# CS473 - Algorithms I

## Lecture 7
## Medians and Order Statistics

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# Medians and Order Statistics

*$i^{th}$ order statistic*: $i^{th}$ smallest element of a set of $n$ elements

*minimum*: first order statistic

*maximum*: $n^{th}$ order statistic

*median*: "halfway point" of the set

$$i = \lfloor (n+1)/2 \rfloor \text{ or } \lceil (n+1)/2 \rceil$$

# Selection Problem

- *Selection problem*: Select the $i^{th}$ smallest of n elements

- *Naïve algorithm*: Sort the input array A; then return A[i]

  $T(n) = \Theta(n\lg n)$

  *using e.g. merge sort (but not quicksort)*

- Can we do any better?

# Selection in Expected Linear Time

- Randomized algorithm using divide and conquer

- Similar to randomized quicksort
  - *Like quicksort*: Partitions input array recursively
  - *Unlike quicksort*: Makes a single recursive call

    *Reminder: Quicksort makes two recursive calls*

- Expected runtime: $\Theta(n)$

  *Reminder: Expected runtime of quicksort: $\Theta(n\lg n)$*

# Selection in Expected Linear Time: Example 1

Select the $2^{nd}$ smallest element:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | 10 | 13 | 5 | 8 | 3 | 2 | 11 |

$i = 2$

Partition the input array:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 13 | 8 | 10 | 6 | 11 |

make a recursive call to select the $2^{nd}$ smallest element in left subarray

# Selection in Expected Linear Time: Example 2

Select the 7th smallest element:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | 10 | 13 | 5 | 8 | 3 | 2 | 11 |

$i = 7$

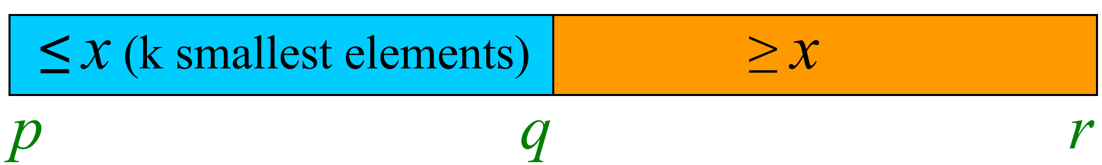
Partition the input array:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 13 | 8 | 10 | 6 | 11 |

make a recursive call to select the 4th smallest element in right subarray

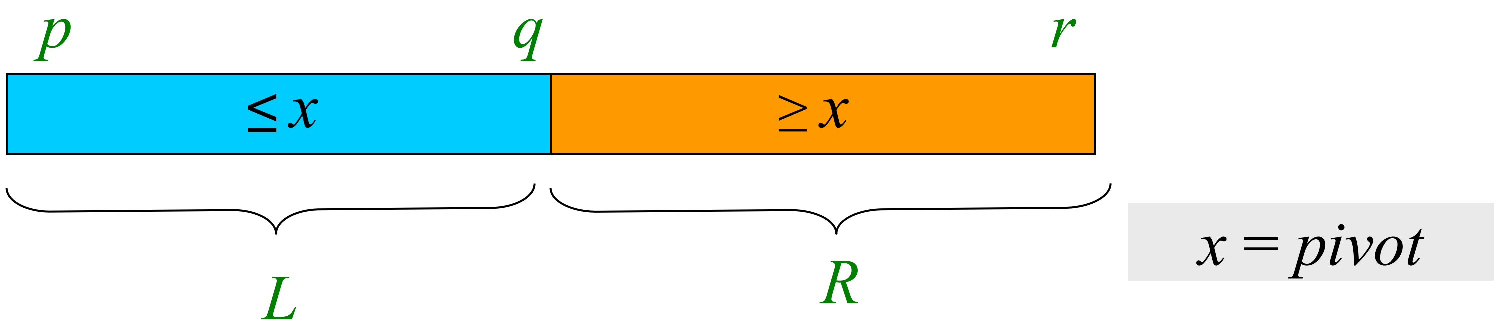
# Selection in Expected Linear Time

R-SELECT($A, p, r, i$)
    if $p = r$ then
        return $A[p]$
  $q \leftarrow$ R-PARTITION($A, p, r$)
  $k \leftarrow q - p + 1$
  if $i \leq k$ then
        return R-SELECT($A, p, q, i$)
else
        return R-SELECT($A, q+1, r, i-k$)




$x = pivot$

# Selection in Expected Linear Time




- All elements in L ≤ all elements in R
- L contains $|L| = q-p+1 = k$ smallest elements of A[$p \ldots r$]

  if $i \leq |L| = k$ then

  search L recursively for its $i$-th smallest element

  else

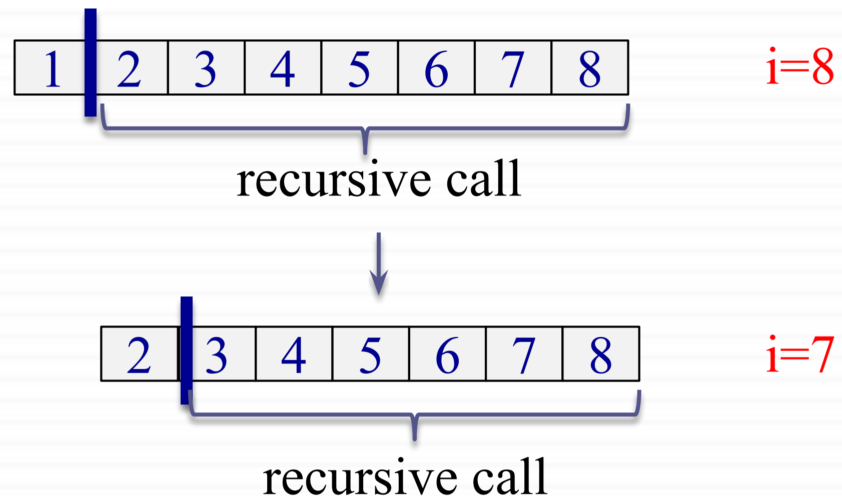
  search R recursively for its $(i-k)$-th smallest element

# Runtime Analysis

- ***Worst case:***

Imbalanced partitioning at every level

and the recursive call always to the larger partition



# Runtime Analysis

- ***Worst case:***

  $T(n) = T(n-1) + \Theta(n)$

  $\Rightarrow T(n) = \Theta(n^2)$

  Worse than the naïve method (based on sorting)

- ***Best case***: Balanced partitioning at every recursive level

  $T(n) = T(n/2) + \Theta(n)$

  $\Rightarrow T(n) = \Theta(n)$

- ***Avg case***: Expected runtime – need analysis

# Reminder: Various Outcomes of H-PARTITION

$\mathbf{P}(\text{rank}(x) = i) = 1/n$ for $1 \le i \le n$

x: pivot

|L|: size of left region

**if** rank(x) = 1 **then** |L| = 1

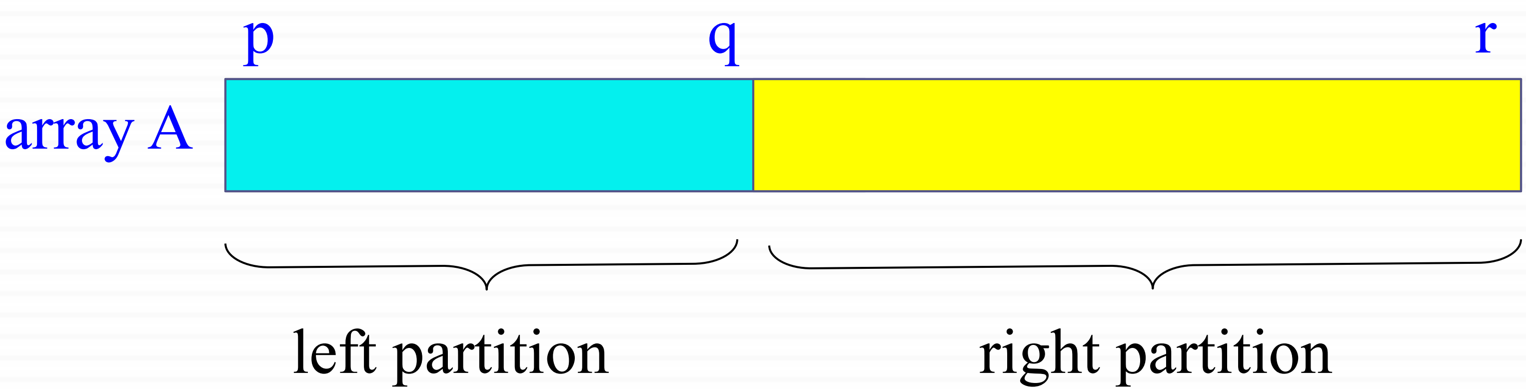
**if** rank(x) > 1 **then** |L| = rank(x) - 1

$\mathbf{P}(|L| = 1) = \mathbf{P}(\text{rank}(x) = 1) + \mathbf{P}(\text{rank}(x) = 2)$ ⇒ $\mathbf{P}(|L| = 1) = 2/n$

$\mathbf{P}(|L| = i) = \mathbf{P}(\text{rank}(x) = i+1)$ for $1 < i < n$ ⇒ $\mathbf{P}(|L| = i) = 1/n$ for $1 < i < n$

# Average Case Analysis of Randomized Select

- To compute the upper bound for the avg case, assume that the $i^{th}$ element always falls into the larger partition.

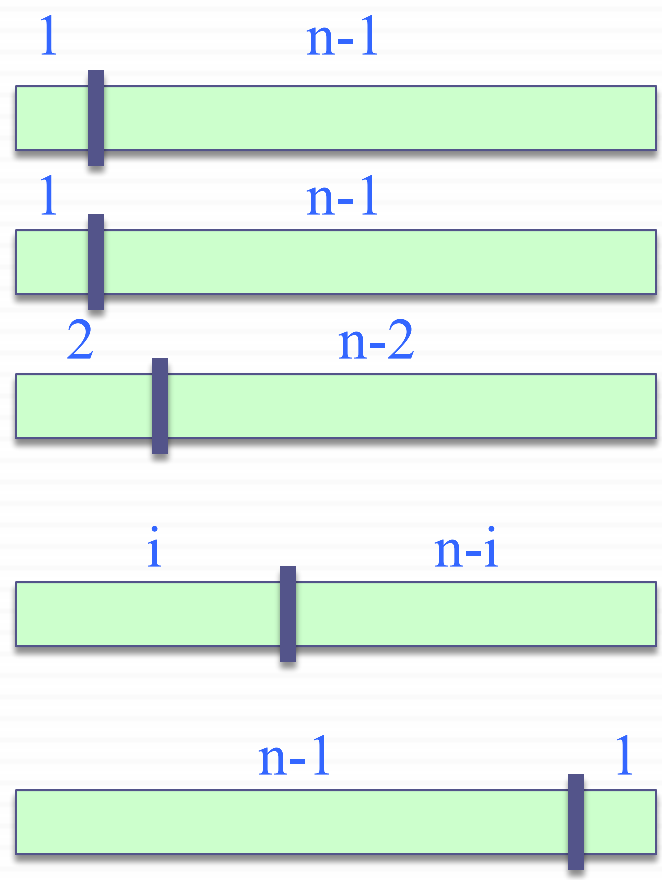



We will analyze the case where the recursive call is always made to the larger partition

→ this will give us an upper bound for the avg case

# Various Outcomes of H-PARTITION

| rank(x) | prob. | T(n) |
|---|---|---|
| 1 | 1/n | $\le T(\max(1, n-1)) + \Theta(n)$ |
| 2 | 1/n | $\le T(\max(1, n-1)) + \Theta(n)$ |
| 3 | 1/n | $\le T(\max(2, n-2)) + \Theta(n)$ |
| . | . | . |
| . | . | . |
| . | . | . |
| i+1 | 1/n | $\le T(\max(i, n-i)) + \Theta(n)$ |
| . | . | . |
| . | . | . |
| . | . | . |
| n | 1/n | $\le T(\max(n-1, 1)) + \Theta(n)$ |



# Average-Case Analysis of Randomized Select

Recall: $P(|L|=i) = \begin{cases} 2/n & \text{for } i = 1 \\ 1/n & \text{for } i = 2,3,\ldots,n-1 \end{cases}$

Upper bound: Assume *i*-th element always falls into the larger part

$$T(n) \le \frac{1}{n} T(\max(1, n-1)) + \frac{1}{n} \sum_{q=1}^{n-1} T(\max(q, n-q)) + O(n)$$

Note: $\frac{1}{n} T(\max(1, n-1)) = \frac{1}{n} T(n-1) = \frac{1}{n} O(n^2) = O(n)$

$$\therefore \quad T(n) \le \frac{1}{n} \sum_{q=1}^{n-1} T(\max(q, n-q)) + O(n)$$

# Average-Case Analysis of Randomized Select

$$\therefore\ T(n) \le \frac{1}{n} \sum_{q=1}^{n-1} T(\max(q, n-q)) + O(n)$$

$$\max(q, n-q) = \begin{cases} q & \text{if } q \ge \lceil n/2 \rceil \\ n-q & \text{if } q < \lceil n/2 \rceil \end{cases}$$

$n$ is odd: $T(k)$ appears twice for $k = \lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \ldots, n-1$

$n$ is even: $T(\lceil n/2 \rceil)$ appears once T($k$) appears twice for

$k = \lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \ldots, n-1$

Hence, in both cases: $\sum_{q=1}^{n-1} T(\max(q, n-q)) + O(n) \le 2 \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$

$$\therefore\ T(n) \le \frac{2}{n} \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$$

# Average-Case Analysis of Randomized Select

$$T(n) \le \frac{2}{n} \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$$

By substitution guess $T(n) = O(n)$
Inductive hypothesis: $T(k) \le ck, \ \forall\, k < n$

$$T(n) \le (2/n) \sum_{k=\lceil n/2 \rceil}^{n-1} ck + O(n)$$

$$= \frac{2c}{n}\left( \sum_{k=1}^{n-1} k - \sum_{k=1}^{\lceil n/2 \rceil - 1} k \right) + O(n)$$

$$\frac{2c}{n}\left( \frac{1}{2}\, n\,(n-1) - \frac{1}{2}\left\lceil \frac{n}{2} \right\rceil \left( \frac{n}{2} - 1 \right) \right) + O(n)$$

# Average-Case Analysis of Randomized Select

$$T(n) \le \frac{2c}{n}\left(\frac{1}{2}\,n(n\text{-}1) - \frac{1}{2}\left\lceil\frac{n}{2}\right\rceil\left(\frac{n}{2} - 1\right)\right) + \mathrm{O}(n)$$

$$\le c(n\text{-}1) - \frac{c}{4}n + \frac{c}{2} + \mathrm{O}(n)$$

$$= cn - \frac{c}{4}\,n - \frac{c}{2} + \mathrm{O}(n)$$

$$= cn - \left(\left(\frac{c}{4}\,n + \frac{c}{2}\right) - \mathrm{O}(n)\right)$$

$$\le cn$$

***since we can choose c large enough so that*** **( $cn/4+c/2$ ) dominates O($n$)**

# Summary of Randomized Order-Statistic Selection

- Works fast: linear expected time
- Excellent algorithm in practise
- But, the worst case is very bad: $\Theta(n^2)$

Q: Is there an algorithm that runs in linear time in the worst case?

A: Yes, due to Blum, Floyd, Pratt, Rivest & Tarjan[1973]

Idea: Generate a good pivot recursively..

# Selection in Worst Case Linear Time

SELECT(S, $n$, $i$) ▷ return $\boldsymbol{i}$-th element in set S with $\boldsymbol{n}$ elements

    if $n \leq 5$ then

        SORT S and return the $i$-th element

    DIVIDE S into $\lceil n/5 \rceil$ groups

    ▷ first $\lceil n/5 \rceil$ groups are of size 5, last group is of size $n$ mod 5

    FIND median set $M=\{m_1, \ldots, m_{\lceil n/5 \rceil}\}$ ▷ $m_j$: median of $j$-th group

    $x \leftarrow$ SELECT(M, $\lceil n/5 \rceil$, $(\lfloor \lceil n/5 \rceil + 1)/2 \rfloor$)

    PARTITION set S around the pivot $x$ into L and R

    if $i \leq |L|$ then

        return SELECT(L, $|L|$, $i$)

    else

        return SELECT(R, $n-|L|$, $i-|L|$)

# Selection in Worst Case Linear Time - Example

*Input*: Array S and index i

*Output*: The $i^{th}$ smallest value

S = {25 9 16 8 11 27 39 42 15 6 32 14 36 20 33 22 31 4 17 3 30 41
2 13 19 7 21 10 34 1 37 23 40 5 29 18 24 12 38 28 26 35 43}

# Selection in Worst Case Linear Time - Example

Step 1: Divide the input array into groups of size 5

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 25 | 27 | 32 | 22 | 30 | 7 | 37 | 18 | 26 |
| 9 | 39 | 14 | 31 | 41 | 21 | 23 | 24 | 35 |
| 16 | 42 | 36 | 4 | 2 | 10 | 40 | 12 | 43 |
| 8 | 15 | 20 | 17 | 13 | 34 | 5 | 38 | |
| 11 | 6 | 33 | 3 | 19 | 1 | 29 | 28 | |

# Selection in Worst Case Linear Time - Example

Step 2: Compute the median of each group $\Rightarrow \Theta(n)$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 9 | 15 | 14 | 4 | 2 | 7 | 5 | 18 | |
| 8 | 6 | 20 | 3 | 13 | 1 | 23 | 12 | 26 |
| **11** | **27** | **32** | **17** | **19** | **10** | **29** | **24** | **35** |
| 16 | 42 | 33 | 31 | 30 | 34 | 40 | 28 | 43 |
| 25 | 39 | 36 | 22 | 41 | 21 | 37 | 38 | |

Let M be the set of the medians computed:

$M = \{11, 27, 32, 17, 19, 10, 29, 24, 35\}$

# Selection in Worst Case Linear Time - Example

Step 3: Compute the median of the median group M

$x \leftarrow \text{SELECT}\left(M, |M|, \left\lfloor (|M|+1)/2 \right\rfloor\right)$ where $|M| = \left\lceil n/5 \right\rceil$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 9 | 15 | 14 | 4 | 2 | 7 | 5 | 18 | |
| | 8 | 6 | 20 | 3 | 13 | 1 | 23 | 12 | 26 |
| M | 11 | 27 | 32 | 17 | 19 | 10 | 29 | 24 | 35 |
| | 16 | 42 | 33 | 31 | 30 | 34 | 40 | 28 | 43 |
| | 25 | 39 | 36 | 22 | 41 | 21 | 37 | 38 | |

$\Rightarrow$ median = 24

The runtime of the recursive call: $T(|M|) = T\left(\left\lceil n/5 \right\rceil\right)$

# Selection in Worst Case Linear Time - Example

***Step 4***: Partition the input array S around the median-of-medians x

S = {25 9 16 8 11 27 39 42 15 6 32 14 36 20 33 22 31 4 17 3 30 41
2 13 19 7 21 10 34 1 37 23 40 5 29 18 24 12 38 28 26 35 43}

Partition S around x = 24

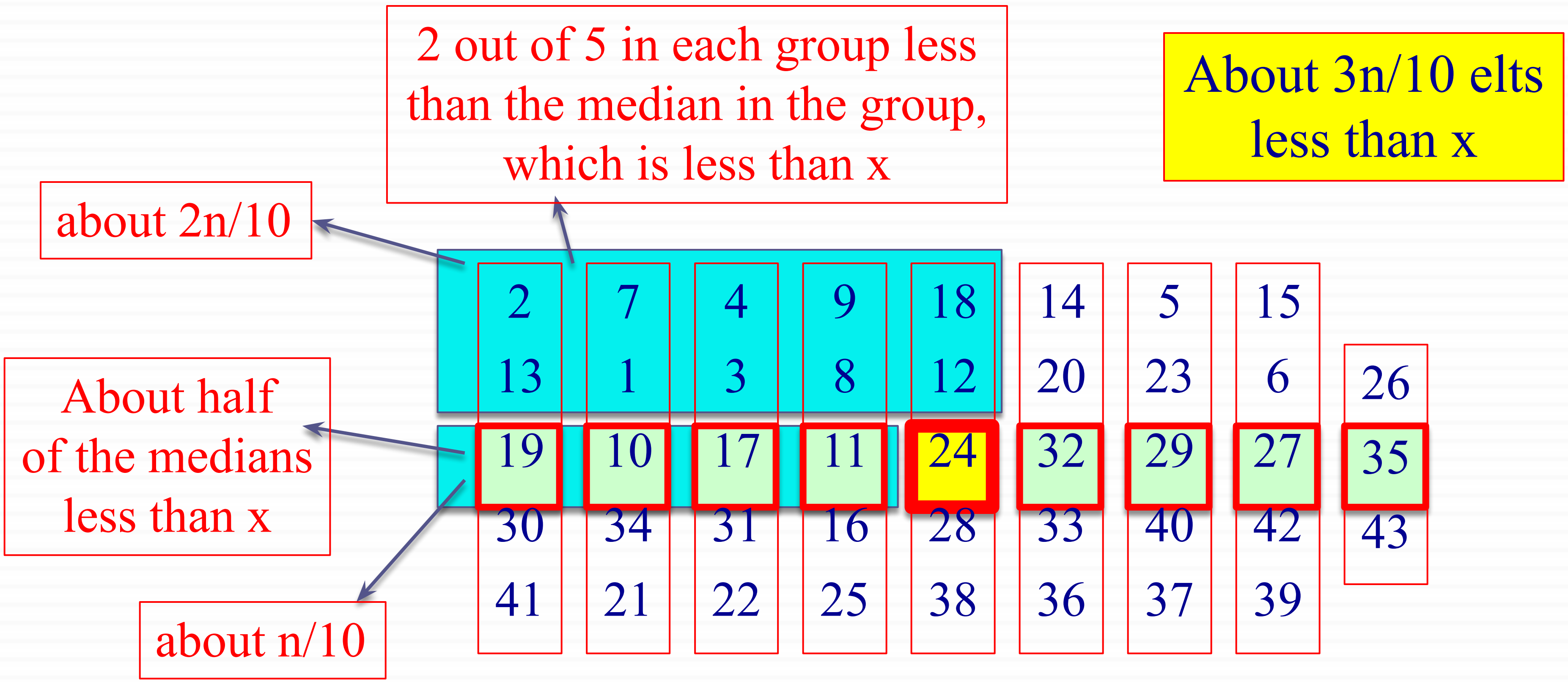
*Claim*: Partitioning around x is guaranteed to be *well-balanced*.

# Selection in Worst Case Linear Time - Example

*Claim*: Partitioning around x=24 is guaranteed to be *well-balanced*.

# Selection in Worst Case Linear Time - Example

*Claim*: Partitioning around x=24 is guaranteed to be *well-balanced*.



# Selection in Worst Case Linear Time - Example

S = {25 9 16 8 11 27 39 42 15 6 32 14 36 20 33 22 31 4 17 3 30 41 2 13 19 7 21 10 34 1 37 23 40 5 29 18 24 12 38 28 26 35 43}

Partitioning S around $x = 24$ will lead to partitions of sizes ~3n/10 and ~7n/10 in the worst case.

***Step 5***: Make a recursive call to one of the partitions

if $i \leq |L|$ then
    return SELECT(L, $|L|$, $i$)
else
    return SELECT(R, $n-|L|$, $i-|L|$)

# Selection in Worst Case Linear Time

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DIVIDE S into $\lceil n/5 \rceil$ groups

▷ first $\lceil n/5 \rceil$ groups are of size 5, last group is of size $n$ mod 5

FIND median set $M=\{m_1, \ldots, m_{\lceil n/5 \rceil}\}$ ▷ $m_j$: median of $j$-th group

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PARTITION set S around the pivot $x$ into L and R

if $i \le |L|$ then

return SELECT(L, $|L|$, $i$)

else

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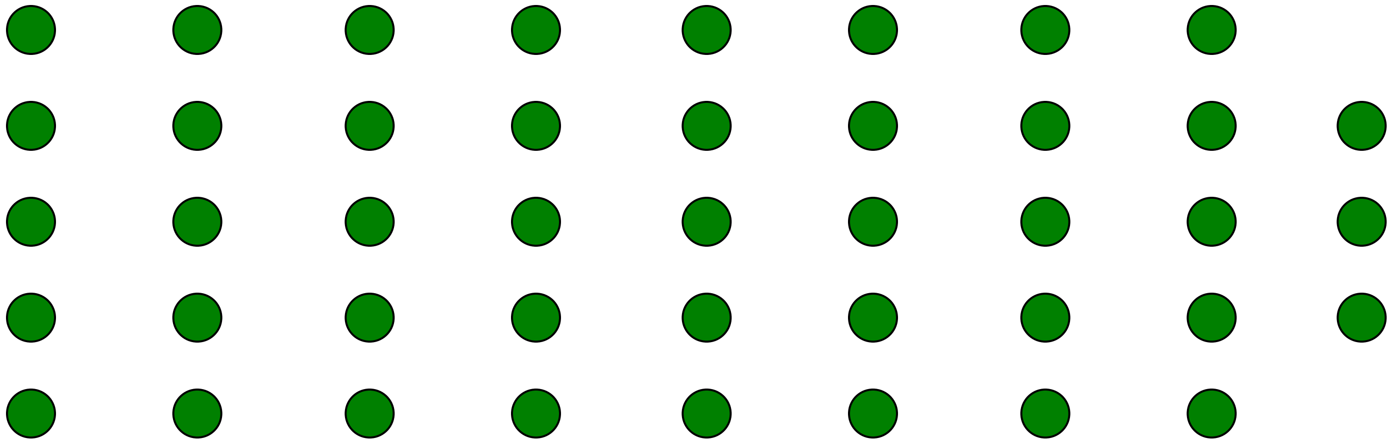
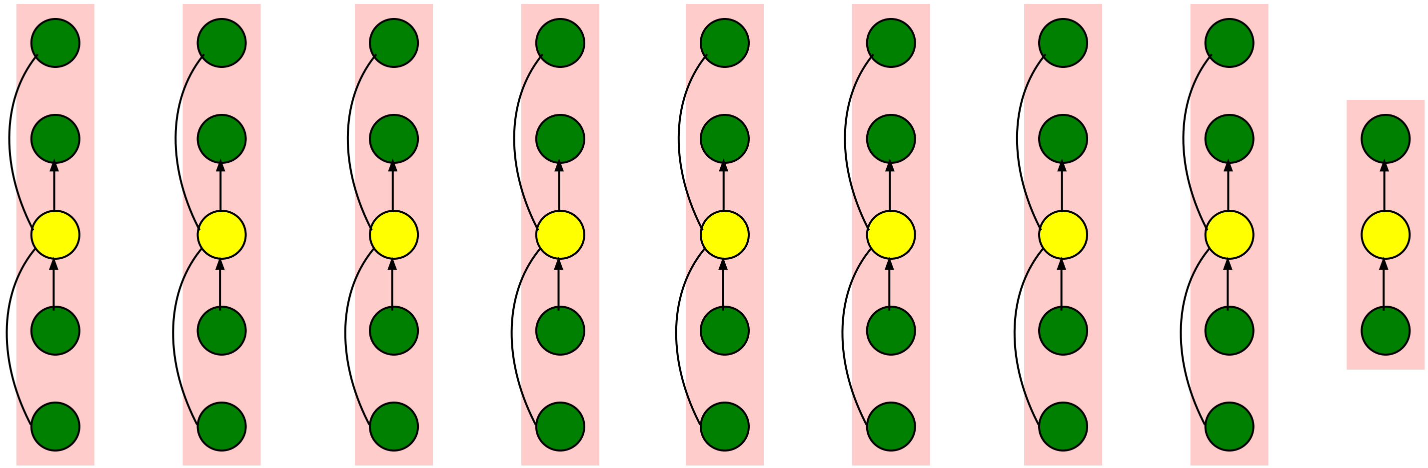
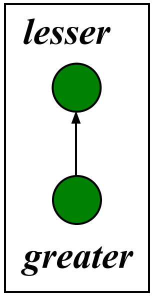
# Choosing the Pivot



1. Divide S into groups of size 5

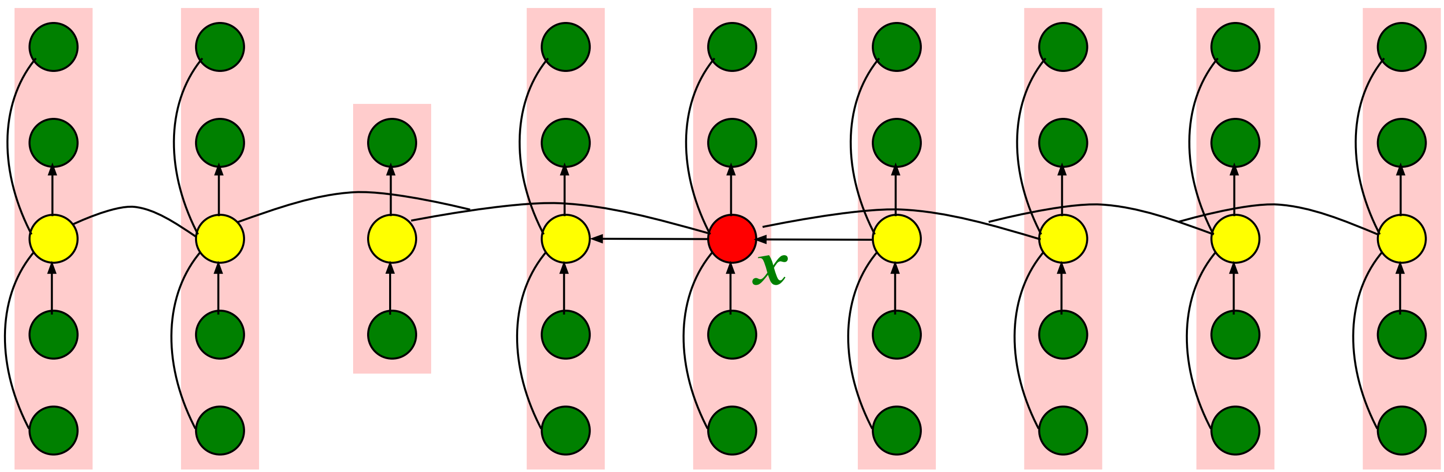
# Choosing the Pivot

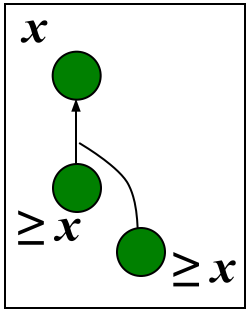





1. Divide S into groups of size 5
2. Find the median of each group

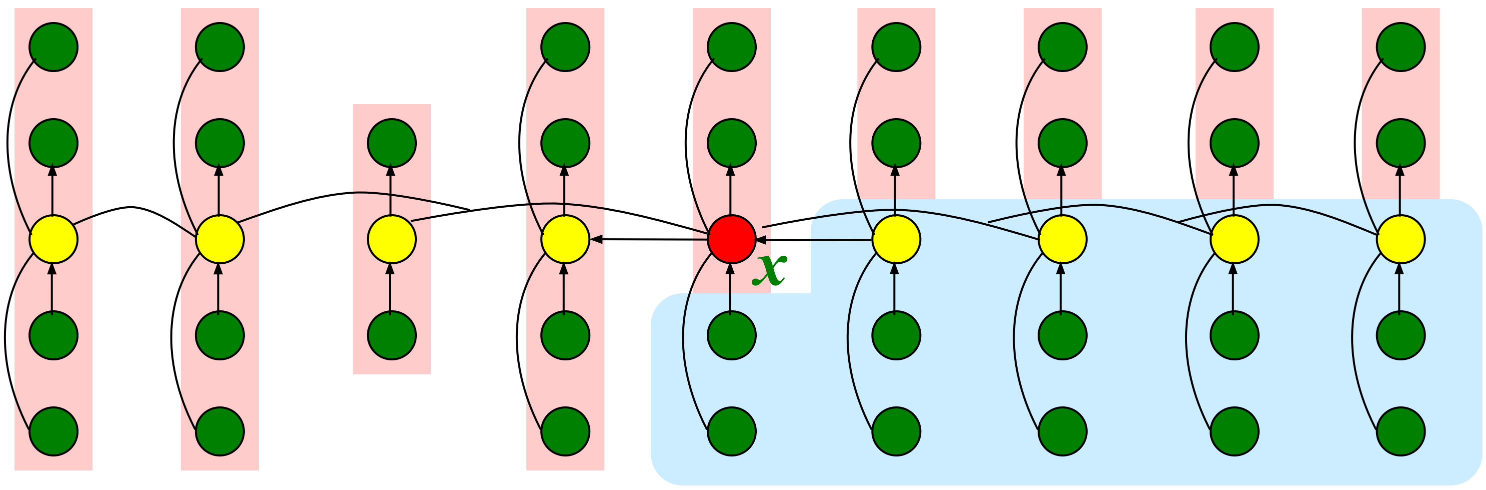
# Choosing the Pivot




1. Divide S into groups of size 5
2. Find the median of each group
3. Recursively select the median $x$ of the medians

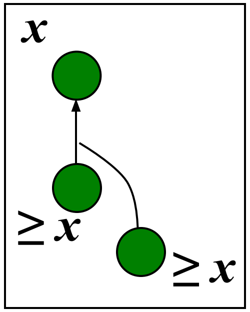
# Choosing the Pivot



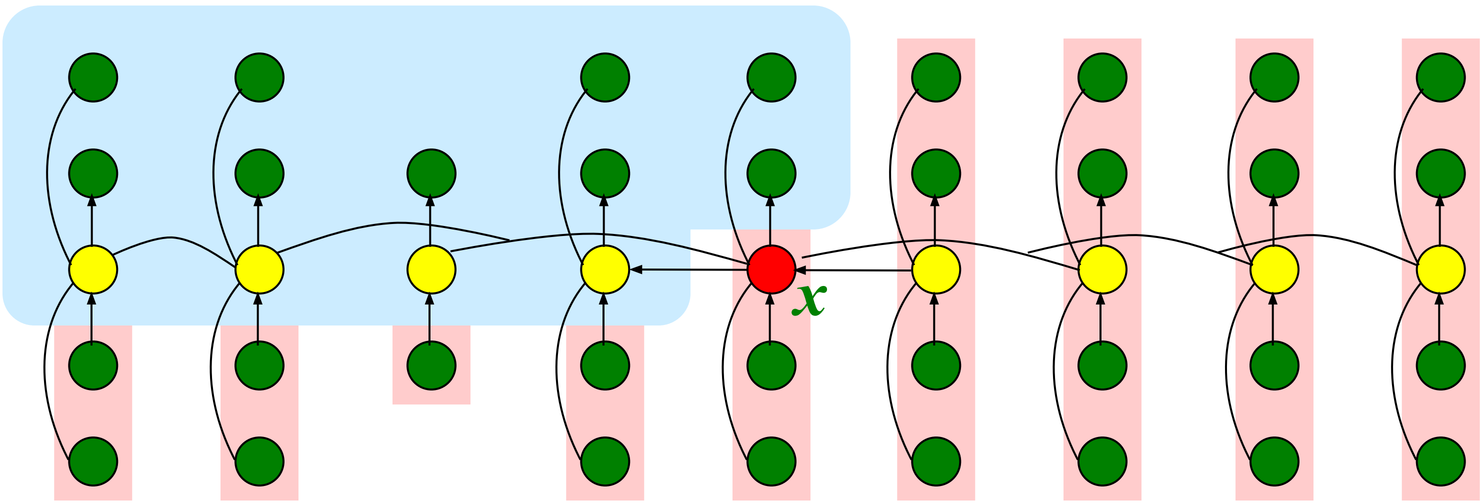

At least half of the medians $\geq x$

Thus $m = \left\lceil \lceil n/5 \rceil / 2 \right\rceil$ groups contribute 3 elements to R except possibly the last group and the group that contains $x$

$$|R| \geq 3\left(m - 2\right) \geq \frac{3n}{10} - 6$$



# Analysis

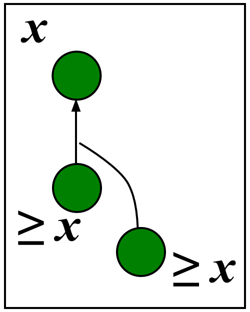



Similarly

$|L| \geq \frac{3n}{10} - 6$

Therefore, SELECT is recursively called on at most

$n - \left(\frac{3n}{10} - 6\right) = \frac{7n}{10} + 6$ elements



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if $n \le 5$ then

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$\Theta(n)$ DIVIDE S into $\lceil n/5 \rceil$ groups

▷ first $\lceil n/5 \rceil$ groups are of size 5, last group is of size $n$ mod 5

$\Theta(n)$ FIND median set $M=\{m_1, \ldots, m_{\lceil n/5 \rceil}\}$ ▷ $m_j$: median of $j$-th group

$T(\lceil n/5 \rceil)$ $x \leftarrow$ SELECT(M, $\lceil n/5 \rceil$, $(\lfloor \lceil n/5 \rceil + 1)/2 \rfloor$)

$\Theta(n)$ PARTITION set S around the pivot $x$ into L and R

$T(\frac{7n}{10}+6)$

if $i \le |L|$ then

return SELECT(L, $|L|$, $i$)

else

return SELECT(R, $n-|L|$, $i-|L|$)

# Selection in Worst Case Linear Time

Thus recurrence becomes

$$T(n) \le T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\frac{7n}{10} + 6\right) + \Theta(n)$$

Guess $T(n) = O(n)$ and prove by induction

Inductive step: $T(n) \le c\lceil n/5 \rceil + c\,(7n/10+6) + \Theta(n)$

$\le cn/5 + c + 7cn/10 + 6c + \Theta(n)$

$= 9cn/10 + 7c + \Theta(n)$

$= cn - [c(n/10 - 7) - \Theta(n)] \le cn$ for large $c$

Work at each level of recursion is a constant factor (9/10) smaller