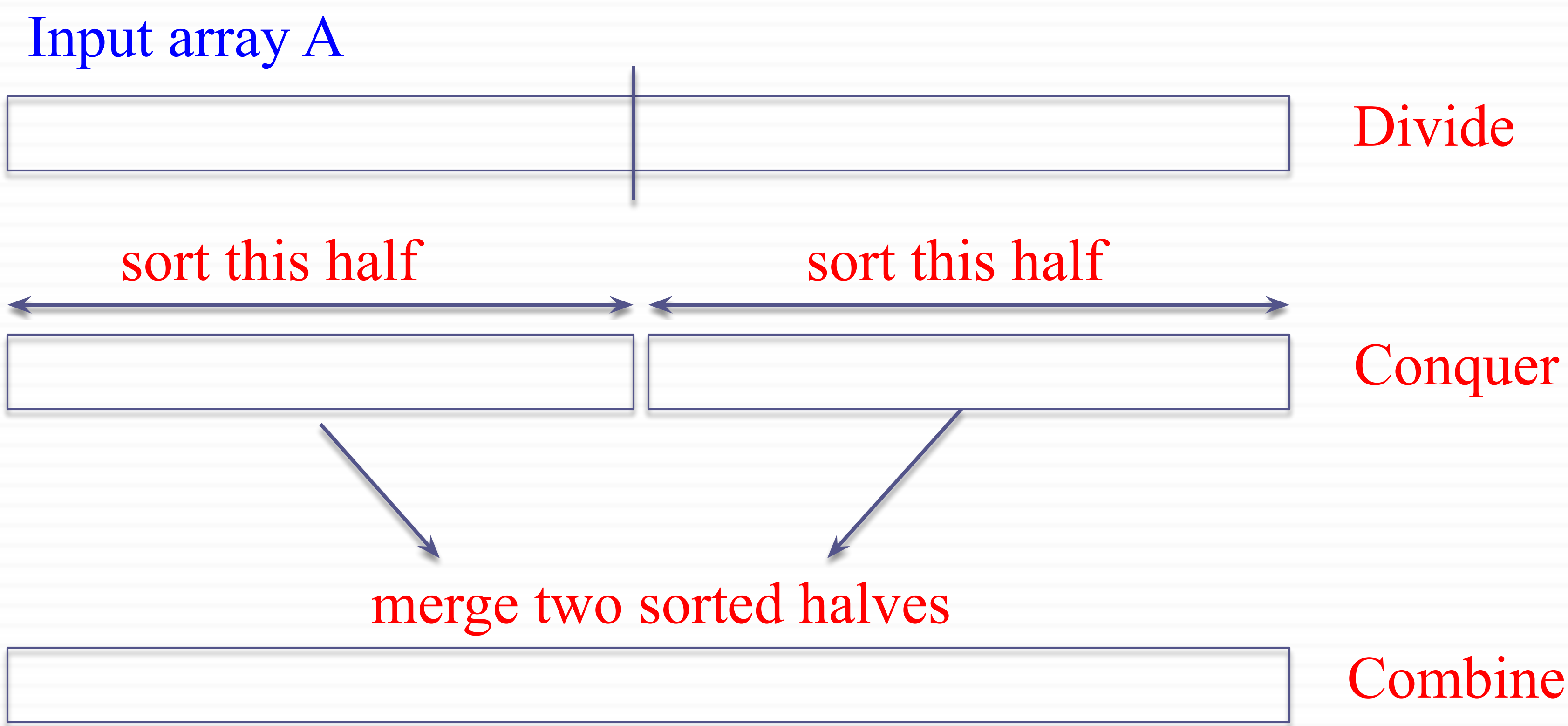
CS473 - Algorithms I

Lecture 4
The Divide-and-Conquer Design Paradigm

View in slide-show mode

Cevdet Aykanat and Mustafa Ozdal
Computer Engineering Department, Bilkent University

Reminder: Merge Sort



The Divide-and-Conquer Design Paradigm

1. ***Divide*** **the problem (instance) into subproblems.**
2. ***Conquer*** **the subproblems by solving them recursively.**
3. ***Combine*** **subproblem solutions.**

Example: Merge Sort

1. ***Divide***: Trivial.
2. ***Conquer***: Recursively sort 2 subarrays.
3. ***Combine***: Linear- time merge.

$$T(n) = \underbrace{2}_{\text{\# subproblems}}\, T(\underbrace{n/2}_{\text{subproblem size}}) + \underbrace{\Theta(n)}_{\text{work dividing and combining}}$$

Master Theorem: Reminder

T(n) = aT(n/b) + f(n)

Case 1: $\dfrac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon})$ → $T(n) = \Theta(n^{\log_b a})$

Case 2: $\dfrac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)$ → $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

Case 3: $\dfrac{f(n)}{n^{\log_b a}} = \Omega(n^{\varepsilon})$

and $a\, f(n/b) \le c\, f(n)$ for $c < 1$ → $T(n) = \Theta(f(n))$

Merge Sort: Solving the Recurrence

$$T(n) = 2\,T(n/2) + \Theta(n)$$

$\Rightarrow$ $a = 2$, $b = 2$, $f(n) = \Theta(n)$, $n^{\log_b a} = n$

Case 2: $\dfrac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)$ $\Rightarrow$ $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

holds for k = 0

$\Rightarrow$ $T(n) = \Theta\,(n\lg n)$

Binary Search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.

2. ***Conquer:*** Recursively search 1 subarray.

3. ***Combine:*** Trivial.

Example: Find **9**

3 5 7 8 9 12 15

Recurrence for Binary Search

$$T(n) = 1\, T(n/2) + \Theta(1)$$

 # subproblems

subproblem size

work dividing and combining

Binary Search: Solving the Recurrence

$$T(n) = T(n/2) + \Theta(1)$$

$a = 1, \quad b = 2, \quad f(n) = \Theta(1), \quad n^{\log_b a} = n^0 = 1$

Case 2: $\dfrac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n) \Rightarrow T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

holds for k = 0

$T(n) = \Theta(\lg n)$

Powering a Number

- Problem: Compute a^n, where n is a natural number

```
Naive-Power (a, n)
    powerVal ← 1
    for i ← 1 to n
        powerVal ← powerVal . a
    return powerVal
```

- What is the complexity? $T(n) = \Theta(n)$

Powering a Number: Divide & Conquer

Basic idea:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if n is \underline{even}} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if n is \underline{odd}} \end{cases}$$

Powering a Number: Divide & Conquer

```
POWER (a, n)
    if n = 0 then return 1

    else if n is even then
        val ← POWER (a, n/2)
        return val * val

    else if n is odd then
        val ← POWER (a, (n-1)/2)
        return val * val * a
```

Powering a Number: Solving the Recurrence

$$T(n) = T(n/2) + \Theta(1)$$

➡ $a = 1, \quad b = 2, \quad f(n) = \Theta(1), \quad n^{\log_b a} = n^0 = 1$

Case 2: $\dfrac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)$ ➡ $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

holds for k = 0

➡ $T(n) = \Theta(\lg n)$

Matrix Multiplication

$$\left.\begin{array}{l}\textbf{Input} \quad : A = [a_{ij}],\ B = [b_{ij}] \\ \textbf{Output:}\ C = [c_{ij}] = A \cdot B\end{array}\right\}\ i\ ,\ j = 1,\ 2,\ \ldots\ ,\ n$$

$$\begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \ldots & & & \\ c_{n1} & c_{n2} & \ldots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & & & \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{pmatrix}$$

$$c_{ij} = \sum_{1 \leq k \leq n} a_{ik} . b_{kj}$$

Standard Algorithm

for $i \leftarrow 1$ to n

do for $j \leftarrow 1$ to n

do $c_{ij} \leftarrow 0$

for $k \leftarrow 1$ to n

do $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$

Running time = $\Theta(n^3)$

Matrix Multiplication: Divide & Conquer

IDEA: Divide the $n \times n$ matrix into

2×2 matrix of $(n/2) \times (n/2)$ submatrices

$$
\begin{array}{ccccc}
C & & A & & B \\
\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} & = & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & \cdot & \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}
\end{array}
$$

$$c_{11} = a_{11} b_{11} + a_{12} b_{21}$$

Matrix Multiplication: Divide & Conquer

IDEA: Divide the n x n matrix into

2x2 matrix of (n/2)x(n/2) submatrices

$$
\begin{array}{ccccc}
C & & A & & B \\
\left(\begin{array}{c|c} c_{11} & c_{12} \\ \hline c_{21} & c_{22} \end{array}\right) & = & \left(\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right) & \cdot & \left(\begin{array}{c|c} b_{11} & b_{12} \\ \hline b_{21} & b_{22} \end{array}\right)
\end{array}
$$

$$c_{12} = a_{11} b_{12} + a_{12} b_{22}$$

Matrix Multiplication: Divide & Conquer

IDEA: Divide the n x n matrix into

2x2 matrix of (n/2)x(n/2) submatrices

$$\begin{array}{ccccc} C & & A & & B \\ \left(\begin{array}{c|c} c_{11} & c_{12} \\ \hline c_{21} & c_{22} \end{array}\right) & = & \left(\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right) & \cdot & \left(\begin{array}{c|c} b_{11} & b_{12} \\ \hline b_{21} & b_{22} \end{array}\right) \end{array}$$

$$c_{21} = a_{21}\, b_{11} + a_{22}\, b_{21}$$

Matrix Multiplication: Divide & Conquer

IDEA: Divide the n x n matrix into

2x2 matrix of (n/2)x(n/2) submatrices

$$\underset{C}{\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & \boxed{c_{22}} \end{pmatrix}} = \underset{A}{\begin{pmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & \boxed{a_{22}} \end{pmatrix}} \cdot \underset{B}{\begin{pmatrix} b_{11} & \boxed{b_{12}} \\ b_{21} & \boxed{b_{22}} \end{pmatrix}}$$

$$c_{22} = a_{21}\, b_{12} + a_{22}\, b_{22}$$

Matrix Multiplication: Divide & Conquer

$$\overset{C}{\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}} = \overset{A}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \cdot \overset{B}{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}}$$

$$\left.\begin{aligned} c_{11} &= a_{11} b_{11} + a_{12} b_{21} \\ c_{12} &= a_{11} b_{12} + a_{12} b_{22} \\ c_{21} &= a_{21} b_{11} + a_{22} b_{21} \\ c_{22} &= a_{21} b_{12} + a_{22} b_{22} \end{aligned}\right\}$$

8 mults of $(n/2)\times(n/2)$ submatrices

4 adds of $(n/2)\times(n/2)$ submatrices

Matrix Multiplication: Divide & Conquer

MATRIX-MULTIPLY (A, B)

// Assuming that both A and B are nxn matrices

if $n = 1$ **then return** $A * B$

else

partition A, B, and C as shown before

c_{11} = MATRIX-MULTIPLY (a_{11}, b_{11}) + MATRIX-MULTIPLY (a_{12}, b_{21})

c_{12} = MATRIX-MULTIPLY (a_{11}, b_{12}) + MATRIX-MULTIPLY (a_{12}, b_{22})

c_{21} = MATRIX-MULTIPLY (a_{21}, b_{11}) + MATRIX-MULTIPLY (a_{22}, b_{21})

c_{22} = MATRIX-MULTIPLY (a_{21}, b_{12}) + MATRIX-MULTIPLY (a_{22}, b_{22})

return C

Matrix Multiplication: Divide & Conquer Analysis

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

8 recursive calls

each subproblem has size n/2

submatrix addition

Matrix Multiplication: Solving the Recurrence

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

$\Rightarrow$ $a = 8,\quad b = 2,\quad f(n) = \Theta(n^2),\quad n^{\log_b a} = n^3$

Case 1: $\dfrac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon})$ $\Rightarrow$ $T(n) = \Theta(n^{\log_b a})$

$\Rightarrow$ $T(n) = \Theta\,(n^3)$

No better than the ordinary algorithm!

Matrix Multiplication: Strassen's Idea

$$\overset{\textstyle C}{\left(\begin{array}{c|c} c_{11} & c_{12} \\ \hline c_{21} & c_{22} \end{array}\right)} = \overset{\textstyle A}{\left(\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right)} \cdot \overset{\textstyle B}{\left(\begin{array}{c|c} b_{11} & b_{12} \\ \hline b_{21} & b_{22} \end{array}\right)}$$

Compute $\mathbf{c_{11}}$, $\mathbf{c_{12}}$, $\mathbf{c_{21}}$, and $\mathbf{c_{22}}$ using **7** recursive multiplications

Matrix Multiplication: Strassen's Idea

$$P_1 = a_{11} \mathbf{x} (b_{12} - b_{22})$$
$$P_2 = (a_{11} + a_{12}) \mathbf{x} b_{22}$$
$$P_3 = (a_{21} + a_{22}) \mathbf{x} b_{11}$$
$$P_4 = a_{22} \mathbf{x} (b_{21} - b_{11})$$
$$P_5 = (a_{11} + a_{22}) \mathbf{x} (b_{11} + b_{22})$$
$$P_6 = (a_{12} - a_{22}) \mathbf{x} (b_{21} + b_{22})$$
$$P_7 = (a_{11} - a_{21}) \mathbf{x} (b_{11} + b_{12})$$

Reminder: Each submatrix is of size $(n/2)x(n/2)$

Each add/sub operation takes $\Theta(n^2)$ time

Compute $P_1..P_7$ using 7 recursive calls to matrix-multiply

How to compute c_{ij} using $P_1.. P_7$?

Matrix Multiplication: Strassen's Idea

$$P_1 = a_{11} \mathbf{x} (b_{12} - b_{22})$$
$$P_2 = (a_{11} + a_{12}) \mathbf{x} b_{22}$$
$$P_3 = (a_{21} + a_{22}) \mathbf{x} b_{11}$$
$$P_4 = a_{22} \mathbf{x} (b_{21} - b_{11})$$
$$P_5 = (a_{11} + a_{22}) \mathbf{x} (b_{11} + b_{22})$$
$$P_6 = (a_{12} - a_{22}) \mathbf{x} (b_{21} + b_{22})$$
$$P_7 = (a_{11} - a_{21}) \mathbf{x} (b_{11} + b_{12})$$

$$c_{11} = P_5 + P_4 - P_2 + P_6$$
$$c_{12} = P_1 + P_2$$
$$c_{21} = P_3 + P_4$$
$$c_{22} = P_5 + P_1 - P_3 - P_7$$

7 recursive multiply calls
18 add/sub operations

Does not rely on commutativity of multiplication

Matrix Multiplication: Strassen's Idea

$P_1 = a_{11} \mathbf{x} (b_{12} - b_{22})$

$P_2 = (a_{11} + a_{12}) \mathbf{x} b_{22}$

$P_3 = (a_{21} + a_{22}) \mathbf{x} b_{11}$

$P_4 = a_{22} \mathbf{x} (b_{21} - b_{11})$

$P_5 = (a_{11} + a_{22}) \mathbf{x} (b_{11} + b_{22})$

$P_6 = (a_{12} - a_{22}) \mathbf{x} (b_{21} + b_{22})$

$P_7 = (a_{11} - a_{21}) \mathbf{x} (b_{11} + b_{12})$

e.g. Show that $c_{12} = P_1 + P_2$

$$
\begin{aligned}
c_{12} &= P_1 + P_2 \\
&= a_{11}(b_{12} - b_{22}) + (a_{11} + a_{12})b_{22} \\
&= a_{11}b_{12} - a_{11}b_{22} + a_{11}b_{22} + a_{12}b_{22} \\
&= a_{11}b_{12} + a_{12}b_{22}
\end{aligned}
$$

Strassen's Algorithm

1. **Divide**: Partition A and B into (n/2) x (n/2) submatrices. Form terms to be multiplied using + and –.

2. **Conquer**: Perform 7 multiplications of (n/2) x (n/2) submatrices recursively.

3. **Combine**: Form C using + and – on (n/2) x (n/2) submatrices.

Recurrence: $T(n) = 7\,T(n/2) + \Theta(n^2)$

Strassen's Algorithm: Solving the Recurrence

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

$\Rightarrow$ $a = 7$, $b = 2$, $f(n) = \Theta(n^2)$, $n^{\log_b a} = n^{\lg 7}$

Case 1: $\frac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon})$ $\Rightarrow$ $T(n) = \Theta(n^{\log_b a})$

$\Rightarrow$ $T(n) = \Theta\,(n^{\lg 7})$

Note: $\lg 7 \approx 2.81$

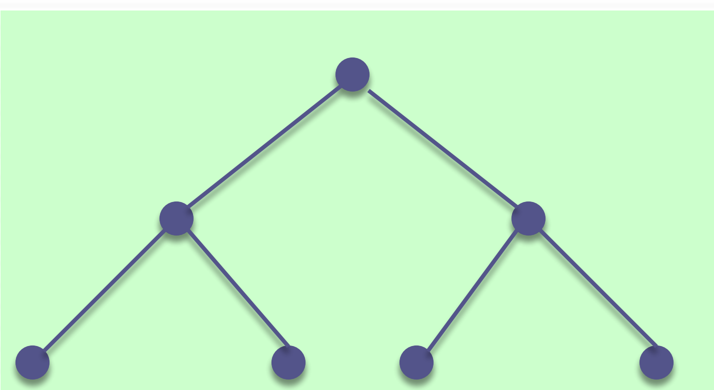
Strassen's Algorithm

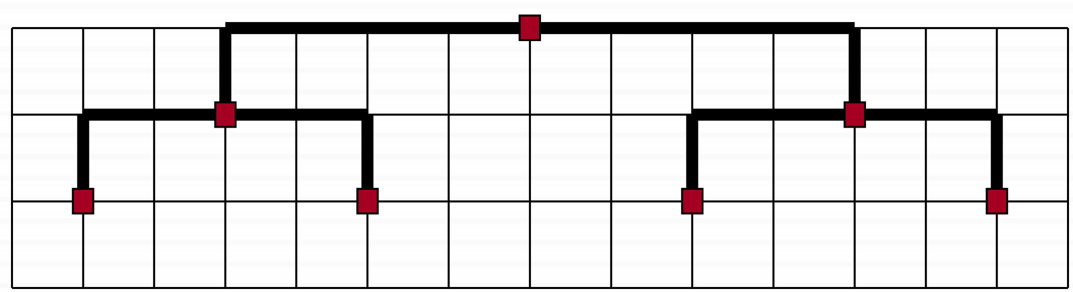
- The number 2.81 may not seem much smaller than 3
- But, it is significant because the difference is in the exponent.
- Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 30$ or so.
- Best to date: $\Theta(n^{2.376...})$ *(of theoretical interest only)*

VLSI Layout: Binary Tree Embedding

- Problem: Embed a complete binary tree with n leaves into a 2D grid with minimum area.

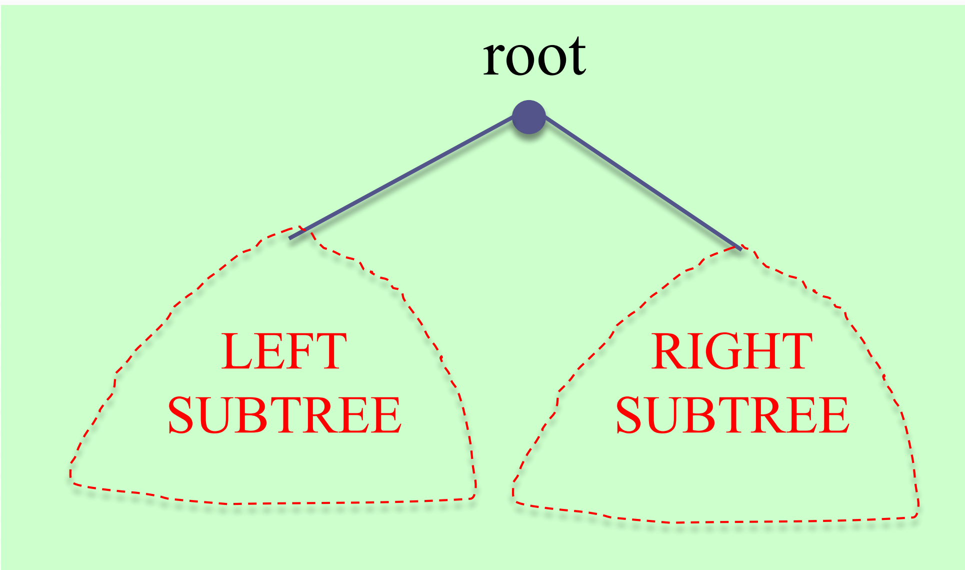
- Example:





Binary Tree Embedding

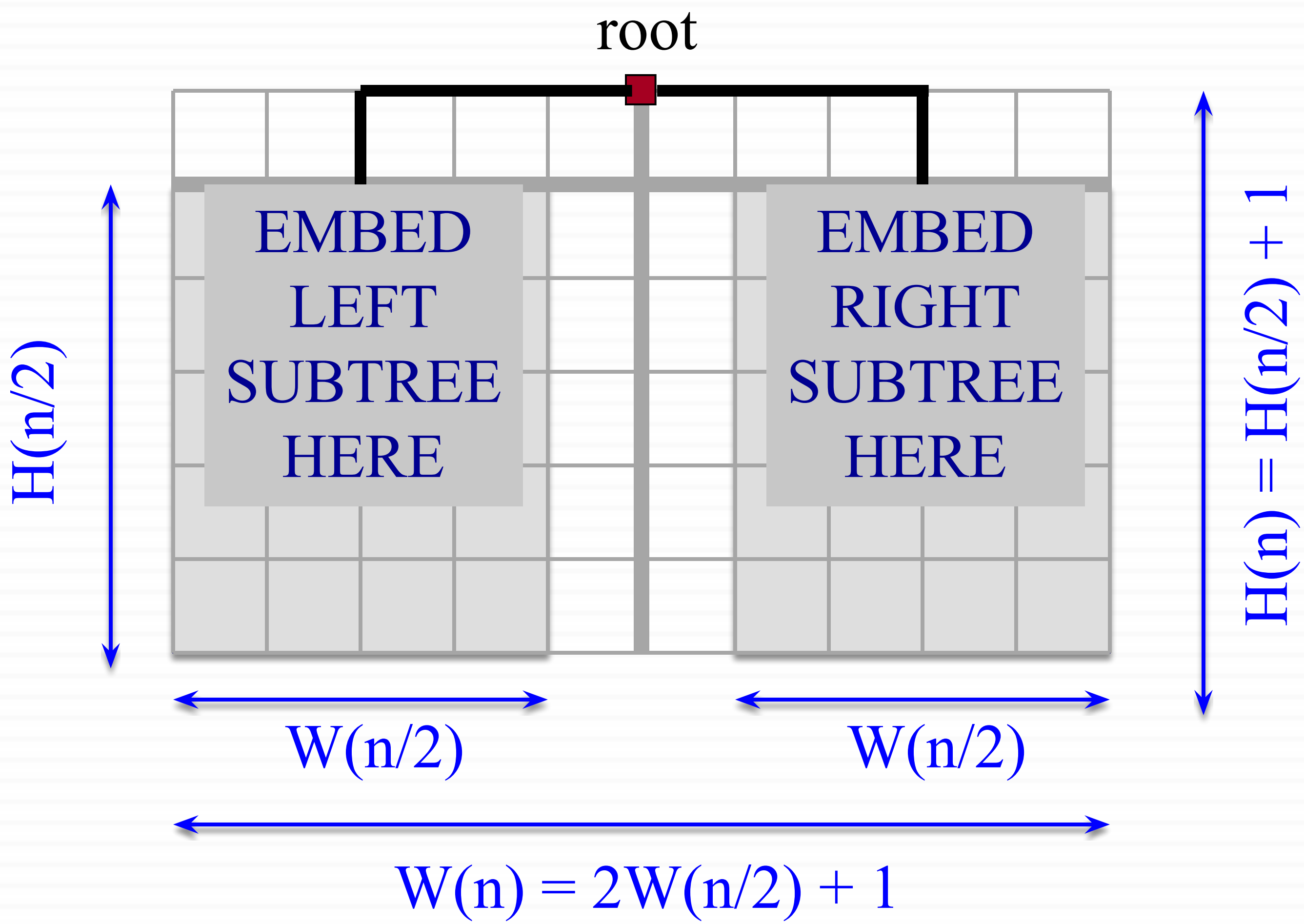
- Use divide and conquer



1. Embed the root node
2. Embed the left subtree
3. Embed the right subtree

What is the min-area required for n leaves?

Binary Tree Embedding



Binary Tree Embedding

- Solve the recurrences:

  $W(n) = 2W(n/2) + 1$

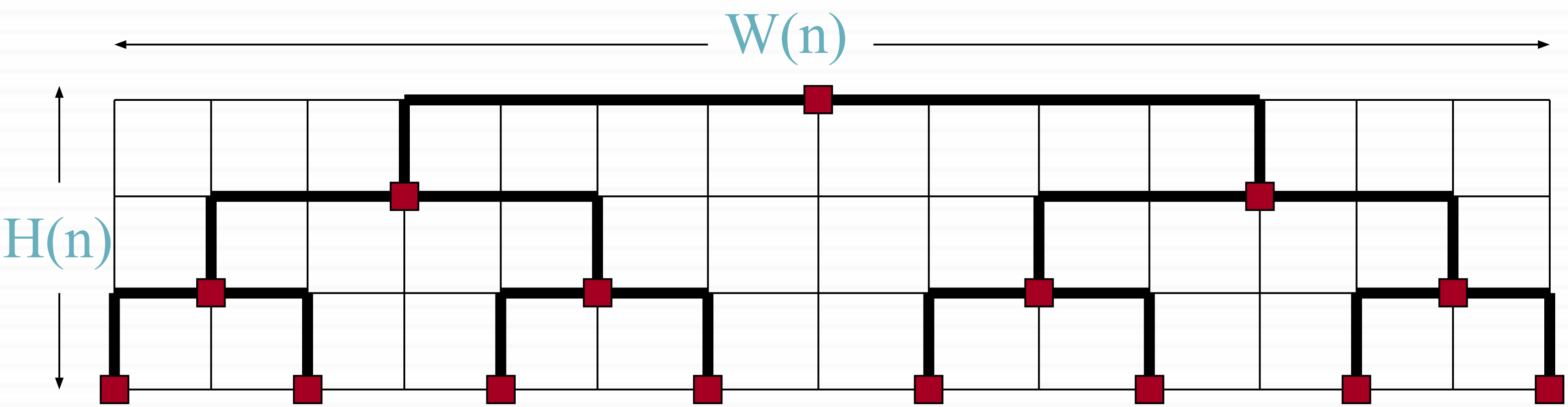
  $H(n) = H(n/2) + 1$

  - $W(n) = \Theta(n)$
  - $H(n) = \Theta(\lg n)$

- $Area(n) = \Theta(n \lg n)$

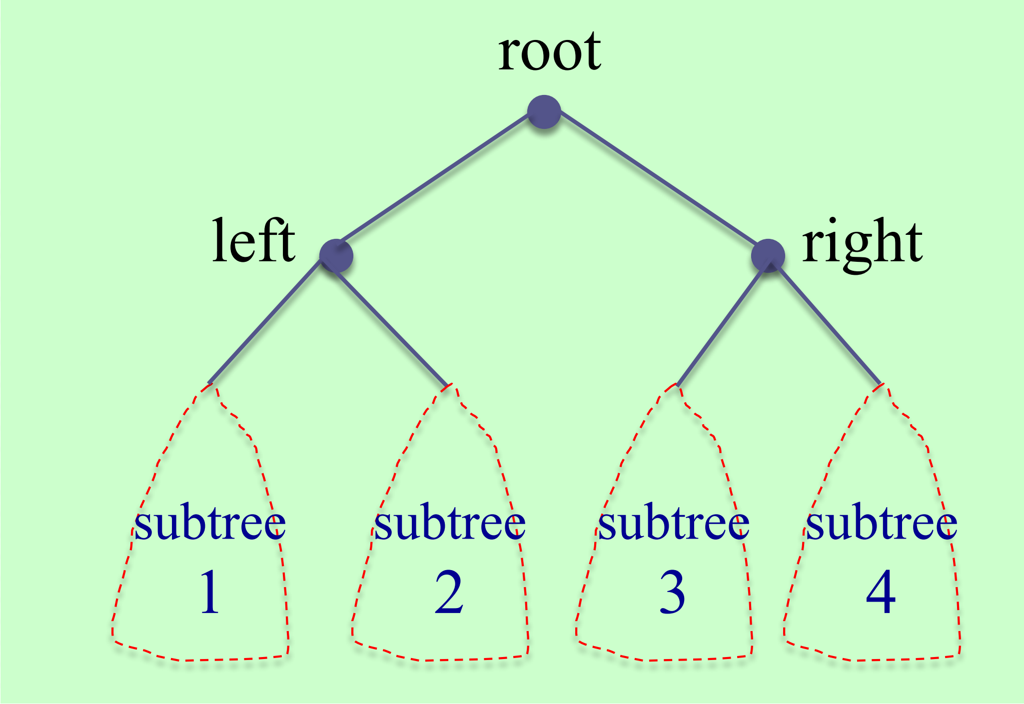
Binary Tree Embedding

Example:



Binary Tree Embedding: H-Tree

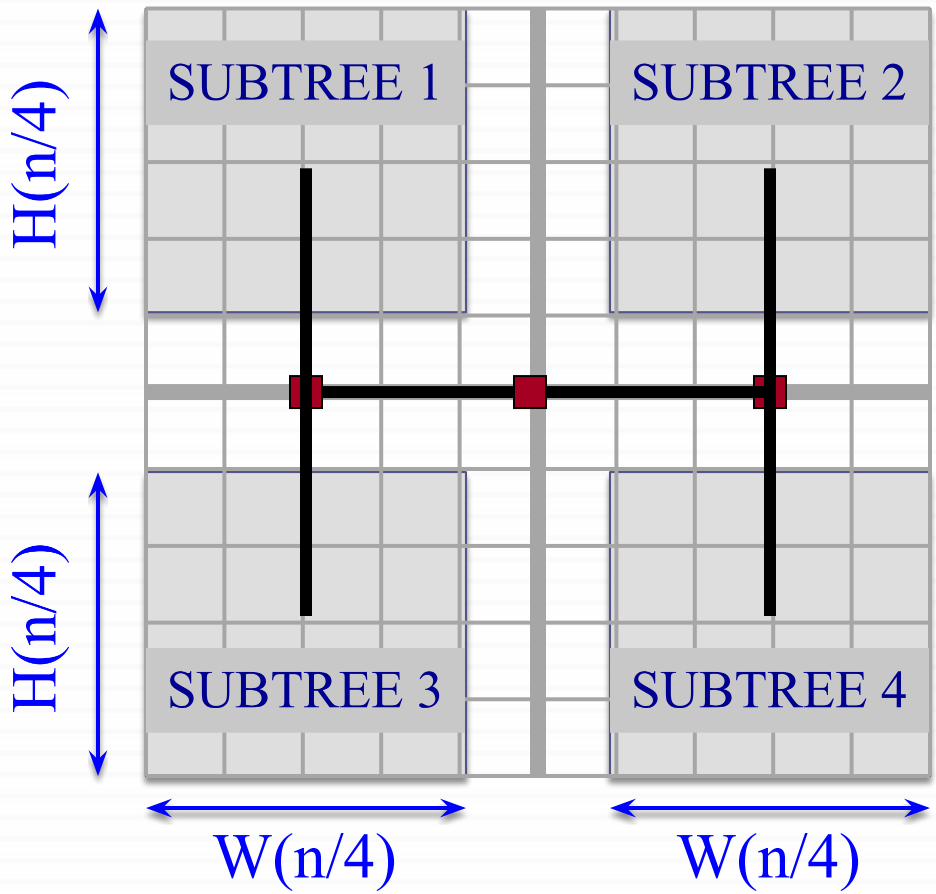
- Use a different divide and conquer method



1. Embed root, left, right nodes
2. Embed subtree 1
3. Embed subtree 2
4. Embed subtree 3
5. Embed subtree 4

What is the min-area required for n leaves?

Binary Tree Embedding: H-Tree



$$W(n) = 2W(n/4) + 1$$

$$H(n) = 2H(n/4) + 1$$

Binary Tree Embedding: H-Tree

- Solve the recurrences:

  $W(n) = 2W(n/4) + 1$

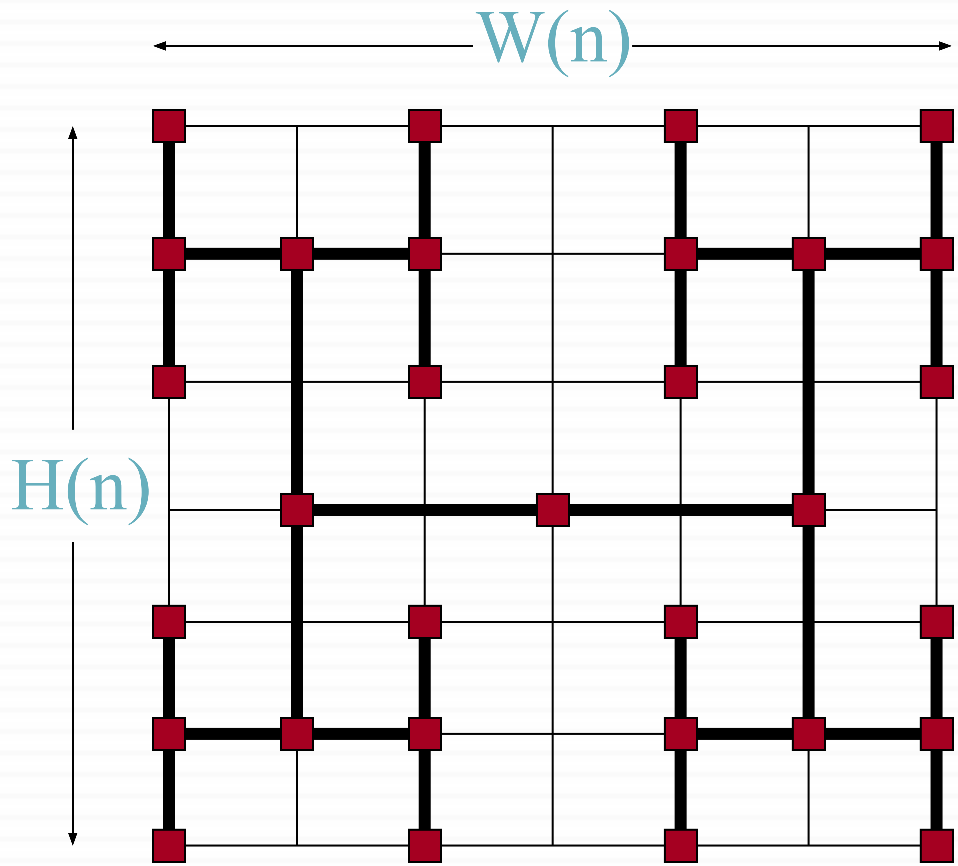
  $H(n) = 2H(n/4) + 1$

  - $W(n) = \Theta(\sqrt{n})$
  - $H(n) = \Theta(\sqrt{n})$

- $\text{Area}(n) = \Theta(n)$

Binary Tree Embedding: H-Tree

Example:



Correctness Proofs

- ***Proof by induction*** commonly used for D&C algorithms

- *Base case*: Show that the algorithm is correct when the recursion bottoms out (i.e., for sufficiently small n)

- *Inductive hypothesis*: Assume the alg. is correct for any recursive call on any smaller subproblem of size k ($k < n$)

- *General case*: Based on the inductive hypothesis, prove that the alg. is correct for any input of size n

Example Correctness Proof: Powering a Number

```
POWER (a, n)
    if n = 0 then return 1

    else if n is even then
        val ← POWER (a, n/2)
        return val * val

    else if n is odd then
        val ← POWER (a, (n-1)/2)
        return val * val * a
```

Example Correctness Proof: Powering a Number

- *Base case*: POWER (a, 0) is correct, because it returns 1
- *Ind. hyp*: Assume POWER (a, k) is correct for any $k < n$
- *General case*:

In POWER (a, n) function:

If n is even:

$val = a^{n/2}$ *(due to ind. hyp.)*

it returns $val \,.\, val = a^n$

If n is odd:

$val = a^{(n-1)/2}$ *(due to ind. hyp.)*

it returns $val.\, val \,.\, a = a^n$

- *The correctness proof is complete*

Maximum Subarray Problem

- *Input*: An array of values
- *Output*: The contiguous subarray that has the largest sum of elements

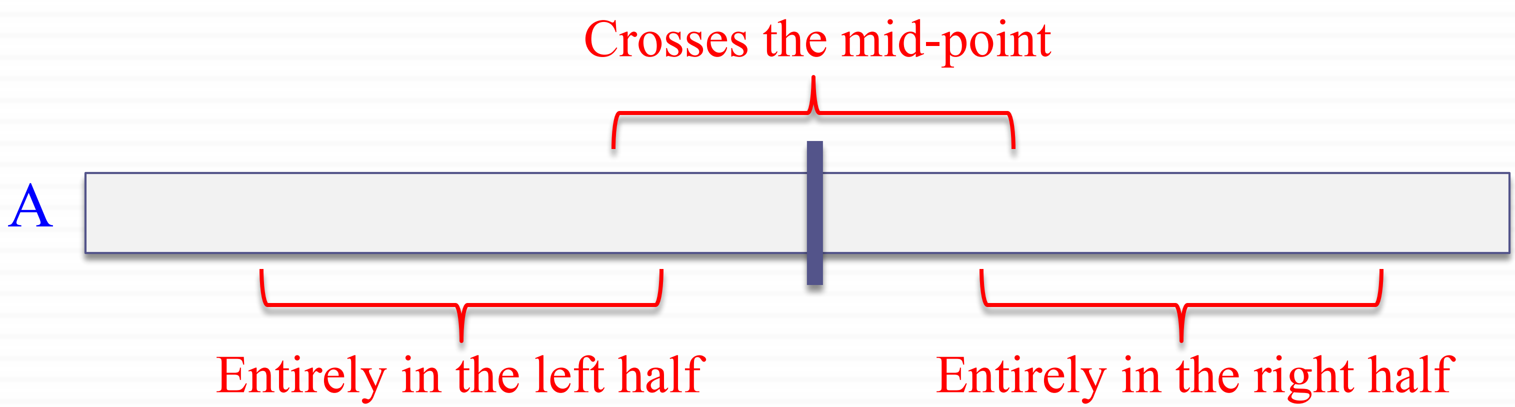
Input array:

13	-3	-25	20	-3	-16	-23	18	20	-7	12	-22	-4	7

the maximum contiguous subarray

Maximum Subarray Problem: Divide & Conquer

- ***Basic idea***:
  - Divide the input array into 2 from the middle
  - Pick the best solution among the following:
    1. The max subarray of the left half
    2. The max subarray of the right half
    3. The max subarray crossing the mid-point



Maximum Subarray Problem: Divide & Conquer

- *Divide*: Trivial (divide the array from the middle)
- *Conquer*: Recursively compute the max subarrays of the left and right halves
- *Combine*: Compute the max-subarray crossing the mid-point *(can be done in $\Theta(n)$ time)*. Return the max among the following:
  1. the max subarray of the left subarray
  2. the max subarray of the right subarray
  3. the max subarray crossing the mid-point

See textbook for the detailed solution.

Conclusion

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- Can lead to more efficient algorithms