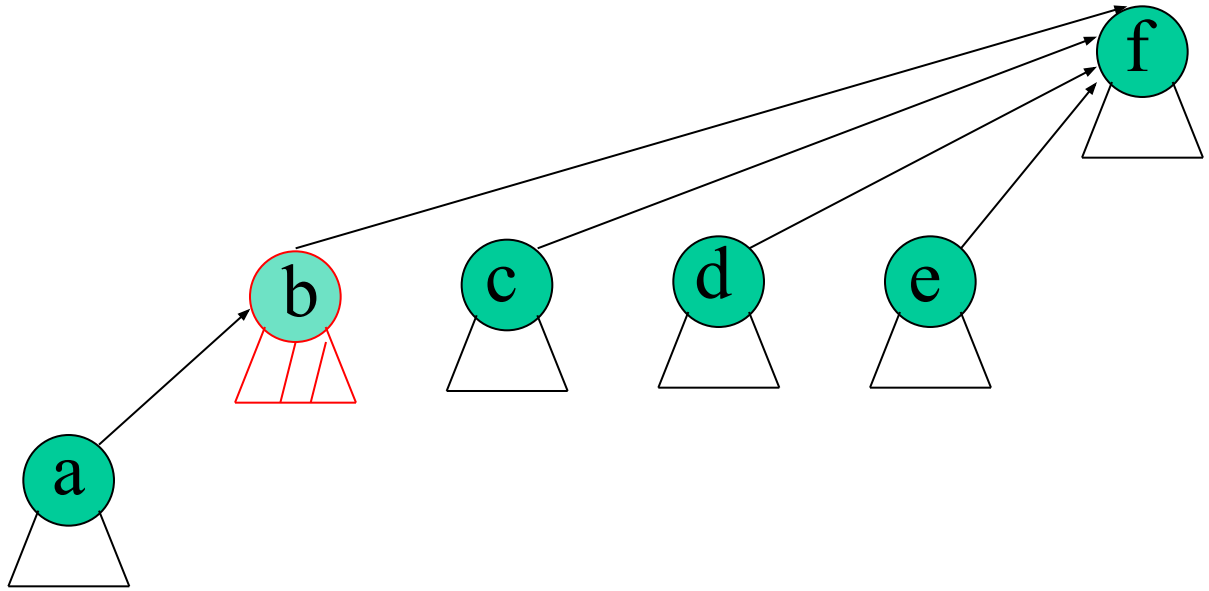
Second Heuristic : Path Compression

- Use it during the FIND-SET operations
- Make each node on the FIND-PATH to point directly to the root



Heuristics To Improve the Running Time

Path Compression During FIND-SET(b) Operation



Pseudocodes For the Heuristics

Implementation of UNION-BY-RANK Heuristic

p[x] : Pointer to the parent of the node x
rank[x] : An upper bound on the height of node x in the tree

```
MAKE-SET(x)
    p[x] ← x
    rank[x] ← 0

LINK(x,y)
    if rank[x] > rank[y] then
        p[y] ← x
    else
        p[x] ← y
    if rank[x] = rank[y]  then
        rank[y] = rank[y] + 1
```

```
UNION(x,y)
    LINK(FIND-SET(x),FIND-SET(y))
```

Implementation of UNION-BY-RANK Heuristic

– When a singleton set is created by a MAKE-SET
the initial rank of the single node in the tree is zero

– Each FIND-SET operation leaves all ranks unchanged

– When applying a UNION to two trees,
we make the root of tree with higher rank
the parent of the root of lower rank

Ties are broken arbitrarily

Implementation of the Path-Compression Heuristic

The FIND-SET procedure with Path-Compression

Iterative Version:

```
FIND-SET(x)
    y ← x
    while y ≠ p[y]  do
        y ← p[y]

    root ← y
    while x ≠ p[x]  do
        parent ← p[x]
        p[x] ← root
        x ← parent
    return root
```

Recursive Version:

```
FIND-SET(x)
    if  x ≠ p[x]  then
        p[x] ← FIND-SET(p[x])
    return p[x]
```