CS473 - Algorithms I

Lecture 10
Dynamic Programming

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Cevdet Aykanat and Mustafa Ozdal
Computer Engineering Department, Bilkent University

Introduction

- An algorithm design paradigm like divide-and-conquer
- "Programming": A tabular method (not writing computer code)

  Older sense of planning or scheduling, typically by filling in a table
- Divide-and-Conquer (DAC): subproblems are independent
- Dynamic Programming (DP): subproblems are not independent
- Overlapping subproblems: subproblems share sub-subproblems
  - In solving problems with overlapping subproblems
    - A DAC algorithm does redundant work
      - Repeatedly solves common subproblems
    - A DP algorithm solves each problem just once
      - Saves its result in a table

Problem 1:
Fibonacci Numbers

Example: Fibonacci Numbers (Recursive Solution)

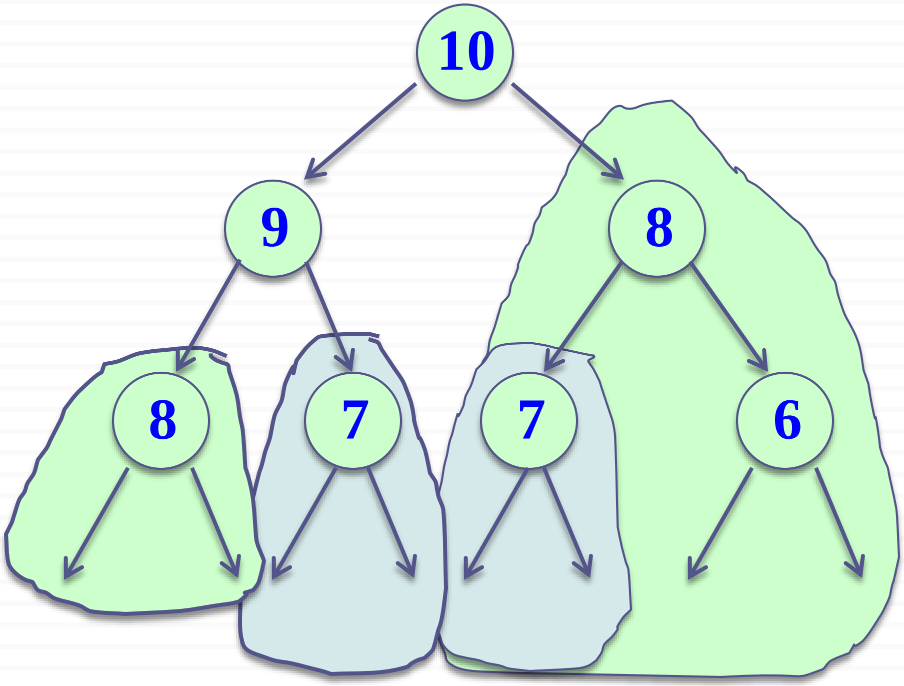
<u>Reminder</u>:

$F(0) = 0$ and $F(1) = 1$

$F(n) = F(n-1) + F(n-2)$

<u>**REC-FIBO(n)**</u>

if n < 2
    return n
else
    return REC-FIBO(n-1)
        + REC-FIBO(n-2)



Overlapping subproblems in different recursive calls. Repeated work!

Example: Fibonacci Numbers (Recursive Solution)

Recurrence:

$$T(n) = T(n-1) + T(n-2) + 1$$

➔ exponential runtime

Recursive algorithm inefficient because it recomputes the same F(i) repeatedly in different branches of the recursion tree.

Example: Fibonacci Numbers (Bottom-up Computation)

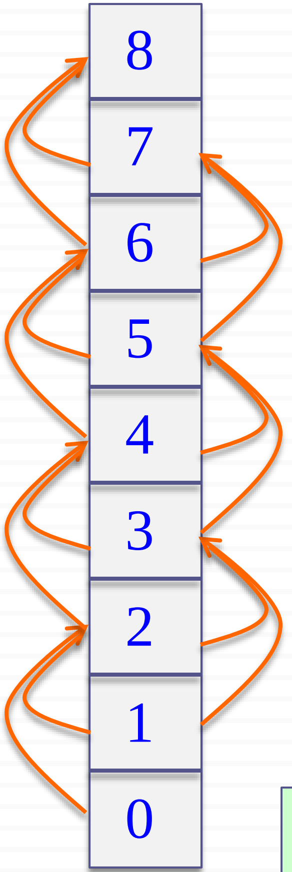
<u>Reminder</u>:

F(0) = 0 and F(1) = 1

F(n) = F(n-1) + F(n-2)

<u>ITER-FIBO(n)</u>

```
F[0] = 0
F[1] = 1
for i = 2 to n do
    F[i] = F[i-1] + F[i-2]
return F[n]
```



Runtime: $\Theta(n)$

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Optimization Problems

- DP typically applied to optimization problems
- In an optimization problem
  - There are many possible solutions (feasible solutions)
  - Each solution has a value
  - Want to find an optimal solution to the problem
    - A solution with the optimal value (min or max value)
  - Wrong to say "the" optimal solution to the problem
    - There may be several solutions with the same optimal value

Development of a DP Algorithm

1. Characterize the structure of an optimal solution
2. Recursively define the value of an optimal solution
3. Compute the value of an optimal solution in a bottom-up fashion
4. Construct an optimal solution from the information computed in Step 3

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Problem 2:
Matric Chain Multiplication

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Example: Matrix-chain Multiplication

- *Input*: a sequence (chain) $\langle A_1, A_2, \ldots, A_n \rangle$ of n matrices
- *Aim*: compute the product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$
- A product of matrices is fully parenthesized if
  - It is either a single matrix
  - Or, the product of two fully parenthesized matrix products surrounded by a pair of parentheses.

$$\big(A_i(A_{i+1}A_{i+2} \ldots A_j)\big)$$

$$\big((A_iA_{i+1}A_{i+2} \ldots A_{j-1})A_j\big)$$

$$\big((A_iA_{i+1}A_{i+2} \ldots A_k)(A_{k+1}A_{k+2} \ldots A_j)\big) \qquad \text{for } i \le k < j$$

  - *All parenthesizations yield the same product; matrix product is associative*

Matrix-chain Multiplication: An Example Parenthesization

- *Input*: $\langle A_1, A_2, A_3, A_4 \rangle$
- 5 distinct ways of full parenthesization

$$(A_1(A_2(A_3A_4)))$$
$$(A_1((A_2A_3)A_4))$$
$$((A_1A_2)(A_3A_4))$$
$$((A_1(A_2A_3))A_4)$$
$$(((A_1A_2)A_3)A_4)$$

- The way we parenthesize a chain of matrices can have a dramatic effect on the cost of computing the product

Reminder: Matrix Multiplication

MATRIX-MULTIPLY(A, B)

if cols[A]≠rows[B] **then**
error(“*incompatible dimensions*”)

for *i* ←1 **to** rows[A] **do**

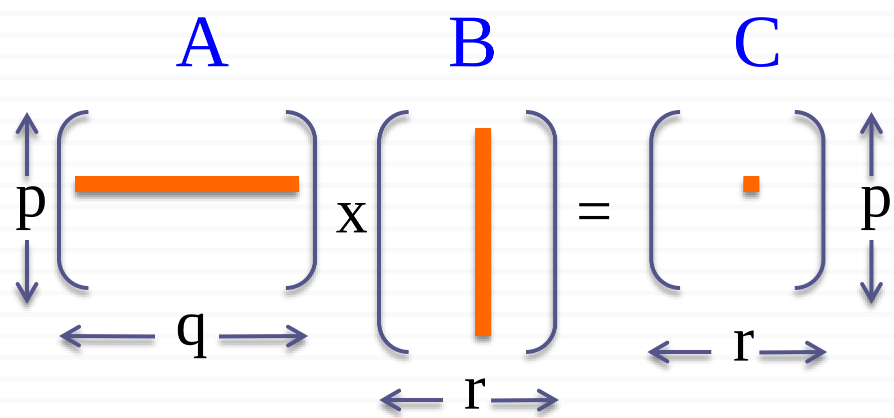
  for *j*←1 **to** cols[B] **do**

    C[i,j] ← 0

    for *k*←1 **to** cols[A] **do**

      C[i,j]← C[i,j]+A[i,k]·B[k,j]

return C



rows(A) = p
cols(A) = q

rows(B) = q
cols(B) = r

rows(C) = p
cols(C) = r

Reminder: Matrix Multiplication

***MATRIX-MULTIPLY**(A, B)*

if cols[A]≠rows[B] **then**
error(“*incompatible dimensions*”)
for *i* ←1 **to** rows[A] **do**
  for *j*←1 **to** cols[B] **do**
    C[i,j] ← 0
    for *k*←1 **to** cols[A] **do**
      C[i,j]← C[i,j]+A[i,k]·B[k,j]
return C

A: p x q
B: q x r

C: p x r

\# of mult-add ops
= rows[A] x cols[B] x cols[A]

\# of mult-add ops = p x q x r

Matrix Chain Multiplication: Example

A_1: 10x100 $\quad$ A_2: 100x5 $\quad$ A_3: 5x50

Which paranthesization is better? $(A_1A_2)A_3$ or $A_1(A_2A_3)$?

$$\begin{array}{cc} & 100 \\ 10 & \begin{bmatrix} \quad A_1 \quad \end{bmatrix} \end{array} \times \begin{array}{cc} & 5 \\ 100 & \begin{bmatrix} \\ A_2 \\ \\ \end{bmatrix} \end{array} = \begin{array}{cc} & 5 \\ 10 & \begin{bmatrix} A_1A_2 \end{bmatrix} \end{array}$$

# of ops: 10 . 100 . 5 = 5000

$$\begin{array}{cc} & 5 \\ 10 & \begin{bmatrix} A_1A_2 \end{bmatrix} \end{array} \times \begin{array}{cc} & 50 \\ 5 & \begin{bmatrix} \quad A_3 \quad \end{bmatrix} \end{array} = \begin{array}{cc} & 50 \\ 10 & \begin{bmatrix} A_1A_2A_3 \end{bmatrix} \end{array}$$

# of ops: 10 . 5 . 50 = 2500

Total # of ops: 7500

Matrix Chain Multiplication: Example

A_1: 10x100 A_2: 100x5 A_3: 5x50

Which paranthesization is better? $(A_1A_2)A_3$ or $A_1(A_2A_3)$?

$$\begin{array}{cc} & 5 \\ 100 & \begin{bmatrix} A_2 \end{bmatrix} \end{array} \times \begin{array}{cc} & 50 \\ 5 & \begin{bmatrix} A_3 \end{bmatrix} \end{array} = \begin{array}{cc} & 50 \\ 100 & \begin{bmatrix} A_2A_3 \end{bmatrix} \end{array}$$

# of ops: 100 . 5 . 50
= 25000

$$\begin{array}{cc} & 100 \\ 10 & \begin{bmatrix} A_1 \end{bmatrix} \end{array} \times \begin{array}{cc} & 50 \\ 100 & \begin{bmatrix} A_2A_3 \end{bmatrix} \end{array} = \begin{array}{cc} & 50 \\ 10 & \begin{bmatrix} A_1A_2A_3 \end{bmatrix} \end{array}$$

# of ops: 10 . 100 . 50
= 50000

Total # of ops: 75000

Matrix Chain Multiplication: Example

A_1: 10x100 A_2: 100x5 A_3: 5x50

Which paranthesization is better? $(A_1A_2)A_3$ or $A_1(A_2A_3)$?

In summary: $(A_1A_2)A_3$ ➔ # of multiply-add ops: 7500

$A_1(A_2A_3)$ ➔ # of multiple-add ops: 75000

➔ First parenthesization yields 10x faster computation

Matrix-chain Multiplication Problem

Input: A chain $\langle A_1, A_2, \ldots, A_n \rangle$ of n matrices,
where A_i is a $p_{i-1} \times p_i$ matrix

Objective: Fully parenthesize the product

$$A_1 \cdot A_2 \cdot \ldots \cdot A_n$$

such that the number of scalar mult-adds is minimized.

Counting the Number of Parenthesizations

- ***Brute force approach***: exhaustively check all parenthesizations
- P(n): # of parenthesizations of a sequence of n matrices
- We can split sequence between k^{th} and $(k+1)^{\text{st}}$ matrices for any $k=1, 2, \ldots, n-1$, then parenthesize the two resulting sequences independently, i.e.,

$$(A_1 A_2 A_3 \ \ldots \ A_k)(A_{k+1} A_{k+2} \ \ldots \ A_n)$$

- We obtain the recurrence

$$\mathrm{P}(1) = 1 \text{ and } \mathrm{P}(n) = \sum_{k=1}^{n-1} \mathrm{P}(k)\mathrm{P}(n-k)$$

Number of Parenthesizations: $\sum_{k=1}^{n-1} P(k)P(n-k)$

- The recurrence generates the sequence of Catalan Numbers
- Solution is $P(n) = C(n-1)$ where

$$C(n) = \frac{1}{n+1}\binom{2n}{n} = \Omega(4^n/n^{3/2})$$

- The number of solutions is exponential in n
- Therefore, brute force approach is a poor strategy

The Structure of Optimal Parenthesization

Notation: $A_{i..j}$: The matrix that results from evaluation of the product: $A_i \, A_{i+1} \, A_{i+2} \, \ldots \, A_j$

Observation: Consider the last multiplication operation in any parenthesization: $(A_1 \, A_2 \, \ldots \, A_k) \, . \, (A_{k+1} \, A_{k+2} \, \ldots \, A_n)$

There is a k value ($1 \leq k < n$) such that:

First, the product $A_{1..k}$ is computed

Then, the product $A_{k+1..n}$ is computed

Finally, the matrices $A_{1..k}$ and $A_{k+1..n}$ are multiplied

Step 1: Characterize the structure of an optimal solution

- An optimal parenthesization of product $A_1 A_2 \ldots A_n$ will be:

  $(A_1\ A_2\ \ldots\ A_k)\ .\ (A_{k+1}\ A_{k+2}\ \ldots\ A_n)$ for some k value

- The cost of this optimal parenthesization will be:

  Cost of computing $A_{1..k}$

  \+ Cost of computing $A_{k+1..n}$

  \+ Cost of multiplying $A_{1..k} \cdot A_{k+1..n}$

Step 1: Characterize the Structure of an Optimal Solution

- ***Key observation***: Given optimal parenthesization

  $$(A_1A_2A_3 \ldots A_k) \cdot (A_{k+1}A_{k+2} \ldots A_n)$$

  – Parenthesization of the subchain $A_1A_2A_3 \ldots A_k$

  – Parenthesization of the subchain $A_{k+1}A_{k+2} \ldots A_n$

  should both be optimal

Thus, optimal solution to an instance of the problem contains optimal solutions to subproblem instances

i.e., optimal substructure within an optimal solution exists.

Step 2: A Recursive Solution

Step 2: Define the value of an optimal solution recursively in terms of optimal solutions to the subproblems

Assume we are trying to determine the min cost of computing $A_{i..j}$

$m_{i,j}$: min # of scalar multiply-add opns needed to compute $A_{i..j}$

Note: The optimal cost of the original problem: $m_{1,n}$

How to compute $m_{i,j}$ recursively?

Step 2: A recursive Solution

Base case: $m_{i,i} = 0$ (single matrix, no multiplication)

Let the size of matrix A_i be $(p_{i-1} \text{ x } p_i)$

Consider an optimal parenthesization of chain $A_i \ldots A_j$:

$$(A_i \ldots A_k) . (A_{k+1} \ldots A_j)$$

The optimal cost:

$$\boxed{m_{i,j} = m_{i,k} + m_{k+1,\, j} + p_{i-1} \text{ x } p_k \text{ x } p_j}$$

where: $m_{i,k}$: Optimal cost of computing $A_{i..k}$

$m_{k+1,j}$: Optimal cost of computing $A_{k+1..j}$

$p_{i-1} \text{ x } p_k \text{ x } p_j$: Cost of multiplying $A_{i..k}$ and $A_{k+1\ldots j}$

Step 2: A Recursive Solution

In an optimal parenthesization:

k must be chosen to minimize m_{ij}

The recursive formulation for m_{ij}:

$$m_{ij} = \begin{cases} 0 & \text{if } i=j \\ \underset{i \le k < j}{\mathrm{MIN}} \{ m_{ik} + m_{k+1,j} + p_{i-1} p_k p_j \} & \text{if } i < j \end{cases}$$

Step 2: A Recursive Solution

- The m_{ij} values give the costs of optimal solutions to subproblems
- In order to keep track of how to construct an optimal solution
  - Define s_{ij} to be the value of k which yields the optimal split of the subchain $A_{i..j}$

    That is, $s_{ij} = k$ such that

    $$m_{ij} = m_{ik} + m_{k+1,j} + p_{i-1} p_k p_j \quad \text{holds}$$

Direct Recursion: Inefficient!

Recursive matrix-chain order

RMC(p, i, j)

if $i = j$ **then**
    return 0

$m[i, j] \leftarrow \infty$

for $k \leftarrow i$ **to** $j - 1$ **do**

    $q \leftarrow \text{RMC}(p, i, k) + \text{RMC}(p, k+1, j) + p_{i-1} p_k p_j$

    if $q < m[i, j]$ **then**

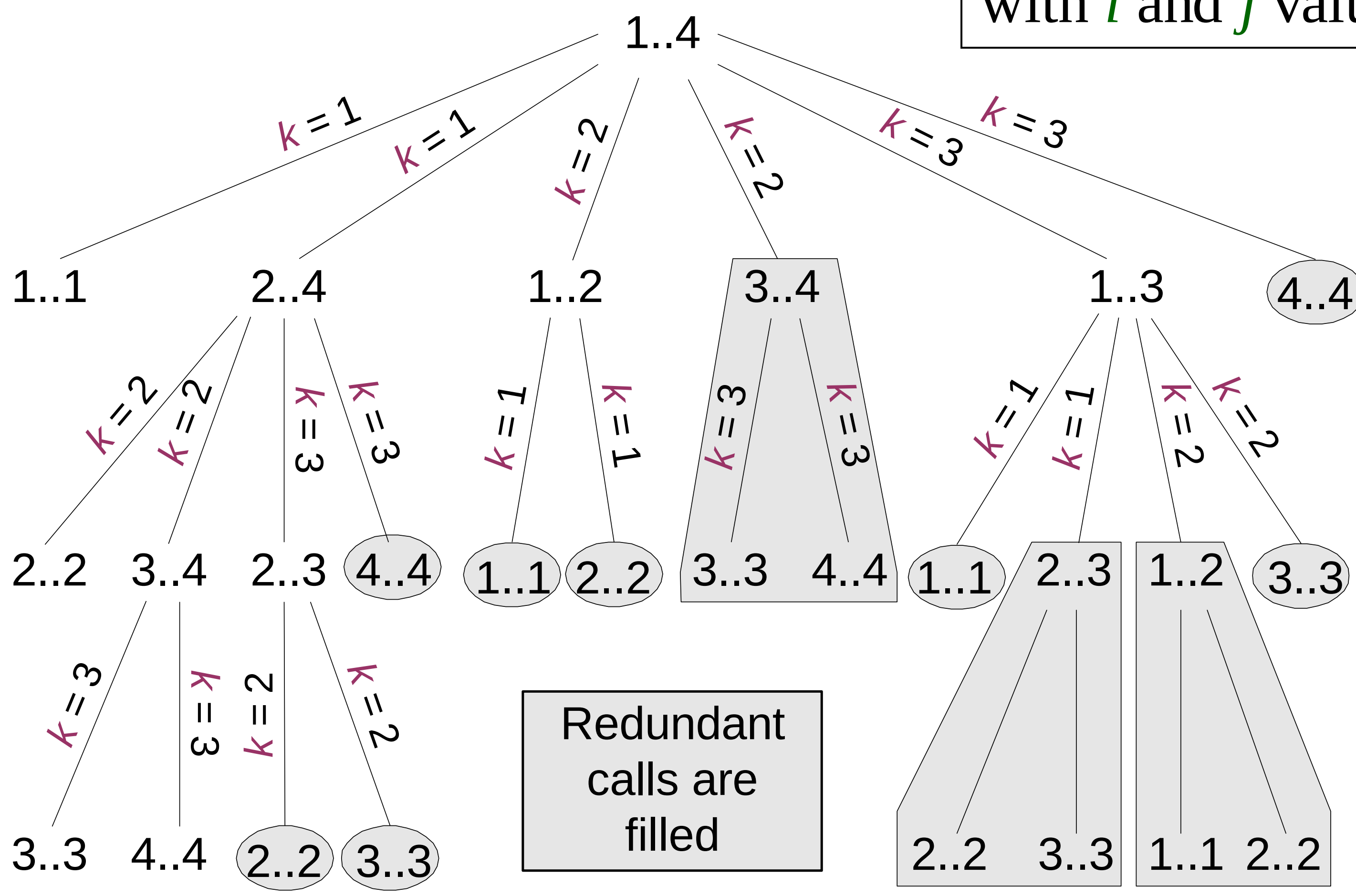
        $m[i, j] \leftarrow q$

return $m[i, j]$

Direct Recursion: Inefficient!

Recursion tree for RMC(p,1,4)

Nodes are labeled with i and j values



Redundant calls are filled

Computing the Optimal Cost (Matrix-Chain Multiplication)

An important observation:

- We have relatively few subproblems
  - one problem for each choice of i and j satisfying $1 \le i \le j \le n$
  - total $n + (n-1) + \ldots + 2 + 1 = \frac{1}{2} n(n+1) = \Theta(n^2)$ subproblems
- We can write a recursive algorithm based on recurrence.
- However, a recursive algorithm may encounter each subproblem many times in different branches of the recursion tree
- This property, overlapping subproblems, is the second important feature for applicability of dynamic programming

Computing the Optimal Cost (Matrix-Chain Multiplication)

Compute the value of an optimal solution in a bottom-up fashion

- matrix A_i has dimensions $p_{i-1} \times p_i$ for $i = 1, 2, \ldots, n$
- the input is a sequence $\langle p_0, p_1, \ldots, p_n \rangle$ where length[p] $= n + 1$

Procedure uses the following auxiliary tables:

- $m[1 \ldots n, 1 \ldots n]$: for storing the $m[i, j]$ costs
- $s[1 \ldots n, 1 \ldots n]$: records which index of k achieved the optimal cost in computing $m[i, j]$

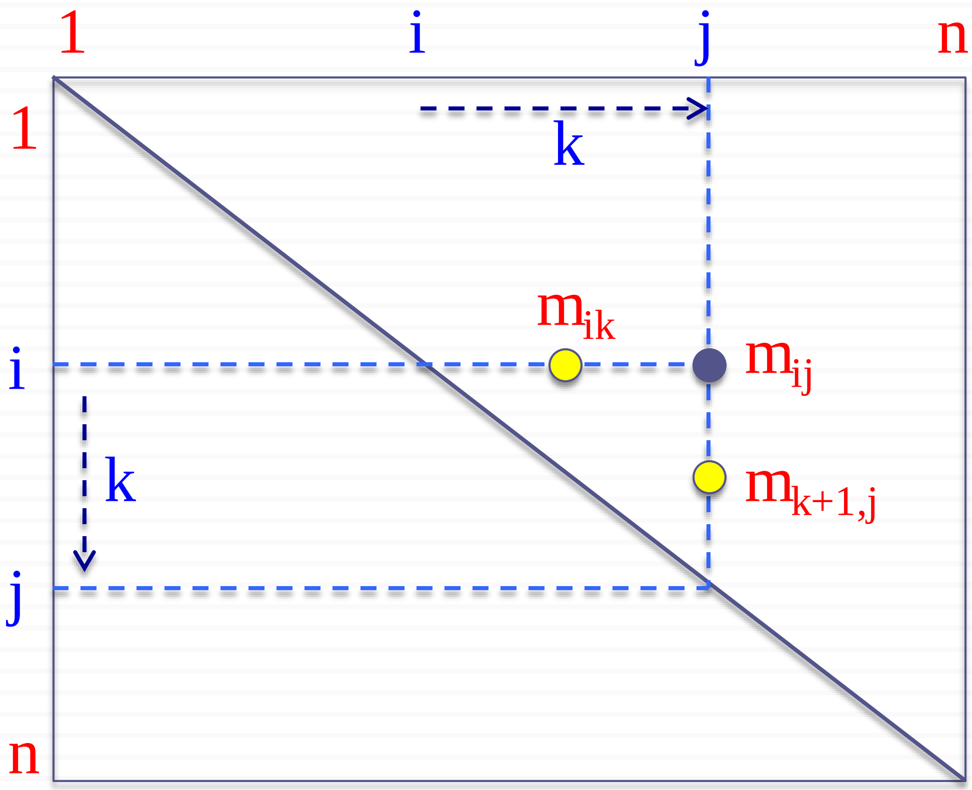
Bottom-up computation

$$m_{ij} = \min_{i \le k < j} \left\{ m_{ik} + m_{k+1,j} + p_{i-1} p_k p_j \right\}$$

How to choose the order in which we process m_{ij} values?

Before computing m_{ij}, we have to make sure that the values for m_{ik} and $m_{k+1,j}$ have been computed for all k.

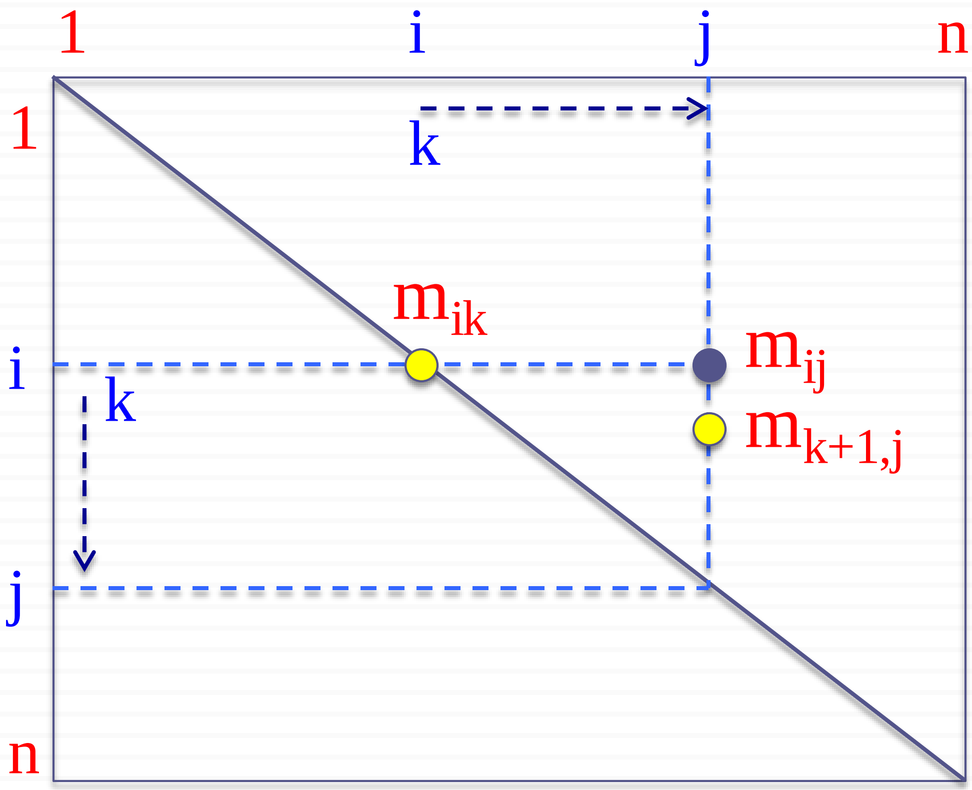
$$m_{ij} = \min_{i \le k < j} \{ m_{ik} + m_{k+1,j} + p_{i-1} p_k p_j \}$$



m_{ij} must be processed after m_{ik} and $m_{j,k+1}$

Reminder: m_{ij} computed only for $j > i$

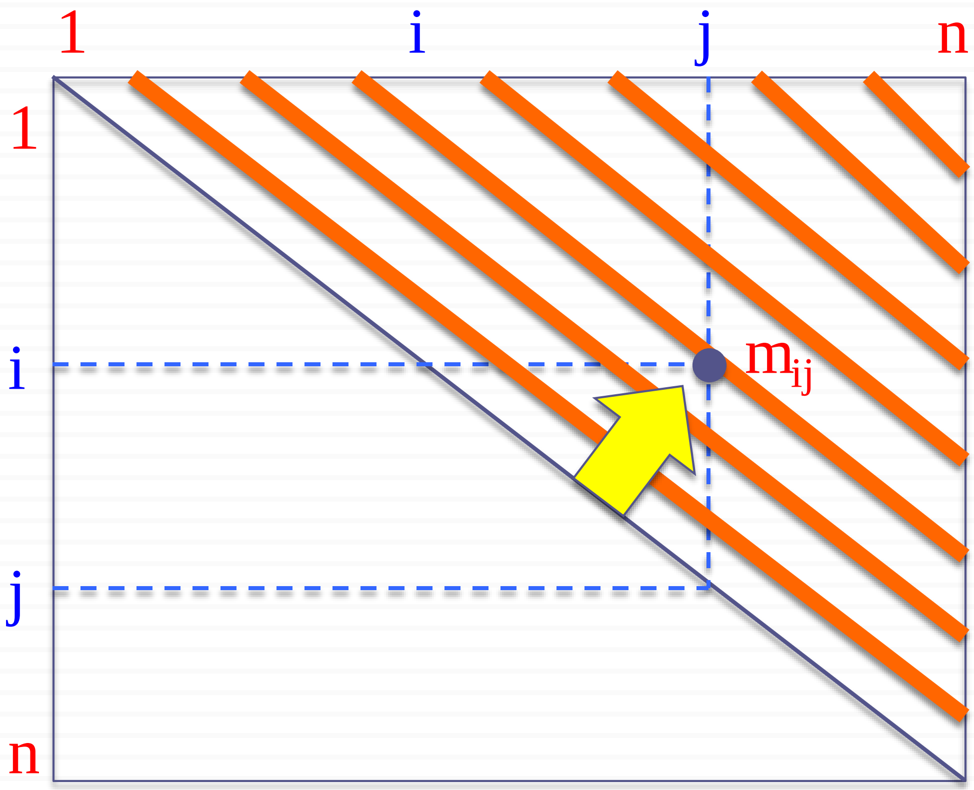
$$m_{ij} = \min_{i \le k < j} \{ m_{ik} + m_{k+1,j} + p_{i-1} p_k p_j \}$$



m_{ij} must be processed after m_{ik} and $m_{j,k+1}$

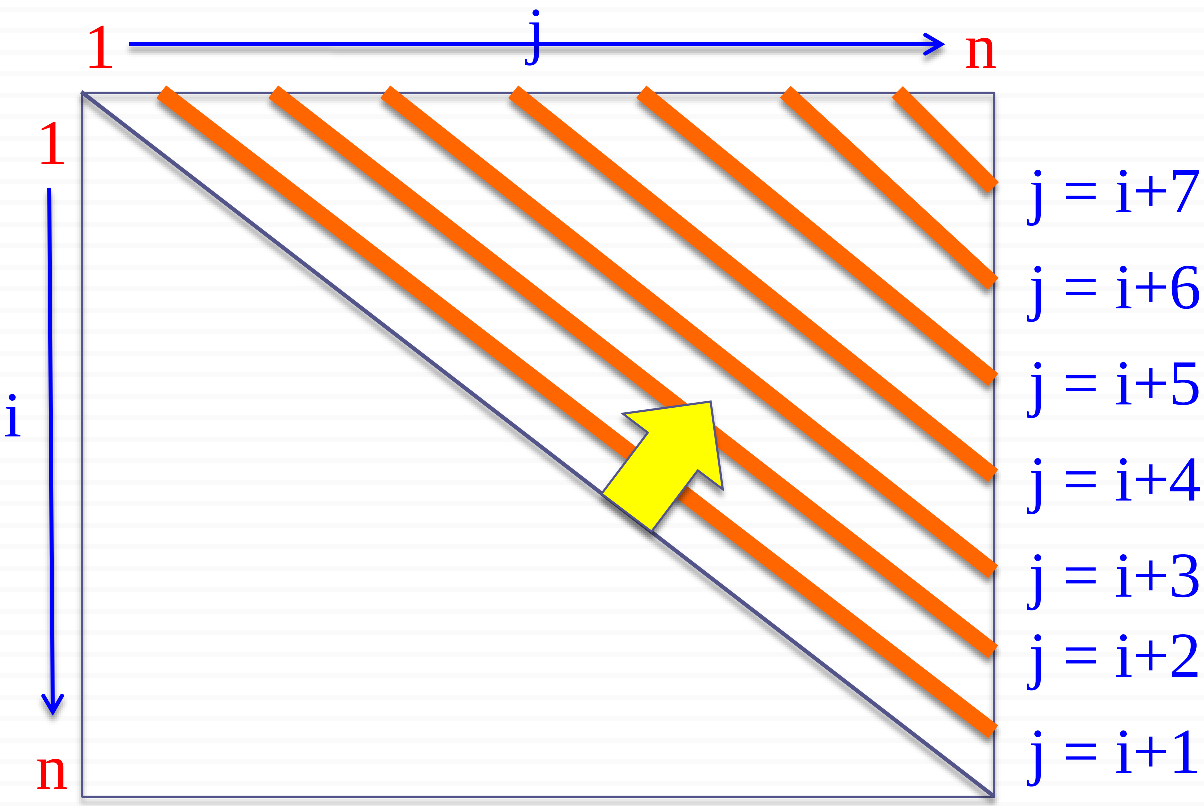
How to set up the iterations over i and j to compute m_{ij}?

$$m_{ij} = \min_{i \leq k < j} \{ m_{ik} + m_{k+1,j} + p_{i-1} p_k p_j \}$$

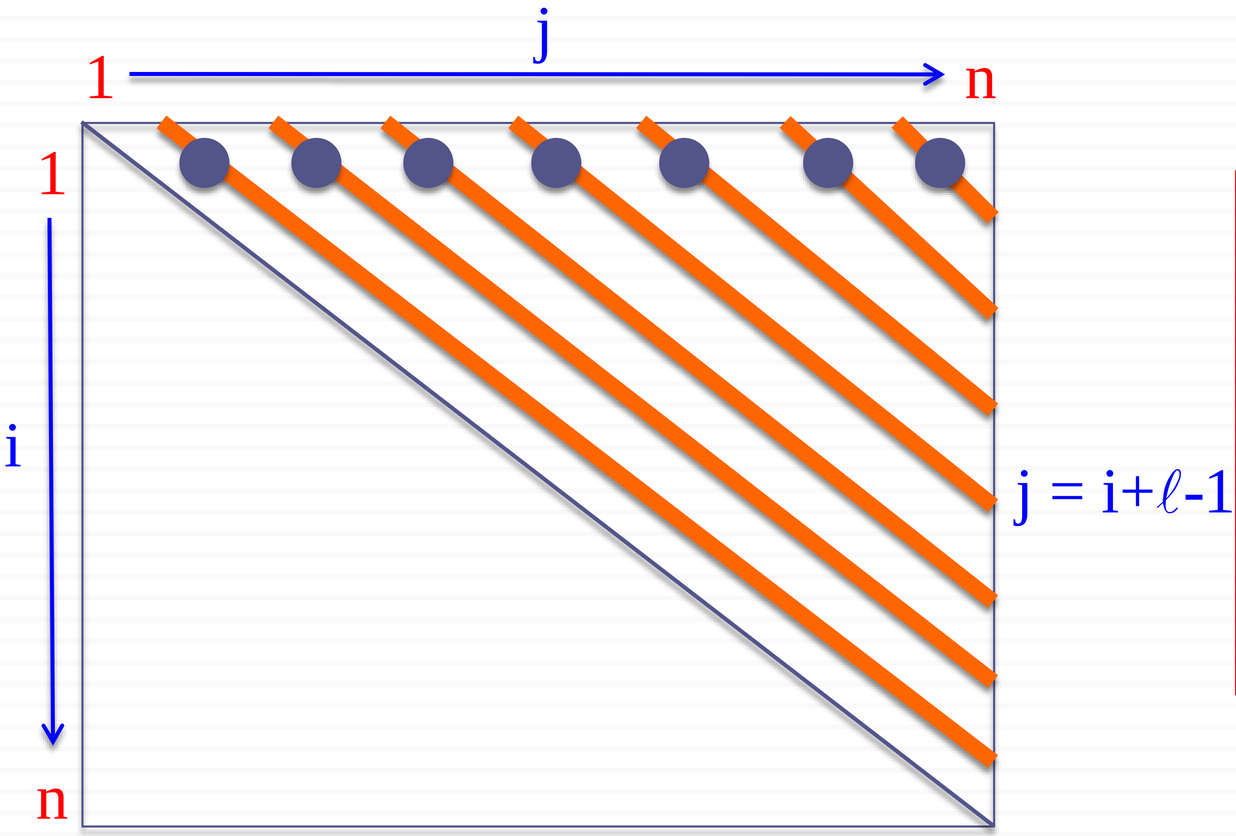


If the entries m_{ij} are computed in the shown order, then m_{ik} and $m_{k+1,j}$ values are guaranteed to be computed before m_{ij}.

$$m_{ij} = \min_{i \leq k < j} \{ m_{ik} + m_{k+1,j} + p_{i-1} p_k p_j \}$$



$$m_{ij} = \min_{i \le k < j} \{ m_{ik} + m_{k+1,j} + p_{i-1} p_k p_j \}$$



for ℓ=2 **to** n

  for i=1 **to** n-ℓ+1

    j = i + ℓ - 1

    ……

    m_{ij} = …

    ……

Algorithm for Computing the Optimal Costs

MATRIX-CHAIN-ORDER(p)

$n \leftarrow$ length[p] -1
for $i \leftarrow 1$ to n do
  $m[i, i] \leftarrow 0$
for $\ell \leftarrow 2$ to n do
  for $i \leftarrow 1$ to $n - \ell + 1$ do
    $j \leftarrow i + \ell - 1$
    $m[i, j] \leftarrow \infty$
    for $k \leftarrow i$ to j–1 do
      $q \leftarrow m[i, k] + m[k+1, j] + p_{i-1} p_k p_j$
      if $q < m[i, j]$ then
        $m[i, j] \leftarrow q$
        $s[i, j] \leftarrow k$
return m and s

Algorithm for Computing the Optimal Costs

- The algorithm first computes
  $m[i, i] \leftarrow 0$ for $i = 1, 2, \ldots, n$ min costs for all chains of length 1

- Then, for $\ell = 2, 3, \ldots, n$ computes
  $m[i, i+\ell-1]$ for $i = 1, \ldots, n-\ell+1$ min costs for all chains of length ℓ

- For each value of $\ell = 2, 3, \ldots, n$,
  $m[i, i+\ell-1]$ depends only on table entries $m[i, k]$ & $m[k+1, i+\ell-1]$
  for $i \le k < i+\ell-1$, which are already computed

Algorithm for Computing the Optimal Costs

$\ell = 2$
for $i = 1$ to $n - 1$
  $m[i, i+1] = \infty$
  for $k = i$ to i do
  .
  .

compute $m[i, i+1]$

$\underbrace{\{m[1, 2], m[2, 3], \ldots, m[n-1, n]\}}_{(n-1) \text{ values}}$

$\ell = 3$
for $i = 1$ to $n - 2$
  $m[i, i+2] = \infty$
  for $k = i$ to $i+1$ do
  .
  .

compute $m[i, i+2]$

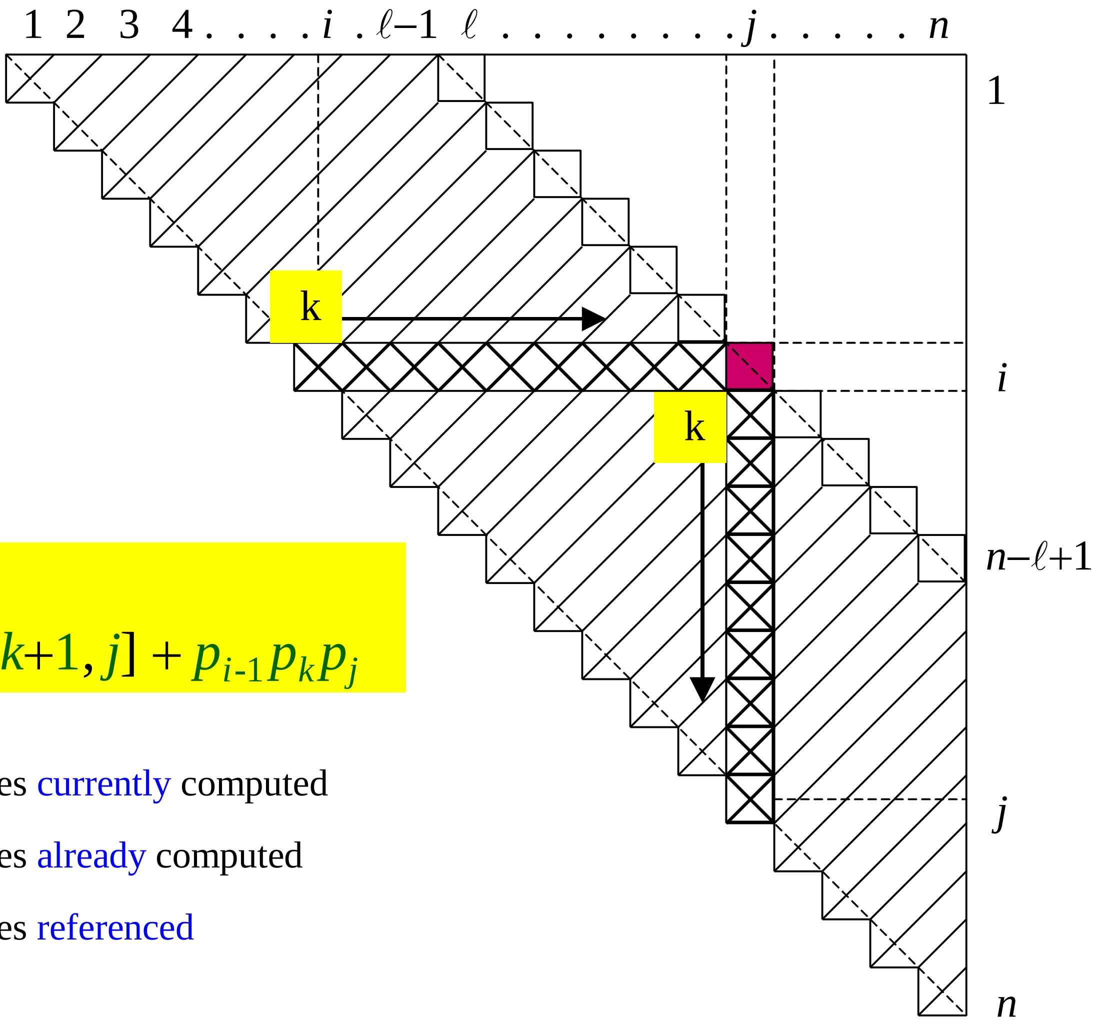
$\underbrace{\{m[1, 3], m[2, 4], \ldots, m[n-2, n]\}}_{(n-2) \text{ values}}$

$\ell = 4$
for $i = 1$ to $n - 3$
  $m[i, i+3] = \infty$
  for $k = i$ to $i+2$ do
  .
  .

compute $m[i, i+3]$

$\underbrace{\{m[1, 4], m[2, 5], \ldots, m[n-3, n]\}}_{(n-3) \text{ values}}$

Table access pattern in computing $m[i, j]$s for $\ell = j - i + 1$



for $k \leftarrow i$ to $j-1$ do
$\quad q \leftarrow m[i, k] + m[k+1, j] + p_{i-1} p_k p_j$

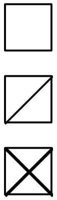
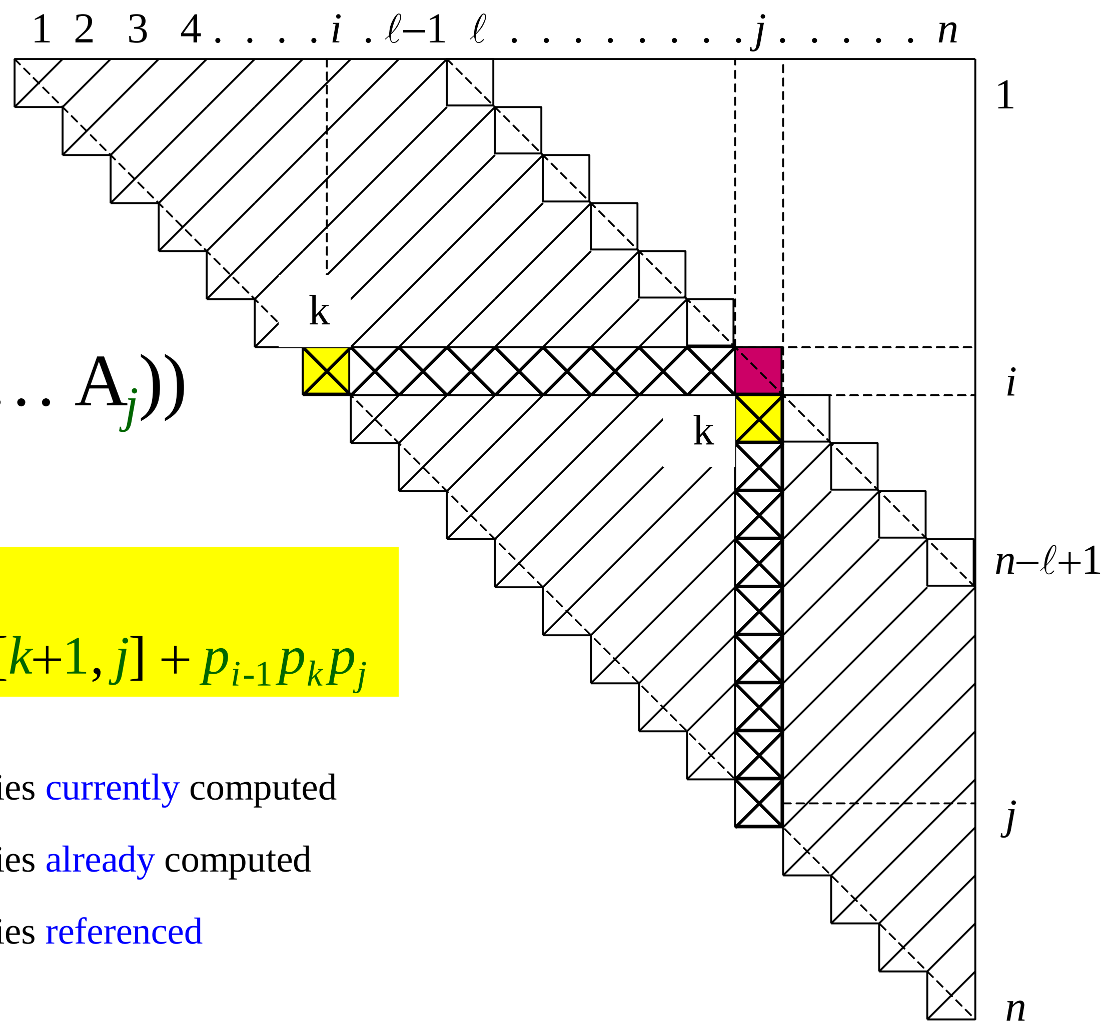


Table entries currently computed

Table entries already computed

Table entries referenced

Table access pattern in computing $m[i, j]$s for $\ell=j-i+1$



$((A_i)\ (A_{i+1}A_{i+2}\ \ldots\ A_j))$

for $k \leftarrow i$ to $j-1$ do

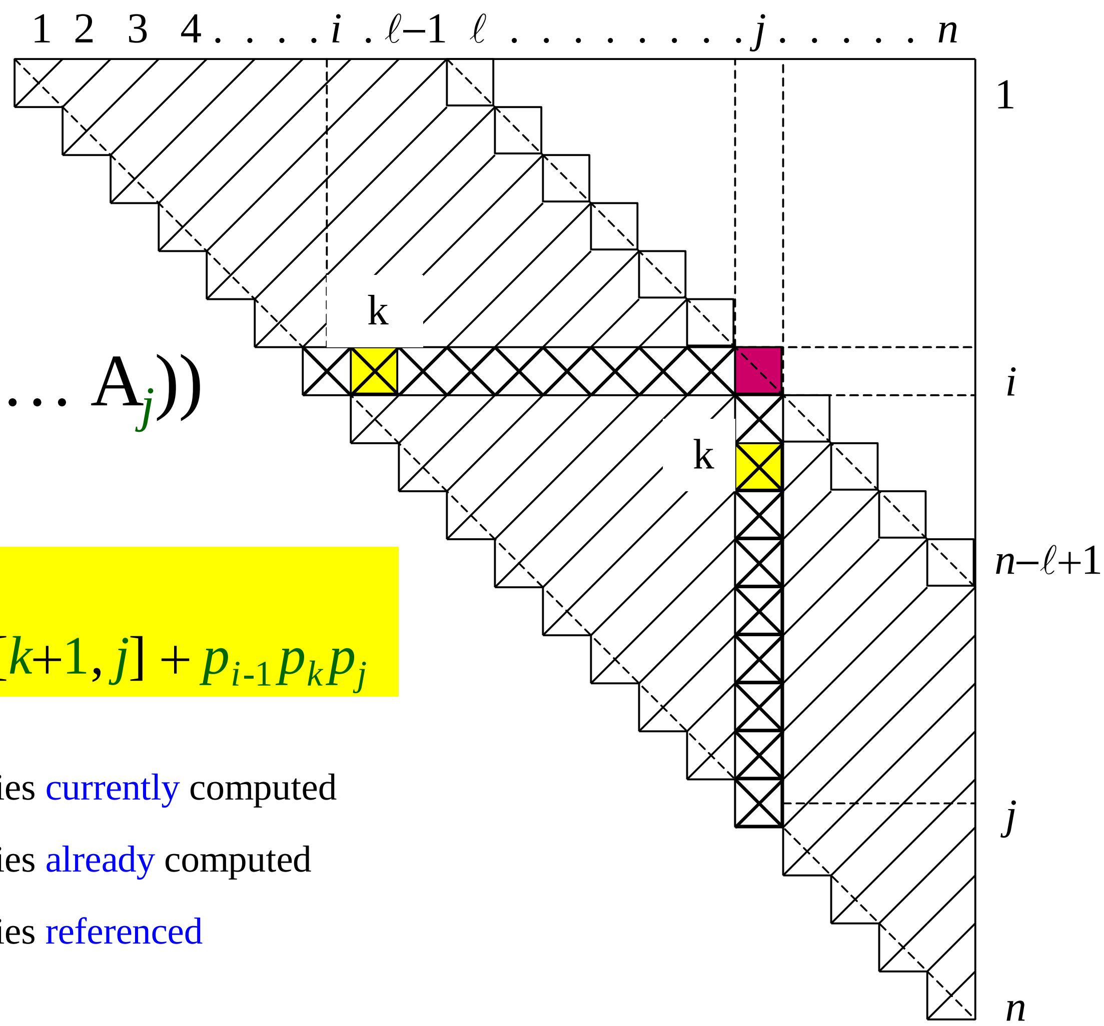
$\quad q \leftarrow m[i, k] + m[k+1, j] + p_{i-1}p_kp_j$

☐ Table entries currently computed

◪ Table entries already computed

☒ Table entries referenced

Table access pattern in computing $m[i, j]$s for $\ell = j-i+1$



$$((A_i A_{i+1})\ (A_{i+2}\ \ldots\ A_j))$$

for $k \leftarrow i$ to $j-1$ do

$\quad q \leftarrow m[i, k] + m[k+1, j] + p_{i-1} p_k p_j$

Table entries currently computed

Table entries already computed

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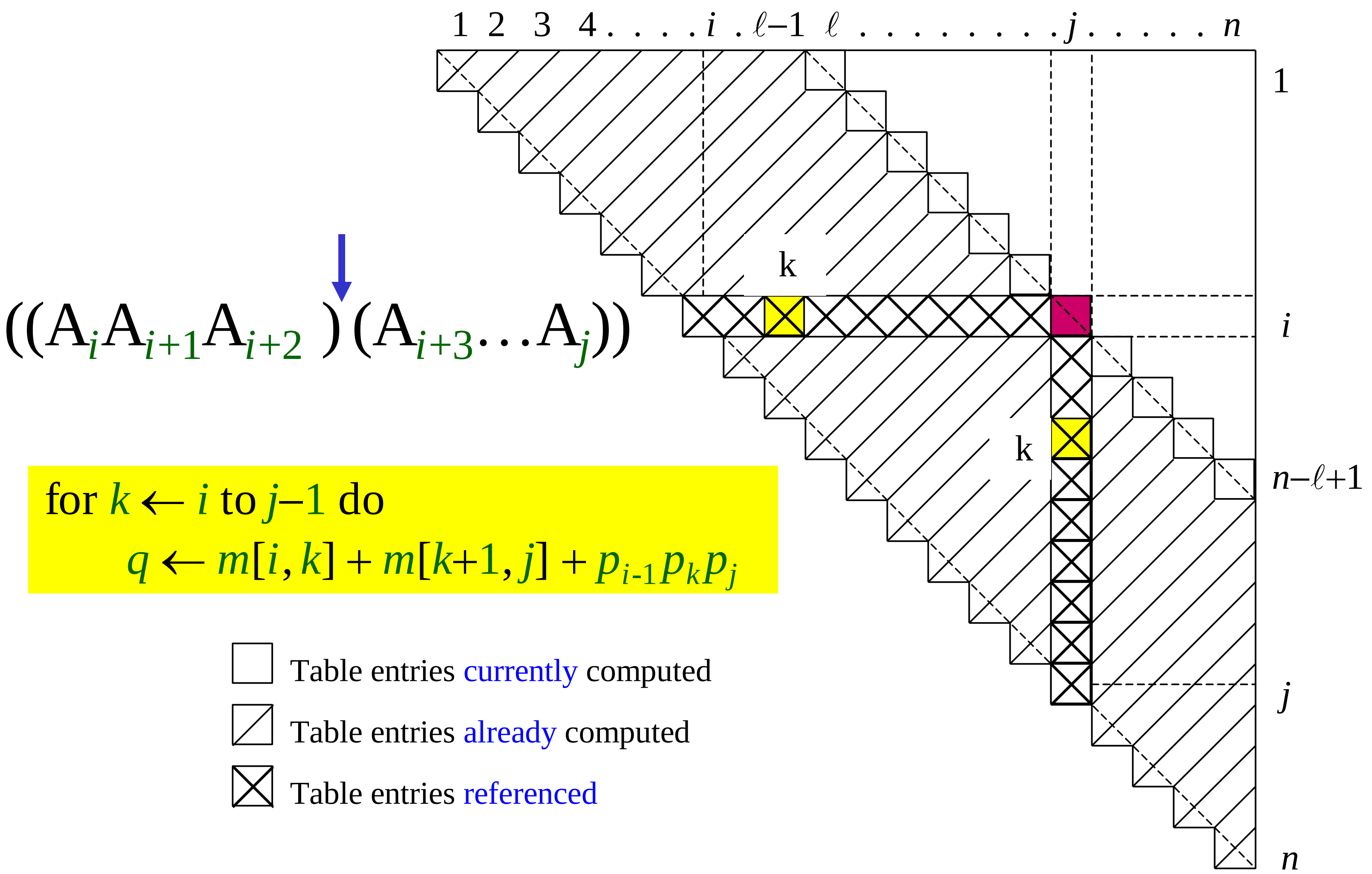
Table access pattern in computing $m[i, j]$s for $\ell=j-i+1$

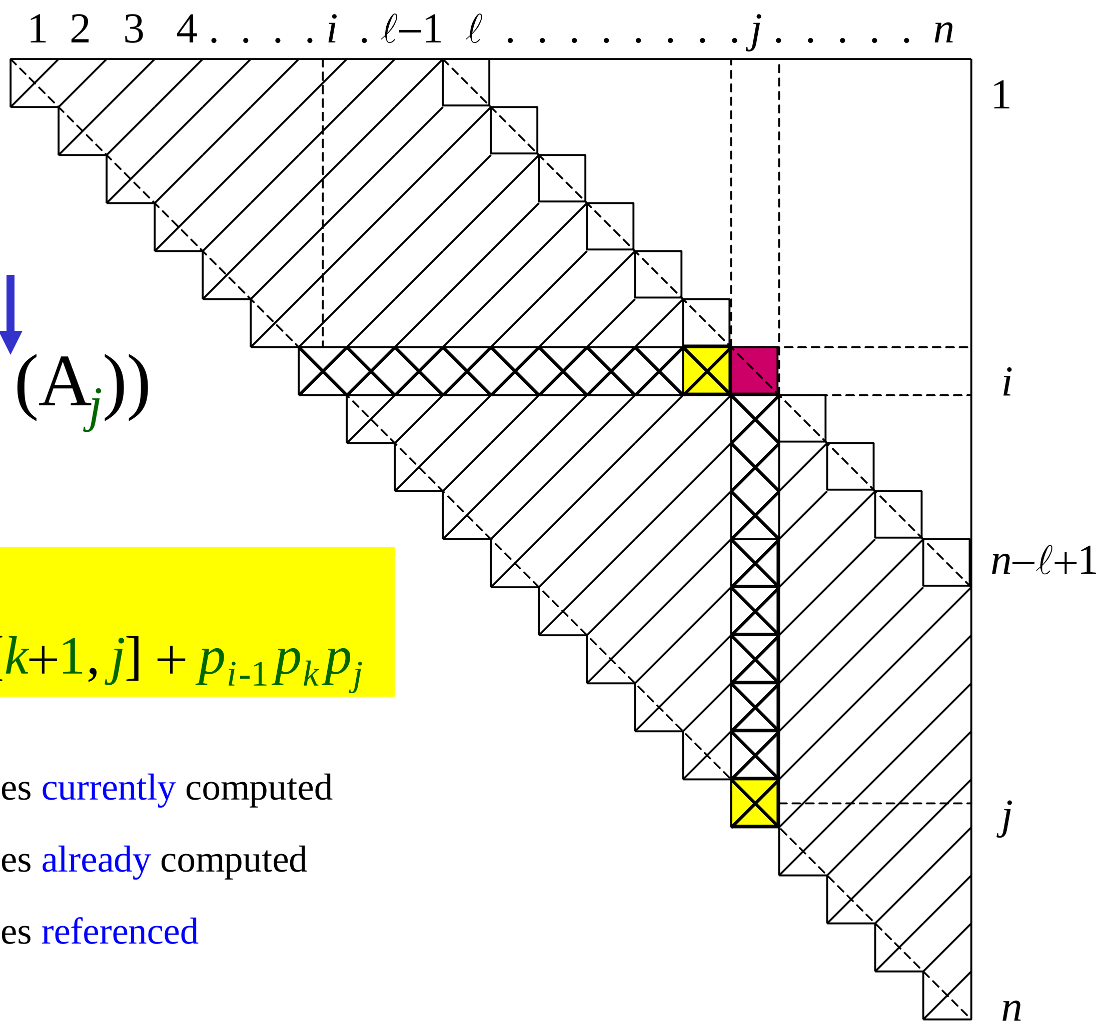


for $k \leftarrow i$ to $j-1$ do

$q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$

- Table entries currently computed
- Table entries already computed
- Table entries referenced

Table access pattern in computing $m[i, j]$s for $\ell = j - i + 1$



$((A_i A_{i+1} \ldots\ A_{j-1})\ (A_j))$

for $k \leftarrow i$ to $j-1$ do

$\quad q \leftarrow m[i, k] + m[k+1, j] + p_{i-1} p_k p_j$

Table entries currently computed

Table entries already computed

Table entries referenced

Example

$$m_{ij} = \min_{i \le k < j} \{ m_{ik} + m_{k+1,j} + p_{i-1} p_k p_j \}$$

1	2	3	4	5	6	
0	15750	7875	9375			1
	0	2625	4375	???		2
		0	750	2500		3
			0	1000	3500	4
				0	5000	5
					0	6

A_1: (30x35)
A_2: (35x15)
A_3: (15x5)
A_4: (5x10)
A_5: (10x20)
A_6: (20x25)

Compute m_{25}

k=2

$(A_2)\ (A_3 A_4 A_5)$

$$\text{cost} = m_{22} + m_{35} + p_1 p_2 p_5$$
$$= 0 + 2500 + 35 \times 15 \times 20$$
$$= 13000$$

Choose the k value that leads to min cost

Example

$$m_{ij} = \min_{i \le k < j} \{ m_{ik} + m_{k+1,j} + p_{i-1} p_k p_j \}$$

A_1: (30x35)
A_2: (35x15)
A_3: (15x5)
A_4: (5x10)
A_5: (10x20)
A_6: (20x25)

Compute m_{25}

k=3

$(A_2A_3)\ (A_4A_5)$

cost $= m_{23} + m_{45} + p_1p_3p_5$
$= 2625 + 1000 + 35 \times 5 \times 20$
$= 7125$

	1	2	3	4	5	6
1	0	15750	7875	9375		
2		0	2625	4375	???	
3			0	750	2500	
4				0	1000	3500
5					0	5000
6						0

Choose the k value that leads to min cost

Example

$$m_{ij} = \min_{i \le k < j} \{ m_{ik} + m_{k+1,j} + p_{i-1} p_k p_j \}$$

A_1: (30x35)
A_2: (35x15)
A_3: (15x5)
A_4: (5x10)
A_5: (10x20)
A_6: (20x25)

Compute m_{25}

	1	2	3	4	5	6
1	0	15750	7875	9375		
2		0	2625	4375	???	
3			0	750	2500	
4				0	1000	3500
5					0	5000
6						0

k=4

$(A_2A_3A_4)\ (A_5)$

cost $= m_{24} + m_{55} + p_1p_4p_5$
$= 4375 + 0 + 35 \times 10 \times 20$
$= 11375$

Choose the k value that leads to min cost

Example

$$m_{ij} = \min_{i \leq k < j}\{m_{ik} + m_{k+1,j} + p_{i-1}p_k p_j\}$$

A_1: (30x35)
A_2: (35x15)
A_3: (15x5)
A_4: (5x10)
A_5: (10x20)
A_6: (20x25)

Compute m_{25}

k=3

$(A_2A_3)\ (A_4A_5)$

$m_{25} = 7125$
$s_{25} = 3$

1	2	3	4	5	6	
0	15750	7875	9375			1
	0	2625	4375	7125		2
		0	750	2500		3
			0	1000	3500	4
				0	5000	5
					0	6

Choose k=3

Constructing an Optimal Solution

- MATRIX-CHAIN-ORDER determines the optimal # of scalar mults/adds
  - needed to compute a matrix-chain product
  - it does not directly show how to multiply the matrices

- That is,
  - it determines the cost of the optimal solution(s)
  - it does not show how to obtain an optimal solution

- Each entry $s[i, j]$ records the value of k such that optimal parenthesization of $A_i \ldots A_j$ splits the product between A_k & A_{k+1}

- We know that the final matrix multiplication in computing $A_{1\ldots n}$ optimally is $A_{1\ldots s[1,n]} \times A_{s[1,n]+1,n}$

Example: Constructing an Optimal Solution

Reminder: s_{ij} is the optimal top-level split of $A_i \ldots A_j$

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

What is the optimal top-level split for:

$$A_1 A_2 A_3 A_4 A_5 A_6$$

$$s_{16} = 3$$

Example: Constructing an Optimal Solution

Reminder: s_{ij} is the optimal top-level split of $A_i \ldots A_j$

2	3	4	5	6	
1	1	3	3	3	1
	2	3	3	3	2
		3	3	3	3
			4	5	4
				5	5

$$(A_1 A_2 A_3) \overset{k=3}{\downarrow} (A_4 A_5 A_6)$$

What is the optimal split for $A_1 \ldots A_3$? $s_{13} = 1$

What is the optimal split for $A_4 \ldots A_6$? $s_{46} = 5$

Cevdet Aykanat and Mustafa Ozdal
Computer Engineering Department, Bilkent University

Example: Constructing an Optimal Solution

2	3	4	5	6	
1	1	3	3	3	1
	2	3	3	3	2
		3	3	3	3
			4	5	4
				5	5

Reminder: s_{ij} is the optimal top-level split of $A_i \ldots A_j$

k=1   k=5

$$((A_1)\ \downarrow\ (A_2A_3))\ ((A_4A_5)\ \downarrow\ (A_6))$$

What is the optimal split for $A_1 \ldots A_3$? $s_{13} = 1$

What is the optimal split for $A_4 \ldots A_6$? $s_{46} = 5$

Example: Constructing an Optimal Solution

Reminder: s_{ij} is the optimal top-level split of $A_i \ldots A_j$

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

$$((A_1)\ (A_2A_3))\ ((A_4A_5)\ (A_6))$$

What is the optimal split for A_2A_3? $s_{23} = 2$

What is the optimal split for A_4A_5? $s_{45} = 4$

Cevdet Aykanat and Mustafa Ozdal
Computer Engineering Department, Bilkent University

Example: Constructing an Optimal Solution

Reminder: s_{ij} is the optimal top-level split of $A_i \ldots A_j$

2	3	4	5	6	
1	1	3	3	3	1
	2	3	3	3	2
		3	3	3	3
			4	5	4
				5	5

k=2 k=4

$$((A_1)\ ((A_2)\ (A_3)))\ (((A_4)\ (A_5))\ (A_6))$$

What is the optimal split for A_2A_3? $s_{23} = 2$

What is the optimal split for A_4A_5? $s_{45} = 4$

Constructing an Optimal Solution

Earlier optimal matrix multiplications can be computed recursively

Given:

- the chain of matrices $A = \langle A_1, A_2, \ldots A_n \rangle$
- the s table computed by MATRIX-CHAIN-ORDER

The following recursive procedure computes the matrix-chain product $A_{i \ldots j}$

```
MATRIX-CHAIN-MULTIPLY(A, s, i, j)
    if j > i then
        X ← MATRIX-CHAIN-MULTIPLY(A, s, i, s[i, j])
        Y ← MATRIX-CHAIN-MULTIPLY(A, s, s[i, j]+1, j)
        return MATRIX-MULTIPLY(X, Y)
    else
        return A_i
```

Invocation: MATRIX-CHAIN-MULTIPLY(A, s, 1, n)

Example: Recursive Construction of an Optimal Solution

$s[1\ldots6,\ 1\ldots6]$

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	4	3
3			3	3	3
4				4	5
5					5

MCM(1,6)
 $X \leftarrow$ MCM(1,3)$=(A_1A_2A_3)$
 $Y \leftarrow$ MCM(4,6)$=(A_4A_5A_6)$
 return (?)

MCM(1,3)
 $X \leftarrow$ MCM(1,1)$=A_1$
 $Y \leftarrow$ MCM(2,3)$=(A_2A_3)$
 return (?)

return A_1

Example: Recursive Construction of an Optimal Solution

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	4	3
3			3	3	3
4				4	5
5					5

$s[1\ldots6,\ 1\ldots6]$

MCM(1,6)
 X $\leftarrow$ MCM(1,3) = $(A_1(A_2A_3))$
 Y $\leftarrow$ MCM(4,6) = $(A_4A_5A_6)$
 return (?)

MCM(1,3)
 X $\leftarrow$ MCM(1,1) = A_1
 Y $\leftarrow$ MCM(2,3) = (A_2A_3)
 return $(A_1(A_2A_3))$

return A_1

MCM(2,3)
 X $\leftarrow$ MCM(2,2) = A_2 return A_2
 Y $\leftarrow$ MCM(3,3) = A_3 return A_3
 return (A_2A_3)

Example: Recursive Construction of an Optimal Solution

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	4	3
3			3	3	3
4				4	5
5					5

$s[1\ldots 6, 1\ldots 6]$

MCM(1,6)
$X \leftarrow$ MCM(1,3)=$(A_1(A_2A_3))$
$Y \leftarrow$ MCM(4,6)=$((A_4A_5)A_6)$
return $(A_1(A_2A_3))((A_4A_5)A_6)$

MCM(1,3)
$X \leftarrow$ MCM(1,1)=A_1
$Y \leftarrow$ MCM(2,3)=(A_2A_3)
return $(A_1(A_2A_3))$

return A_1

MCM(2,3)
$X \leftarrow$ MCM(2,2)=A_2
$Y \leftarrow$ MCM(3,3)=A_3
return (A_2A_3)

return A_2

return A_3

MCM(4,6)
$X \leftarrow$ MCM(4,5)=(A_4A_5)
$Y \leftarrow$ MCM(6,6)=A_6
return $((A_4A_5)A_6)$

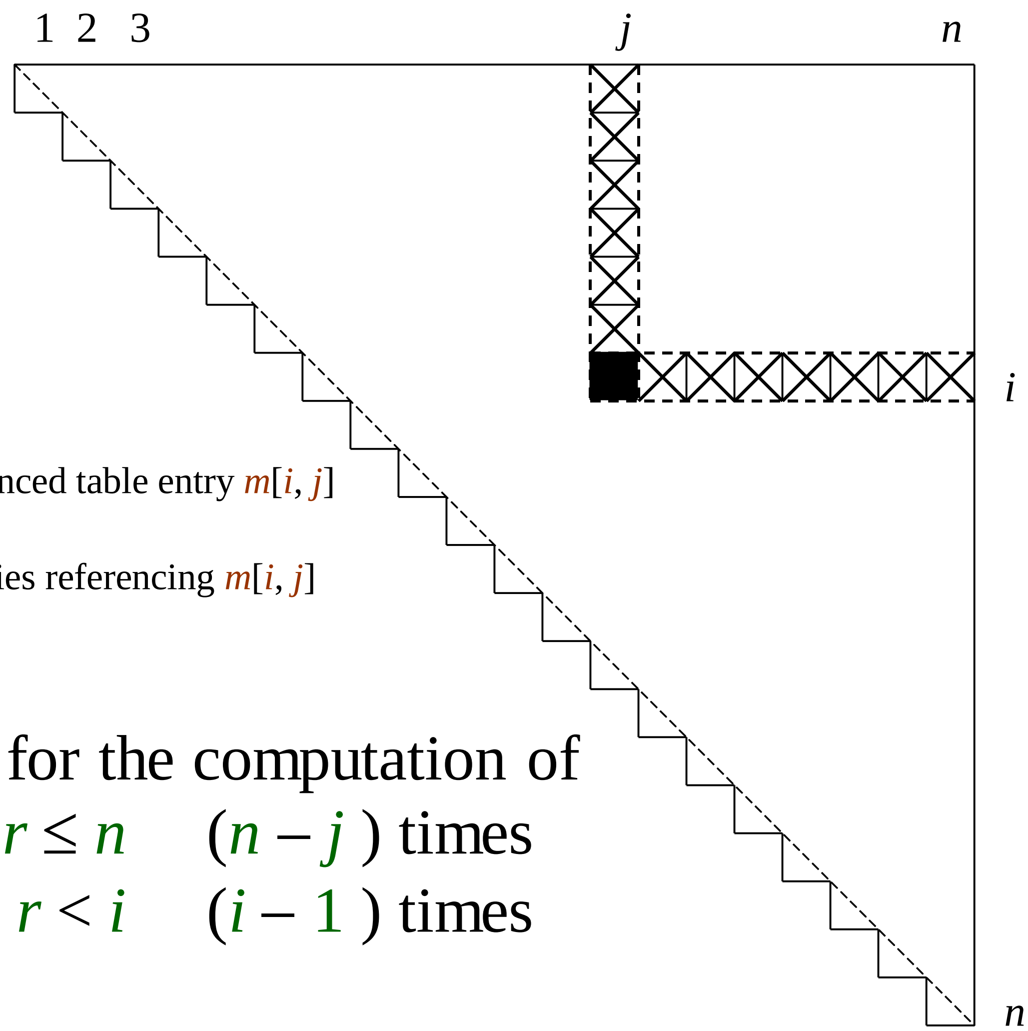
MCM(4,5)
$X \leftarrow$ MCM(4,4)=A_4
$Y \leftarrow$ MCM(5,5)=A_5
return (A_4A_5)

return A_4

return A_5

return A_6

Table reference pattern for $m[i, j]$ $(1 \leq i \leq j \leq n)$



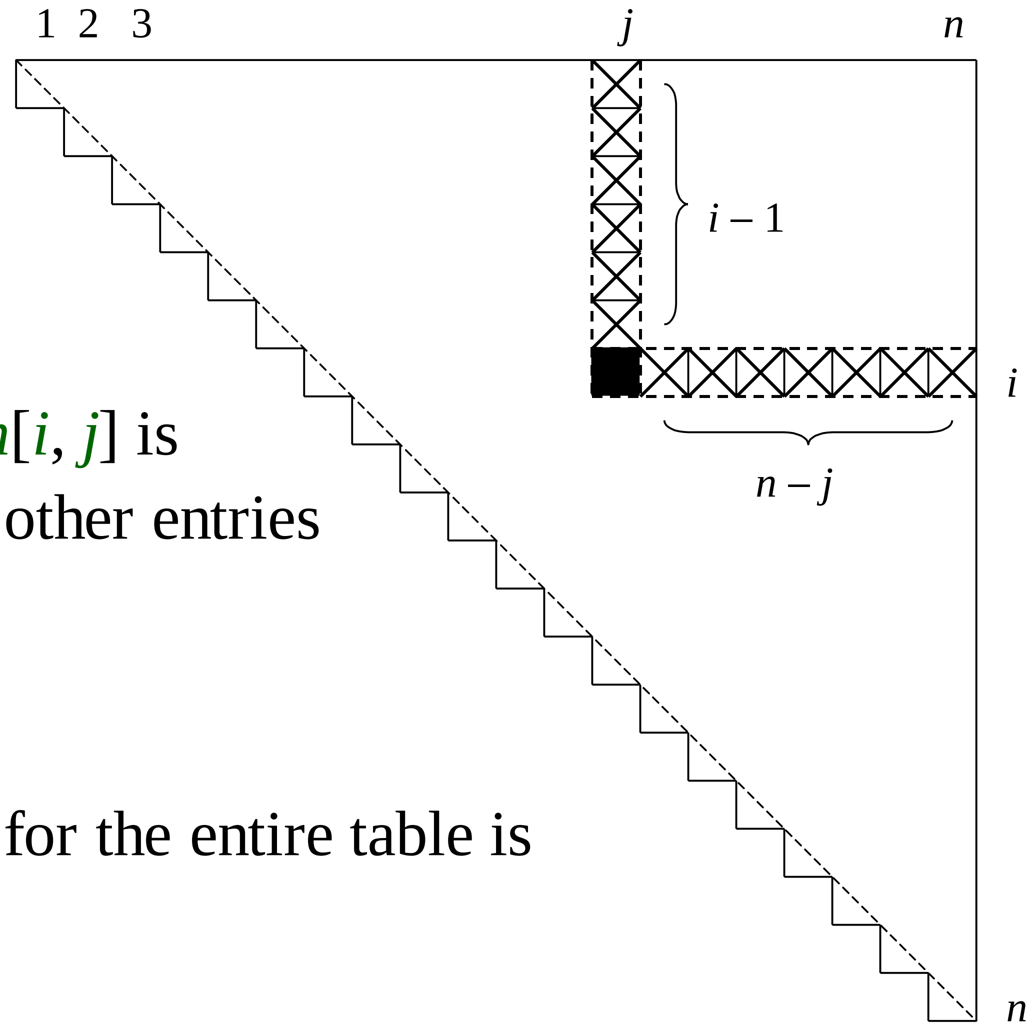
■ The referenced table entry $m[i, j]$

☒ Table entries referencing $m[i, j]$

$m[i, j]$ is referenced for the computation of

- $m[i, r]$ for $j < r \leq n$ $(n - j)$ times
- $m[r, j]$ for $1 \leq r < i$ $(i - 1)$ times

Table reference pattern for $m[i, j]$ $(1 \le i \le j \le n)$



$R(i, j)$ = # of times that $m[i, j]$ is referenced in computing other entries

$$R(i, j) = (n-j) + (i-1)$$
$$= (n-1) - (j-i)$$

The total # of references for the entire table is

$$\sum_{i=1}^{n}\sum_{j=i}^{n} R(i, j) = \frac{n^3 - n}{3}$$

Summary

1. Identification of the optimal substructure property

2. Recursive formulation to compute the cost of the optimal solution

3. Bottom-up computation of the table entries

4. Constructing the optimal solution by backtracing the table entries

Elements of Dynamic Programming

- When should we look for a DP solution to an optimization problem?
- Two key ingredients for the problem
  - Optimal substructure
  - Overlapping subproblems

DP Hallmark #1

Optimal Substructure

- A problem exhibits optimal substructure
  - if an optimal solution to a problem contains within it optimal solutions to subproblems
- Example: matrix-chain-multiplication

Optimal parenthesization of $A_1 A_2 \ldots A_n$ that splits the product between A_k and A_{k+1},

contains within it optimal soln's to the problems of parenthesizing $A_1 A_2 \ldots A_k$ and $A_{k+1} A_{k+2} \ldots A_n$

Optimal Substructure

Finding a suitable space of subproblems

- Iterate on subproblem instances
- Example: matrix-chain-multiplication
  - Iterate and look at the structure of optimal soln's to subproblems, sub-subproblems, and so forth
  - Discover that all subproblems consists of subchains of $\langle A_1, A_2, \ldots, A_n \rangle$
  - Thus, the set of chains of the form

$$\langle A_i, A_{i+1}, \ldots, A_j \rangle \text{ for } 1 \le i \le j \le n$$

  - Makes a natural and reasonable space of subproblems

DP Hallmark #2

Overlapping Subproblems

- Total number of distinct subproblems should be polynomial in the input size
- When a recursive algorithm revisits the same problem over and over again

  we say that the optimization problem has overlapping subproblems

Overlapping Subproblems

- DP algorithms typically take advantage of overlapping subproblems
  - by solving each problem once
  - then storing the solutions in a table where it can be looked up when needed
  - using constant time per lookup

Overlapping Subproblems

Recursive matrix-chain order

RMC(p, i, j)

if $i = j$ **then**
  return 0

$m[i, j] \leftarrow \infty$

for $k \leftarrow i$ **to** $j - 1$ **do**

  $q \leftarrow \mathrm{RMC}(p, i, k) + \mathrm{RMC}(p, k+1, j) + p_{i-1}\, p_k\, p_j$

  if $q < m[i, j]$ **then**

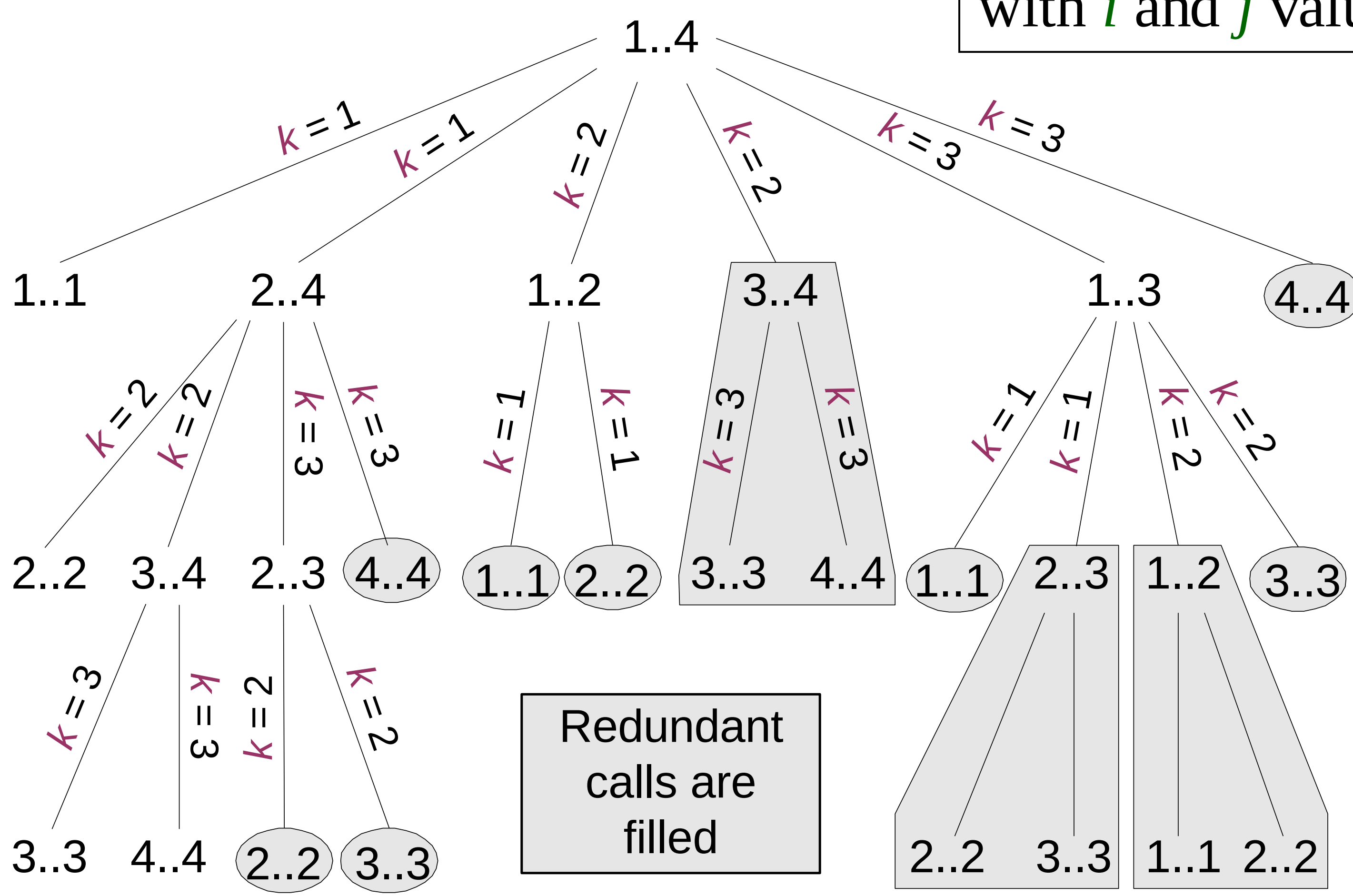
    $m[i, j] \leftarrow q$

return $m[i, j]$

Recursive Matrix-chain Order

Recursion tree for RMC(p,1,4)

Nodes are labeled with i and j values



Running Time of RMC

$T(1) \geq 1$

$$T(n) \geq 1 + \sum_{k=1}^{n-1} \big(T(k) + T(n-k) + 1\big) \text{ for } n > 1$$

- For $i = 1, 2, \ldots, n$ each term $T(i)$ appears twice
  - Once as $T(k)$, and once as $T(n-k)$
- Collect $n-1$ 1's in the summation together with the front 1

$$\boxed{T(n) \geq 2\sum_{i=1}^{n-1} T(i) + n}$$

- Prove that $T(n) = \Omega(2^n)$ using the substitution method

Running Time of RMC: Prove that $T(n) = \Omega(2^n)$

- Try to show that $T(n) \geq 2^{n-1}$ (by substitution)

Base case: $T(1) \geq 1 = 2^0 = 2^{1-1}$ for $n = 1$

IH: $T(i) \geq 2^{i-1}$ for all $i = 1, 2, \ldots, n-1$ and $n \geq 2$

$$T(n) \geq 2 \sum_{i=1}^{n-1} 2^{i-1} + n$$

$$= 2 \sum_{i=0}^{n-2} 2^{i} + n = 2(2^{n-1} - 1) + n$$

$$= 2^{n-1} + (2^{n-1} - 2 + n)$$

$$\Rightarrow T(n) \geq 2^{n-1}$$

Q.E.D.

Running Time of RMC: $T(n) \geq 2^{n-1}$

Whenever

- a recursion tree for the natural recursive solution to a problem contains the same subproblem repeatedly
- the total number of different subproblems is small

it is a good idea to see if DP can be applied

Memoization

- Offers the efficiency of the usual DP approach while maintaining top-down strategy
- Idea is to memoize the natural, but inefficient, recursive algorithm

Memoized Recursive Algorithm

- Maintains an entry in a table for the soln to each subproblem
- Each table entry contains a special value to indicate that the entry has yet to be filled in
- When the subproblem is first encountered its solution is computed and then stored in the table
- Each subsequent time that the subproblem encountered the value stored in the table is simply looked up and returned

Memoized Recursive Matrix-chain Order

MemoizedMatrixChain(p)

$n \leftarrow$ length[p] -1

for $i \leftarrow 1$ **to** n **do**

  for $j \leftarrow 1$ **to** n **do**

    $m[i, j] \leftarrow \infty$

return LookupC(p, 1, n)

LookupC(p, i, j)

if $m[i, j] = \infty$ **then**

  if $i = j$ **then**

    $m[i, j] \leftarrow$ **0**

  else

    for $k \leftarrow i$ **to** $j - 1$ **do**

      $q \leftarrow$ LookupC(p, i, k) + LookupC(p, k+1, j) + $p_{i-1} \, p_k \, p_j$

      if $q < m[i, j]$ **then**

        $m[i, j] \leftarrow q$

return $m[i, j]$

▹Shaded subtrees are looked-up rather than recomputing

Memoized Recursive Algorithm

- The approach assumes that
  - The set of all possible subproblem parameters are known
  - The relation between the table positions and subproblems is established
- Another approach is to memoize
  - by using hashing with subproblem parameters as *key*

Memoization-based solutions will NOT BE ACCEPTED in the exams!

Dynamic Programming vs Memoization Summary

- Matrix-chain multiplication can be solved in $O(n^3)$ time
  - by either a top-down memoized recursive algorithm
  - or a bottom-up dynamic programming algorithm
- Both methods exploit the overlapping subproblems property
  - There are only $\Theta(n^2)$ different subproblems in total
  - Both methods compute the soln to each problem once
- Without memoization the natural recursive algorithm runs in exponential time since subproblems are solved repeatedly

Dynamic Programming vs Memoization Summary

In general practice

- If all subproblems must be solved at once
  - a bottom-up DP algorithm always outperforms a top-down memoized algorithm by a constant factor

  because, bottom-up DP algorithm
    - Has no overhead for recursion
    - Less overhead for maintaining the table
- DP: Regular pattern of table accesses can be exploited to reduce the time and/or space requirements even further
- Memoized: If some problems need not be solved at all, it has the advantage of avoiding solutions to those subproblems

Problem 3:
Longest Common Subsequence

Definitions

- A subsequence of a given sequence is just the given sequence with some elements (possibly none) left out

- Example:

  X = < A, **B**, **C**, B, **D**, A, **B**>

  Z = <**B**, **C**, **D**, **B**>

  ➔ Z is a subsequence of X

Definitions

Formal definition: Given a sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$,

sequence $Z = \langle z_1, z_2, \ldots, z_k \rangle$ is a subsequence of X

if $\exists$ a strictly increasing sequence $\langle i_1, i_2, \ldots, i_k \rangle$ of indices of X such that $x_{i_j} = z_j$ for all $j = 1, 2, \ldots, k$, where $1 \leq k \leq m$

Example: $Z = \langle \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{B} \rangle$ is a subsequence of $X = \langle \overset{1}{A}, \overset{2}{\mathbf{B}}, \overset{3}{\mathbf{C}}, \overset{4}{B}, \overset{5}{\mathbf{D}}, \overset{6}{A}, \overset{7}{\mathbf{B}} \rangle$

with the index sequence $\langle i_1, i_2, i_3, i_4 \rangle = \langle 2, 3, 5, 7 \rangle$

Definitions

If Z is a subsequence of both X and Y, we denote Z as a common subsequence of X and Y.

Example: X = <A, **B**, **C**, B, D, **A**, B> and
Y = <**B**, D, **C**, **A**, B, A>

Z_1 = <**B**, **C**, **A**> is a common subsequence (of length 3) of X and Y.

Two longest common subsequence (LCSs) of X and Y?

Z_2 = <B, C, B, A> of length 4

Z_3 = <B, D, A, B> of length 4

The optimal solution value = 4

Longest Common Subsequence (LCS) Problem

- LCS problem: Given two sequences $X = <x_1, x_2, \ldots, x_m>$ and $Y = <y_1, y_2, \ldots, y_n>$, find the LCS of X & Y

- Brute force approach:
  - Enumerate all subsequences of X
  - Check if each subsequence is also a subsequence of Y
  - Keep track of the LCS
  - What is the complexity?
    - There are 2^m subsequences of X

      → Exponential runtime

Notation

Notation: Let X_i denote the i^{th} prefix of X

i.e. $X_i = \langle x_1, x_2, \ldots, x_i \rangle$

Example: $X = \langle A, B, C, B, D, A, B \rangle$

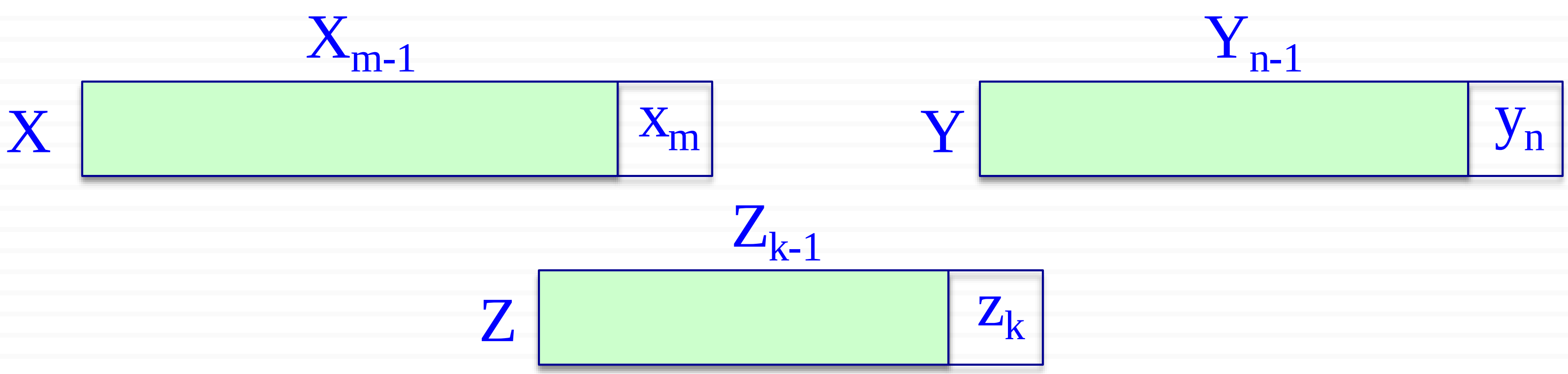
$X_4 = \langle A, B, C, B \rangle$, $X_0 = \langle \ \rangle$

Optimal Substructure of an LCS

Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ are given

Let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be an LCS of X and Y

Question 1: If $x_m = y_n$, how to define the optimal substructure?



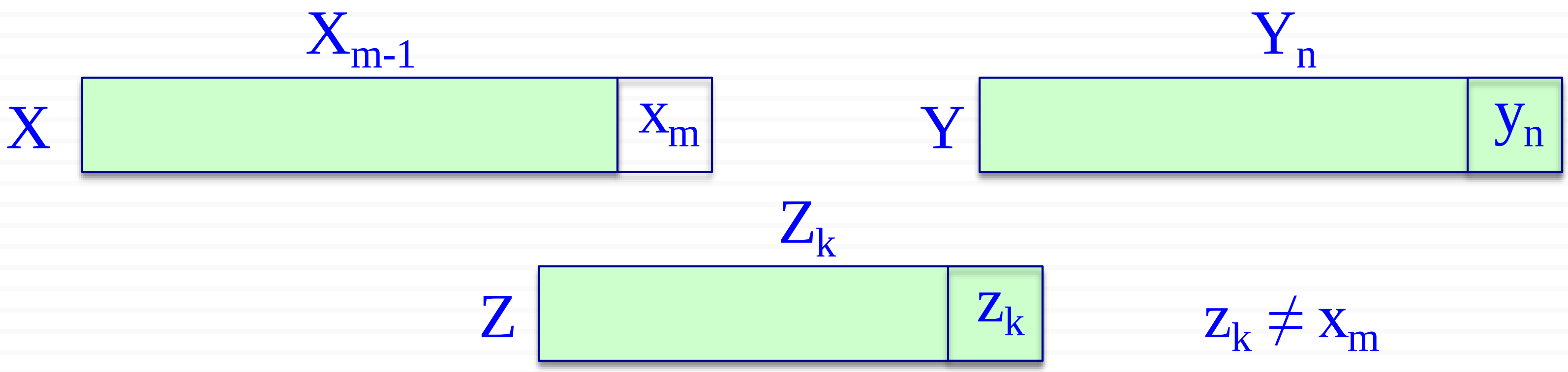
We must have $z_k = x_m = y_n$ and $Z_{k-1} = \text{LCS}(X_{m-1}, Y_{n-1})$

Optimal Substructure of an LCS

Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ are given

Let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be an LCS of X and Y

Question 2: If $x_m \neq y_n$ and $z_k \neq x_m$, how to define the optimal substructure?



$z_k \neq x_m$

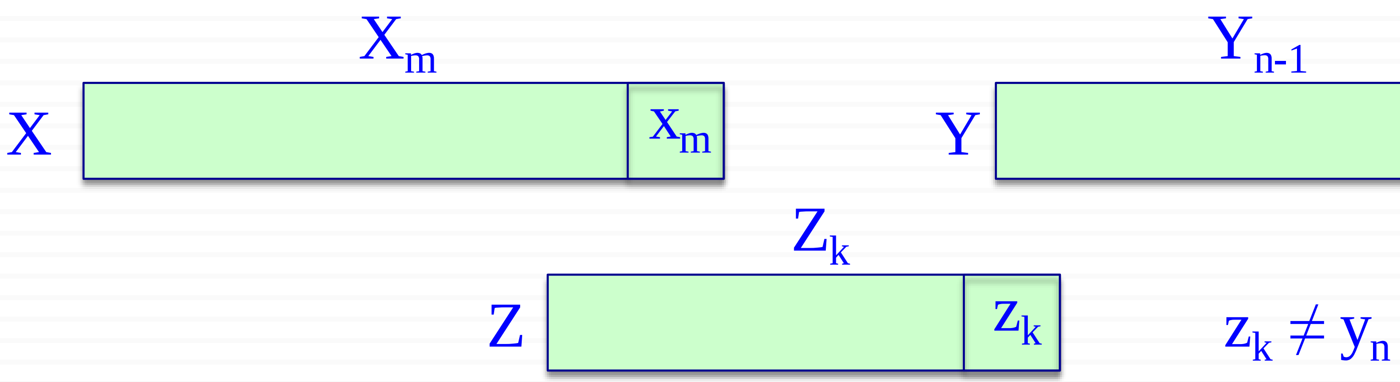
We must have $Z = LCS(X_{m-1}, Y)$

Optimal Substructure of an LCS

Let $X = <x_1, x_2, \ldots, x_m>$ and $Y = <y_1, y_2, \ldots, y_n>$ are given

Let $Z = <z_1, z_2, \ldots, z_k>$ be an LCS of X and Y

Question 3: If $x_m \neq y_n$ and $z_k \neq y_n$, how to define the optimal substructure?



We must have $Z = LCS(X, Y_{n-1})$

Theorem: Optimal Substructure of an LCS

Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ are given

Let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be an LCS of X and Y

Theorem: Optimal substructure of an LCS:

1. **If** $x_m = y_n$

   then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}

2. **If** $x_m \neq y_n$ and $z_k \neq x_m$

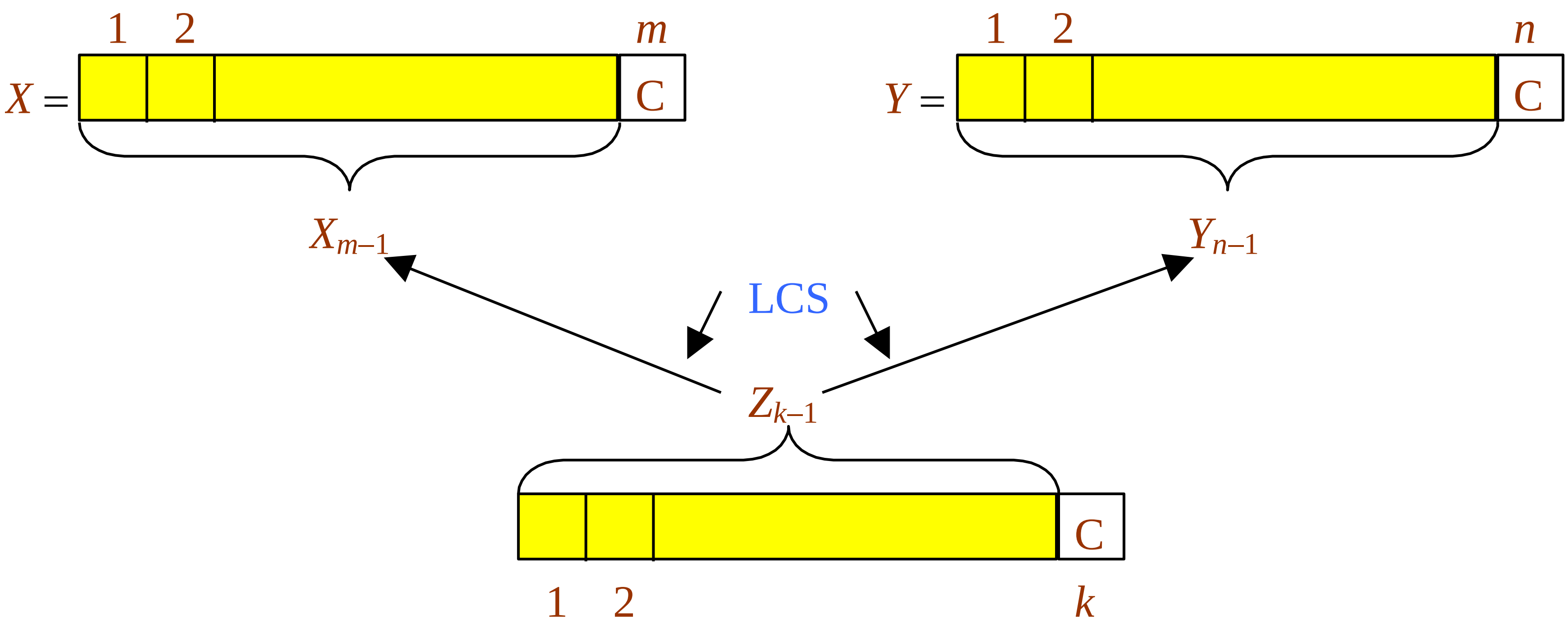
   then Z is an LCS of X_{m-1} and Y

3. **If** $x_m \neq y_n$ and $z_k \neq y_n$

   then Z is an LCS of X and Y_{n-1}

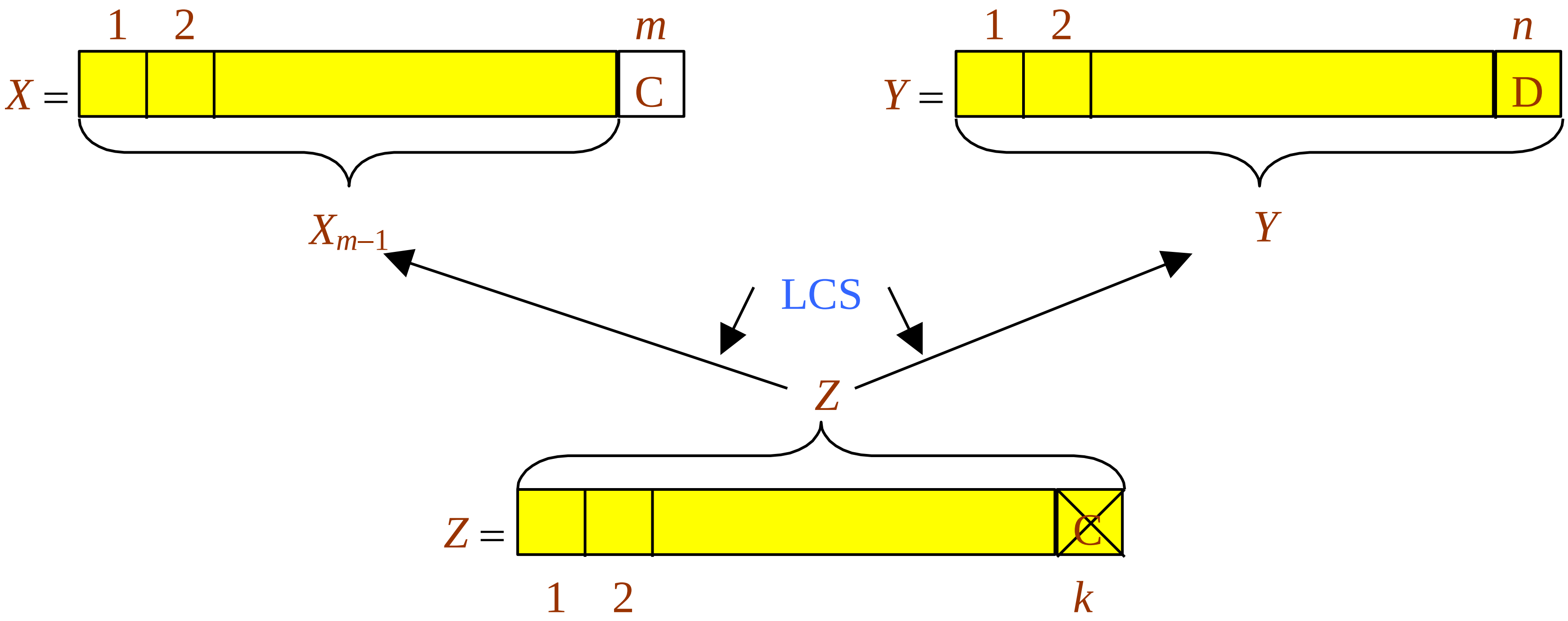
Optimal Substructure Theorem (case 1)

If $x_m = y_n$ then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}



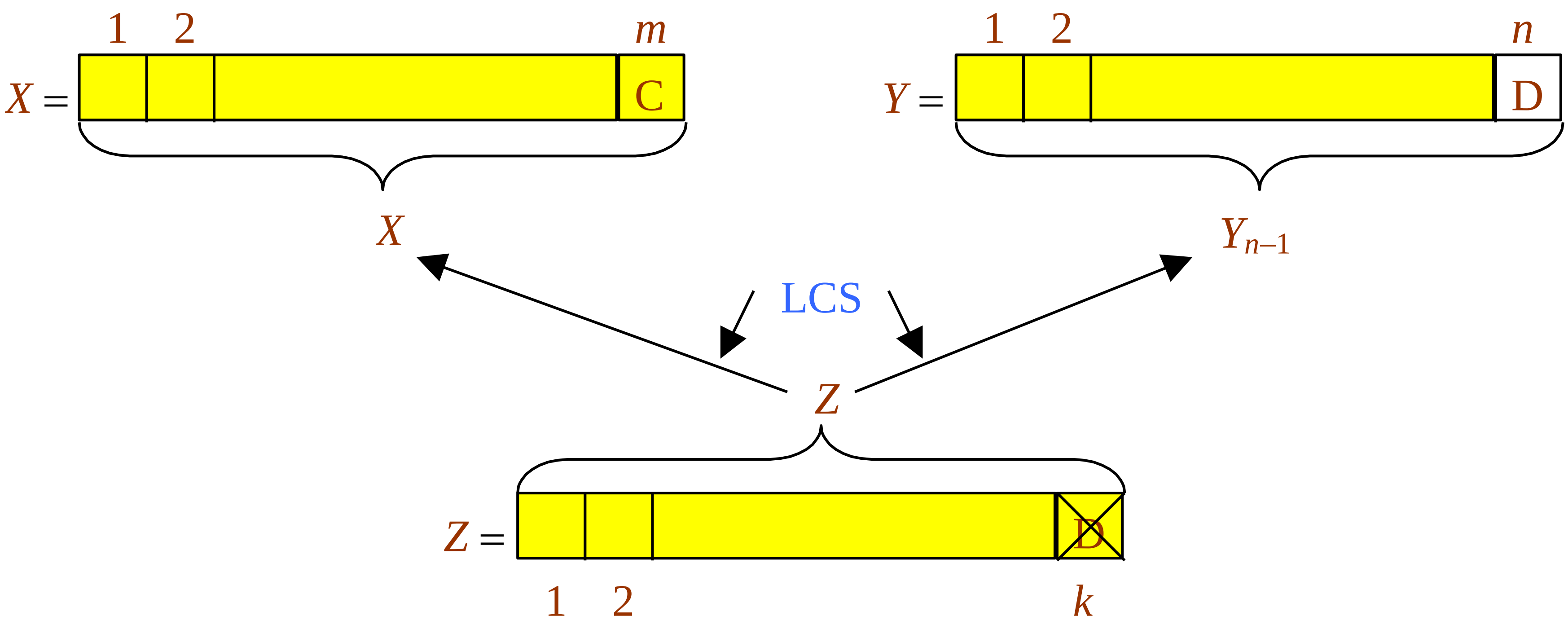
Optimal Substructure Theorem (case 2)

If $x_m \neq y_n$ and $z_k \neq x_m$ then Z is an LCS of X_{m-1} and Y



Optimal Substructure Theorem (case 3)

If $x_m \neq y_n$ and $z_k \neq y_n$ then Z is an LCS of X and Y_{n-1}



Proof of Optimal Substructure Theorem (case 1)

If $x_m = y_n$ then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}

Proof: If $z_k \neq x_m = y_n$ then

we can append $x_m = y_n$ to Z to obtain a common subsequence of length $k+1 \Rightarrow$ contradiction

Thus, we must have $z_k = x_m = y_n$

Hence, the prefix Z_{k-1} is a length-($k-1$) CS of X_{m-1} and Y_{n-1}

We have to show that Z_{k-1} is in fact an LCS of X_{m-1} and Y_{n-1}

Proof by contradiction:

Assume that $\exists$ a CS W of X_{m-1} and Y_{n-1} with $|W| = k$

Then appending $x_m = y_n$ to W produces a CS of length $k+1$

Proof of Optimal Substructure Theorem (case 2)

If $x_m \neq y_n$ and $z_k \neq x_m$ then Z is an LCS of X_{m-1} and Y

Proof : If $z_k \neq x_m$ then Z is a CS of X_{m-1} and Y_n

We have to show that Z is in fact an LCS of X_{m-1} and Y_n

(Proof by contradiction)

Assume that $\exists$ a CS W of X_{m-1} and Y_n with $|W| > k$

Then W would also be a CS of X and Y

Contradiction to the assumption that

Z is an LCS of X and Y with $|Z| = k$

Case 3: Dual of the proof for (case 2)

A Recursive Solution to Subproblems

Theorem implies that there are one or two subproblems to examine

if $x_m = y_n$ then

    we must solve the subproblem of finding an LCS of X_{m-1} & Y_{n-1}

    appending $x_m = y_n$ to this LCS yields an LCS of X & Y

else

    we must solve two subproblems

    – finding an LCS of X_{m-1} & Y

    – finding an LCS of X & Y_{n-1}

    longer of these two LCSs is an LCS of X & Y

endif

Recursive Algorithm (Inefficient!!!)

LCS(X, Y)

  $m \leftarrow$ length[X]

  $n \leftarrow$ length[Y]

  if $x_m = y_n$ then

    $Z \leftarrow$ LCS(X_{m-1}, Y_{n-1}) $\triangleright$ solve one subproblem

    return $<Z, x_m = y_n>$ $\triangleright$ append $x_m = y_n$ to Z

  else

    $Z' \leftarrow$ LCS(X_{m-1}, Y)

    $Z'' \leftarrow$ LCS(X, Y_{n-1})

    $\triangleright$ solve two subproblems

    return longer of Z' and Z''

A Recursive Solution

c[i, j]: length of an LCS of X_i and Y_j

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max\{c[i, j-1], c[i-1, j]\} & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

Computing the Length of an LCS

- We can easily write an **exponential-time recursive algorithm** based on the given recurrence. ➔ **Inefficient!**

- How many distinct subproblems to solve?

  $\Theta(mn)$

- *Overlapping subproblems property*: Many subproblems share the same sub-subproblems.

  e.g. Finding an LCS to X_{m-1} & Y and an LCS to X & Y_{n-1}
  has the sub-subproblem of finding an LCS to X_{m-1} & Y_{n-1}

- Therefore, we can use **dynamic programming.**

Data Structures

Let:

$c[i, j]$: length of an LCS of X_i and Y_j

$b[i, j]$: direction towards the table entry corresponding to the optimal subproblem solution chosen when computing $c[i, j]$. Used to simplify the construction of an optimal solution at the end.

Maintain the following tables:

$c[0 \ldots m, 0 \ldots n]$

$b[1 \ldots m, 1 \ldots n]$

Bottom-up Computation

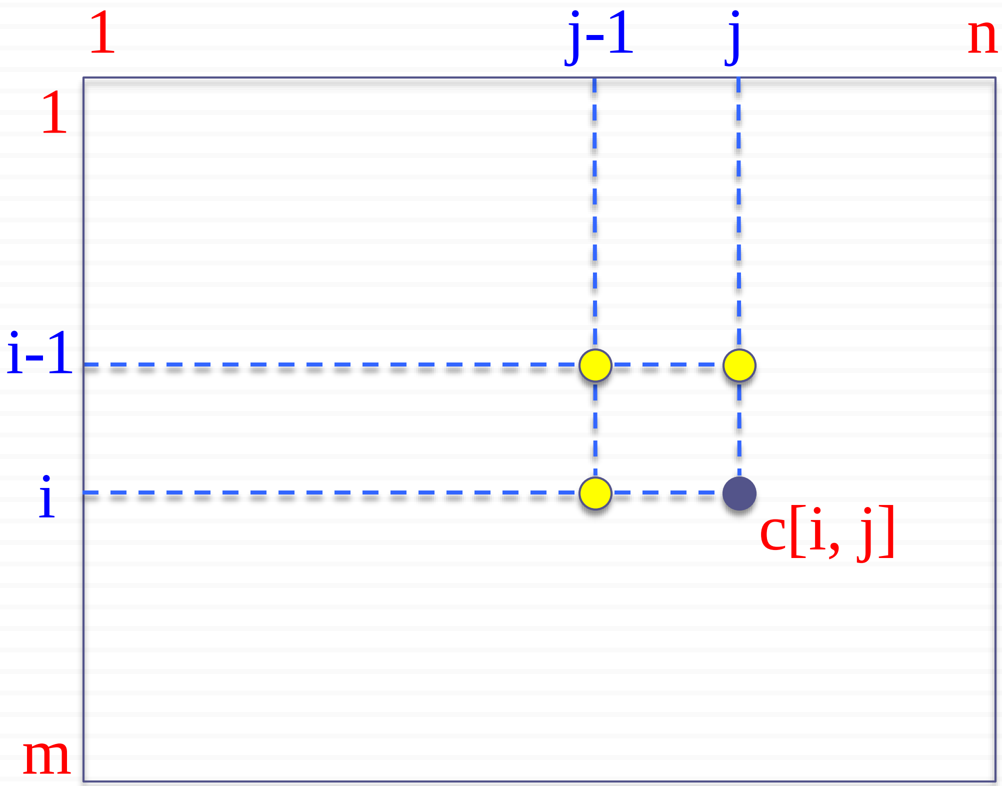
Reminder:

$$c[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max\{c[i, j-1], c[i-1, j]\} & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

How to choose the order in which we process c[i, j] values?

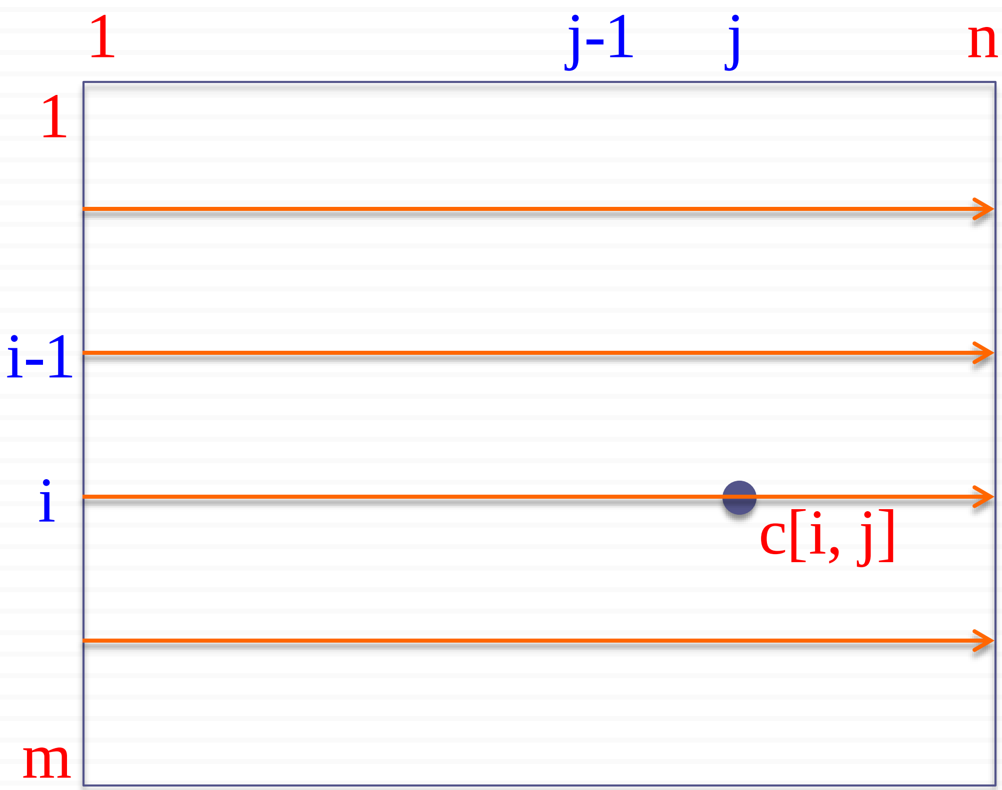
The values for c[i-1, j-1], c[i, j-1], and c[i-1,j] must be computed before computing c[i, j].

$$c[i,j]=\begin{cases}0 & \text{if } i=0 \text{ or } j=0\\ c[i-1,j-1]+1 & \text{if } i,j>0 \text{ and } x_i=y_j\\ \max\{c[i,j-1],c[i-1,j]\} & \text{if } i,j>0 \text{ and } x_i\neq y_j\end{cases}$$



Need to process:
c[i, j]
after computing:
c[i-1, j-1],
c[i, j-1],
c[i-1,j]

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max\{c[i, j-1], c[i-1, j]\} & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$



for i ← 1 **to** m
    for j ← 1 **to** n
        ....
        ....
        c[i, j] =

Computing the Length of an LCS

LCS-LENGTH(X,Y)

```
m ← length[X]; n ← length[Y]
for i ← 0 to m do c[i, 0] ← 0
for j ← 0 to n do c[0, j] ← 0
for i ← 1 to m do
    for j ← 1 to n do
        if x_i = y_j then
            c[i, j] ← c[i–1, j–1]+1
            b[i, j] ← "↖"
        else if c[i – 1, j] ≥ c[i, j–1]
            c[i, j] ← c[i–1, j]
            b[i, j] ← "↑"
        else
            c[i, j] ← c[i, j–1]
            b[i, j] ← "←"
```

Total runtime = Θ(mn)
Total space = Θ(mn)

Computing the Length of an LCS

Operation of LCS-LENGTH on the sequences

1 2 3 4 5 6 7

$X = \langle A, B, C, B, D, A, B \rangle$

$Y = \langle B, D, C, A, B, A \rangle$

1 2 3 4 5 6

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0						
2	B	0						
3	C	0						
4	B	0						
5	D	0						
6	A	0						
7	B	0						

Computing the Length of an LCS

Operation of LCS-LENGTH on the sequences

$$X = \langle \underset{1}{A}, \underset{2}{B}, \underset{3}{C}, \underset{4}{B}, \underset{5}{D}, \underset{6}{A}, \underset{7}{B} \rangle$$

$$Y = \langle \underset{1}{B}, \underset{2}{D}, \underset{3}{C}, \underset{4}{A}, \underset{5}{B}, \underset{6}{A} \rangle$$

i \ j	0 y_j	1 B	2 D	3 C	4 A	5 B	6 A
0 x_i	0	0	0	0	0	0	0
1 A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2 B	0						
3 C	0						
4 B	0						
5 D	0						
6 A	0						
7 B	0						

Computing the Length of an LCS

Operation of LCS-LENGTH on the sequences

1 2 3 4 5 6 7

X = <A, B, C, B, D, A, B>

Y = <B, D, C, A, B, A>

1 2 3 4 5 6

j	0	1	2	3	4	5	6
i	y_j	B	D	C	A	B	A
0 x_i	0	0	0	0	0	0	0
1 A	0	↑ 0	↑ 0	↑ 0	↖ 1	←1	↖ 1
2 B	0	↖ 1	←1	←1	↑ 1	↖ 2	←2
3 C	0						
4 B	0						
5 D	0						
6 A	0						
7 B	0						

Computing the Length of an LCS

Operation of LCS-LENGTH
on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \underset{1}{B}, \underset{2}{D}, \underset{3}{C}, \underset{4}{A}, \underset{5}{B}, \underset{6}{A} \rangle$$

i \ j	0 y_j	1 B	2 D	3 C	4 A	5 B	6 A
0 x_i	0	0	0	0	0	0	0
1 A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2 B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3 C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4 B	0						
5 D	0						
6 A	0						
7 B	0						

Computing the Length of an LCS

Operation of LCS-LENGTH on the sequences

1 2 3 4 5 6 7

X = <A, B, C, B, D, A, B>

Y = <B, D, C, A, B, A>

1 2 3 4 5 6

j	0	1	2	3	4	5	6
i	y_j	B	D	C	A	B	A
0 x_i	0	0	0	0	0	0	0
1 A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2 B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3 C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4 B	0	↖ 1					
5 D	0						
6 A	0						
7 B	0						

Computing the Length of an LCS

Operation of LCS-LENGTH on the sequences

$$X = \langle \underset{1}{A}, \underset{2}{B}, \underset{3}{C}, \underset{4}{B}, \underset{5}{D}, \underset{6}{A}, \underset{7}{B} \rangle$$

$$Y = \langle \underset{1}{B}, \underset{2}{D}, \underset{3}{C}, \underset{4}{A}, \underset{5}{B}, \underset{6}{A} \rangle$$

j	0	1	2	3	4	5	6
i	y_j	B	D	C	A	B	A
0 x_i	0	0	0	0	0	0	0
1 A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2 B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3 C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4 B	0	↖ 1	↑ 1				
5 D	0						
6 A	0						
7 B	0						

Computing the Length of an LCS

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \underset{1}{B}, \underset{2}{D}, \underset{3}{C}, \underset{4}{A}, \underset{5}{B}, \underset{6}{A} \rangle$$

i	x_i \ j	0	1	2	3	4	5	6
	y_j		B	D	C	A	B	A
0		0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2			
5	D	0						
6	A	0						
7	B	0						

Computing the Length of an LCS

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \underset{1}{B}, \underset{2}{D}, \underset{3}{C}, \underset{4}{A}, \underset{5}{B}, \underset{6}{A} \rangle$$

j	0	1	2	3	4	5	6
i	y_j	B	D	C	A	B	A
0 x_i	0	0	0	0	0	0	0
1 A	0	↑ 0	↑ 0	↑ 0	↖ 1	←1	↖ 1
2 B	0	↖ 1	←1	←1	↑ 1	↖ 2	←2
3 C	0	↑ 1	↑ 1	↖ 2	←2	↑ 2	↑ 2
4 B	0	↖ 1	↑ 1	↑ 2	↑ 2		
5 D	0						
6 A	0						
7 B	0						

Computing the Length of an LCS

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \underset{1}{B}, \underset{2}{D}, \underset{3}{C}, \underset{4}{A}, \underset{5}{B}, \underset{6}{A} \rangle$$

i \ j	0 y_j	1 B	2 D	3 C	4 A	5 B	6 A
0 x_i	0	0	0	0	0	0	0
1 A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2 B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3 C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4 B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	
5 D	0						
6 A	0						
7 B	0						

Computing the Length of an LCS

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \underset{1}{B}, \underset{2}{D}, \underset{3}{C}, \underset{4}{A}, \underset{5}{B}, \underset{6}{A} \rangle$$

i \ j	0 y_j	1 B	2 D	3 C	4 A	5 B	6 A
0 x_i	0	0	0	0	0	0	0
1 A	0	↑ 0	↑ 0	↑ 0	↖ 1	←1	↖ 1
2 B	0	↖ 1	←1	←1	↑ 1	↖ 2	←2
3 C	0	↑ 1	↑ 1	↖ 2	←2	↑ 2	↑ 2
4 B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	←3
5 D	0						
6 A	0						
7 B	0						

Computing the Length of an LCS

Operation of LCS-LENGTH on the sequences

1 2 3 4 5 6 7

X = <A, B, C, B, D, A, B>

Y = <B, D, C, A, B, A>

1 2 3 4 5 6

i \ *j*	0	1	2	3	4	5	6
	y_j	B	D	C	A	B	A
0 x_i	0	0	0	0	0	0	0
1 A	0	↑ 0	↑ 0	↑ 0	↖ 1	←1	↖ 1
2 B	0	↖ 1	←1	←1	↑ 1	↖ 2	←2
3 C	0	↑ 1	↑ 1	↖ 2	←2	↑ 2	↑ 2
4 B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	←3
5 D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6 A	0						
7 B	0						

Computing the Length of an LCS

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \underset{1}{B}, \underset{2}{D}, \underset{3}{C}, \underset{4}{A}, \underset{5}{B}, \underset{6}{A} \rangle$$

i \ *j*	0 y_j	1 B	2 D	3 C	4 A	5 B	6 A
0 x_i	0	0	0	0	0	0	0
1 A	0	↑ 0	↑ 0	↑ 0	↖ 1	←1	↖ 1
2 B	0	↖ 1	←1	←1	↑ 1	↖ 2	←2
3 C	0	↑ 1	↑ 1	↖ 2	←2	↑ 2	↑ 2
4 B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	←3
5 D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6 A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7 B	0						

Computing the Length of an LCS

Operation of LCS-LENGTH on the sequences

$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$

$Y = \langle \underset{1}{B}, \underset{2}{D}, \underset{3}{C}, \underset{4}{A}, \underset{5}{B}, \underset{6}{A} \rangle$

Running-time = O(mn) since each table entry takes O(1) time to compute

i \ j	0 y_j	1 B	2 D	3 C	4 A	5 B	6 A
0 x_i	0	0	0	0	0	0	0
1 A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2 B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3 C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4 B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5 D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6 A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7 B	0	↖ 1	↑ 2	↑ 3	↑ 3	↖ 4	↑ 4

Computing the Length of an LCS

Operation of LCS-LENGTH on the sequences

$$X = \langle \underset{1}{A}, \underset{2}{B}, \underset{3}{C}, \underset{4}{B}, \underset{5}{D}, \underset{6}{A}, \underset{7}{B} \rangle$$

$$Y = \langle \underset{1}{B}, \underset{2}{D}, \underset{3}{C}, \underset{4}{A}, \underset{5}{B}, \underset{6}{A} \rangle$$

Running-time $= O(mn)$ since each table entry takes $O(1)$ time to compute

LCS of X & Y = <B, C, B, A>

i \ j	0 y_j	1 B	2 D	3 C	4 A	5 B	6 A
0 x_i	0	0	0	0	0	0	0
1 A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2 B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3 C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4 B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5 D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6 A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7 B	0	↖ 1	↑ 2	↑ 3	↑ 3	↖ 4	↑ 4

Constructing an LCS

The b table returned by LCS-LENGTH can be used to quickly construct an LCS of X & Y

Begin at $b[m, n]$ and trace through the table following arrows

Whenever you encounter a "↖" in entry $b[i, j]$
it implies that $x_i = y_j$ is an element of LCS

The elements of LCS are encountered in reverse order

Constructing an LCS

```
PRINT-LCS(b, X, i, j)
    if i = 0 or j = 0 then
        return
    if b[i, j] = "↖" then
        PRINT-LCS(b, X, i–1, j–1)
        print x_i
    else if b[i, j] = "↑" then
        PRINT-LCS(b, X, i–1, j)
    else
        PRINT-LCS(b, X, i, j–1)
```

The initial invocation:
PRINT-LCS(b, X, length[X], length[Y])

The recursive procedure PRINT-LCS prints out LCS in proper order

This procedure takes $O(m+n)$ time
since at least one of i and j is decremented in each stage of the recursion

Do we really need the b table (back-pointers)?

	Ø	B	D	C	A	B	A
Ø	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0	1	1	1	1	2	2
C	0	1	1	2	2	2	2
B	0	1	1	2	2	3	3
D	0	1	2	2	2	3	3
A	0	1	2	2	3	3	4
B	0	1	2	3	3	4	4

Question: From which neighbor did we expand to the highlighted cell?

Answer: Upper-left neighbor, because X[i] = Y[j].

Do we really need the b table (back-pointers)?

	Ø	B	D	C	A	B	A
Ø	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0	1	1	1	1	2	2
C	0	1	1	2	2	2	2
B	0	1	1	2	2	3	3
D	0	1	2	2	2	3	3
A	0	1	2	2	3	3	4
B	0	1	2	3	3	4	4

Question: From which neighbor did we expand to the highlighted cell?

Answer: Left neighbor, because $X[i] \neq Y[j]$ and $LCS[i, j-1] > LCS[i-1, j]$.

Do we really need the b table (back-pointers)?

	∅	B	D	C	A	B	A
∅	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0	1	1	1	1	2	2
C	0	1	1	2	2	2	2
B	0	1	1	2	2	3	3
D	0	1	2	2	2	3	3
A	0	1	2	2	3	3	4
B	0	1	2	3	3	4	4

Question: From which neighbor did we expand to the highlighted cell?

Answer: Upper neighbor, because $X[i] \neq Y[j]$ and $LCS[i, j-1] = LCS[i-1, j]$.

(See pseudo-code to see how ties are handled.)

Improving the Space Requirements

We can eliminate the b table altogether

- each $c[i, j]$ entry depends only on 3 other c table entries: $c[i-1, j-1]$, $c[i-1, j]$ and $c[i, j-1]$

Given the value of $c[i, j]$:

- We can determine in O(1) time which of these 3 values was used to compute $c[i, j]$ without inspecting table b
- We save $\Theta(mn)$ space by this method
- However, space requirement is still $\Theta(mn)$ since we need $\Theta(mn)$ space for the c table anyway

What if we store the last 2 rows only?

	∅	B	D	C	A	B	A
∅							
A							
B							
C	0	1	1	2	2	2	2
B	0	1	1	2	2	3	3
D							
A							
B							

To compute c[i, j], we only need c[i-1, j-1], c[i-1, j], and c[i-1, j-1]

So, we can store only the last two rows.

Cevdet Aykanat and Mustafa Ozdal
Computer Engineering Department, Bilkent University

What if we store the last 2 rows only?

	Ø	B	D	C	A	B	A
Ø							
A							
B							
C							
B	0	1	1	2	2	3	3
D	0	1	2	2	2	3	3
A							
B							

To compute c[i, j], we only need c[i-1, j-1], c[i-1, j], and c[i-1, j-1]

So, we can store only the last two rows.

What if we store the last 2 rows only?

	Ø	B	D	C	A	B	A
Ø							
A							
B							
C							
B							
D	0	1	2	2	2	3	3
A	0	1	2	2			
B							

To compute c[i, j], we only need c[i-1, j-1], c[i-1, j], and c[i-1, j-1]

So, we can store only the last two rows.

This reduces space complexity from $\Theta(mn)$ to $\Theta(n)$.

Is there a problem with this approach?

Cevdet Aykanat and Mustafa Ozdal
Computer Engineering Department, Bilkent University

What if we store the last 2 rows only?

	∅	B	D	C	A	B	A
∅							
A							
B							
C							
B							
D	0	1	2	2	2	3	3
A	0	1	2	2			
B							

Is there a problem with this approach?

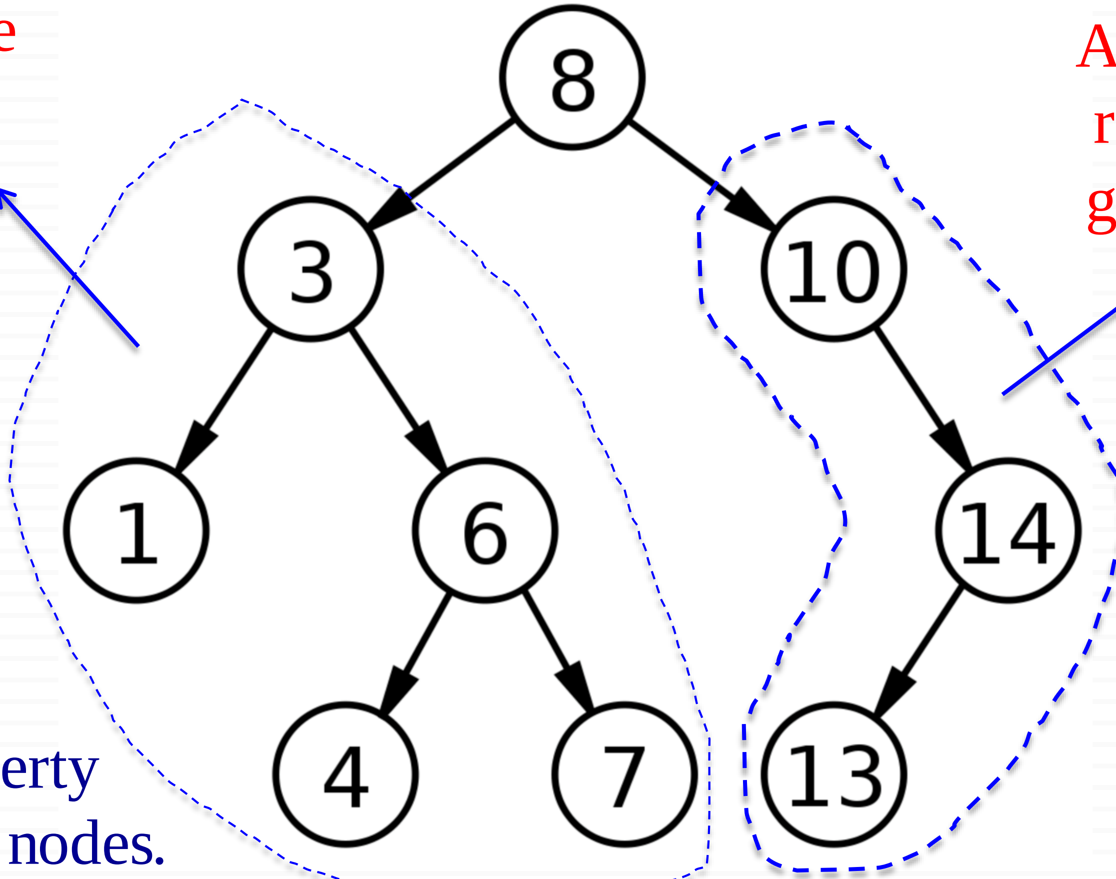
We cannot construct the optimal solution because we cannot backtrace anymore.

This approach works if we only need the length of an LCS, not the actual LCS.

Problem 4
Optimal Binary Search Tree

Reminder: Binary Search Tree (BST)

All keys in the left subtree less than 8



All keys in the right subtree greater than 8

This property holds for all nodes.

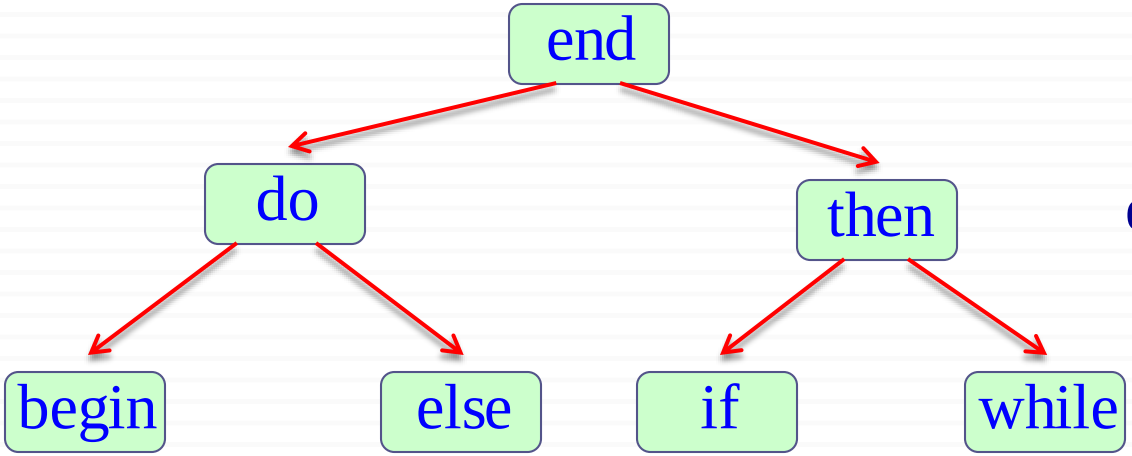
Image from Wikimedia

Binary Search Tree Example

Example: English-to-French translation

Organize (English, French) word pairs in a BST

- Keyword: English word
- Satellite data: French word

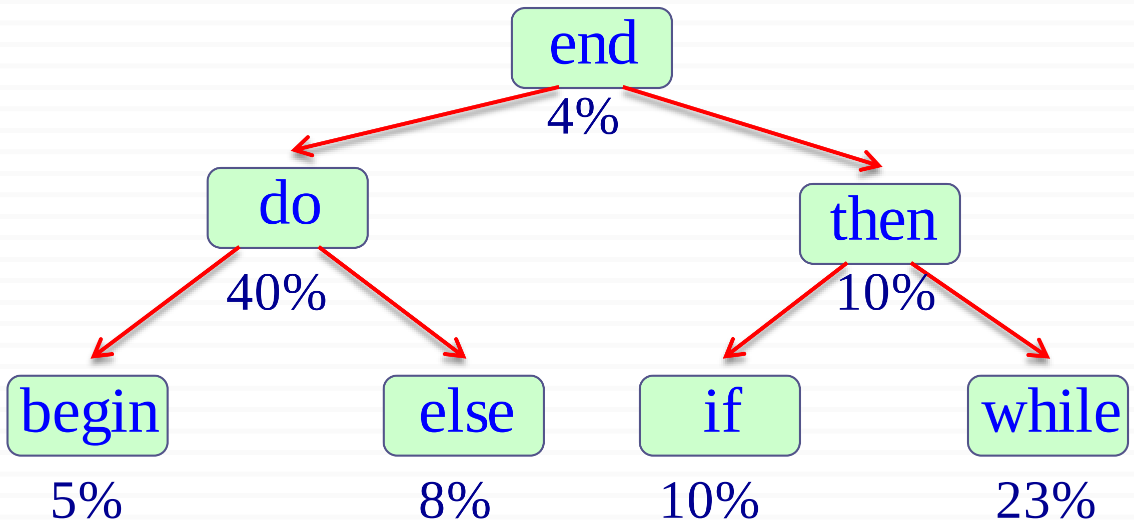


We can search for an English word (node key) efficiently, and return the corresponding French word (satellite data).

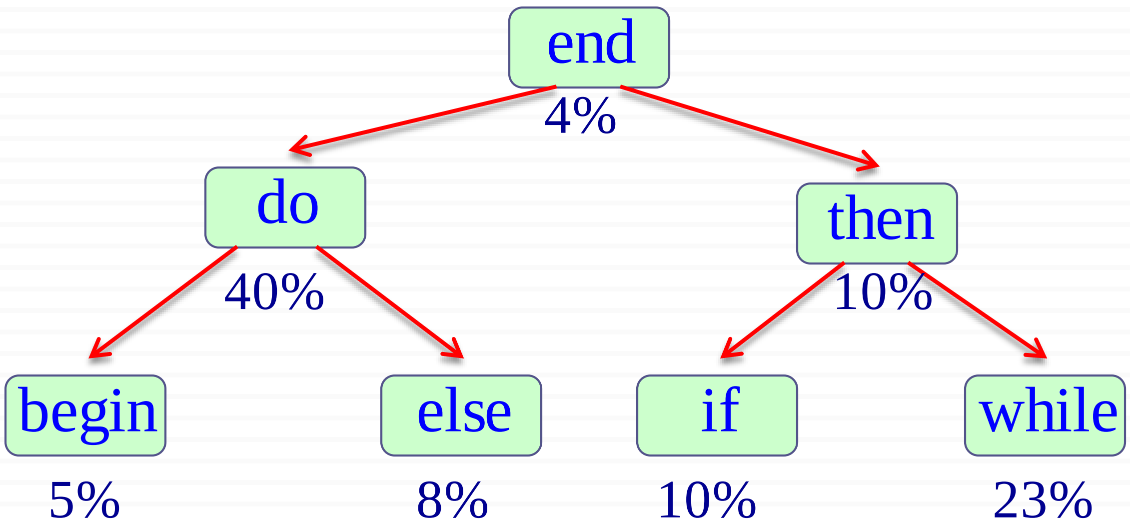
Binary Search Tree Example

Suppose we know the frequency of each keyword in texts:

begin	do	else	end	if	then	while
5%	40%	8%	4%	10%	10%	23%



Cost of a Binary Search Tree

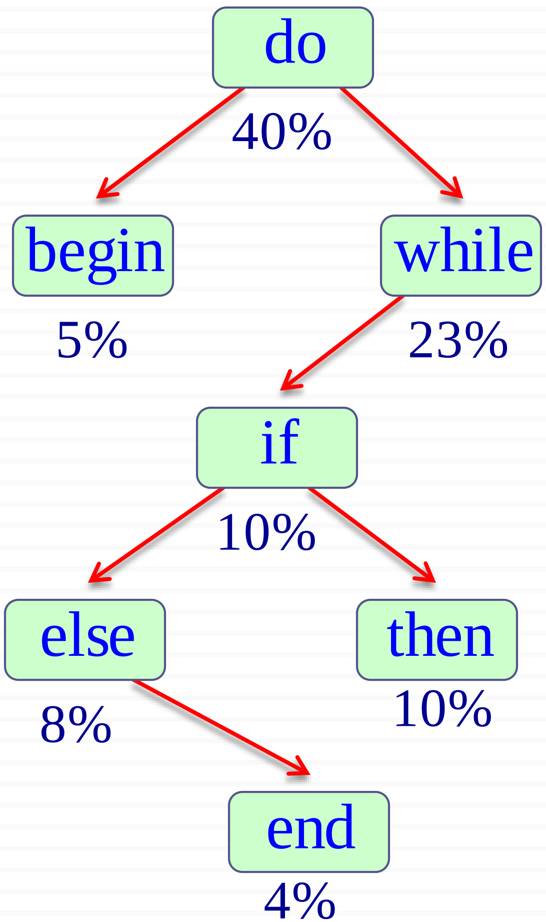


Example: If we search for keyword "while", we need to access 3 nodes. So, 23% of the queries will have cost of 3.

$$\text{Total cost} = \sum_i (depth(i)+1) \times freq(i)$$

= 1x0.04 + 2x0.4 + 2x0.1 + 3x0.05 + 3x0.08 + 3x0.1 + 3x0.23
= 2.42

Cost of a Binary Search Tree



A different binary search tree (BST) leads to a different total cost:

$$\text{Total cost} = 1\text{x}0.4 + 2\text{x}0.05 + 2\text{x}0.23 + 3\text{x}0.1 + 4\text{x}0.08 + 4\text{x}0.1 + 5\text{x}0.04 = 2.18$$

This is in fact an optimal BST.

Optimal Binary Search Tree Problem

Given:

A collection of n keys $K_1 < K_2 < \dots K_n$ to be stored in a BST.

The corresponding p_i values for $1 \le i \le n$

p_i: probability of searching for key K_i

Find:

An optimal BST with minimum total cost:

$$\text{Total cost} = \sum_i (depth(i) + 1) \times freq(i)$$

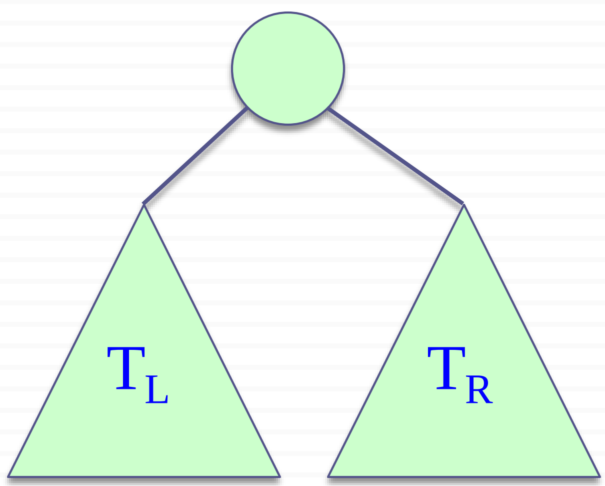
Note: The BST will be static. Only search operations will be performed. No insert, no delete, etc.

Cost of a Binary Search Tree

Lemma 1: Let T_{ij} be a BST containing keys $K_i < K_{i+1} < \ldots < K_j$.

Let T_L and T_R be the left and right subtrees of T. Then we have:

$$\text{cost}(T_{ij}) = \text{cost}(T_L) + \text{cost}(T_R) + \sum_{h=i}^{j} p_h$$



Intuition: When we add the root node, the depth of each node in T_L and T_R increases by 1. So, the cost of node h increases by p_h. In addition, the cost of root node r is p_r. That's why, we have the last term at the end of the formula above.

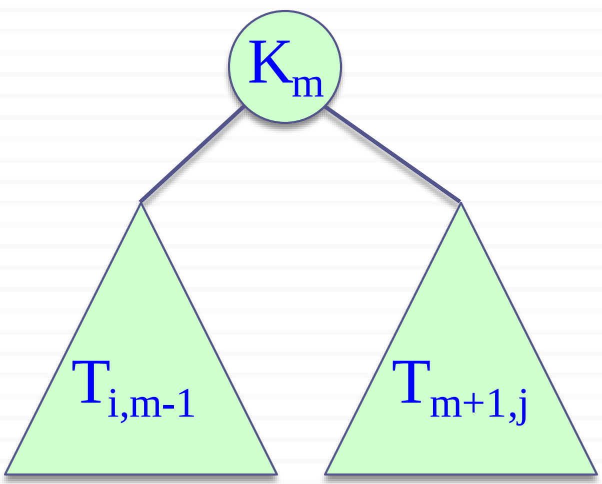
Optimal Substructure Property

__Lemma 2__: Optimal substructure property

Consider an optimal BST T_{ij} for keys $K_i < K_{i+1} < \ldots < K_j$

Let K_m be the key at the root of T_{ij}

Then:



$T_{i,m-1}$ is an optimal BST for subproblem containing keys: $K_i < \ldots < K_{m-1}$

$T_{m+1,j}$ is an optimal BST for subproblem containing keys: $K_{m+1} < \ldots < K_j$

$$\text{cost}(T_{ij}) = \text{cost}(T_{i,m-1}) + \text{cost}(T_{m+1,j}) + \sum_{h=i}^{j} p_h$$

Recursive Formulation

Note: We don't know which root vertex leads to the minimum total cost. So, we need to try each vertex m, and choose the one with minimum total cost.

$c[i, j]$: cost of an optimal BST T_{ij} for the subproblem $K_i < \ldots < K_j$

$$c[i,j] = \begin{cases} 0 & \text{if } i > j \\ \min\limits_{i \le r \le j} \left\{ c[i, r-1] + c[r+1, j] + P_{ij} \right\} & \text{otherwise} \end{cases}$$

where $P_{ij} = \sum_{h=i}^{j} p_h$

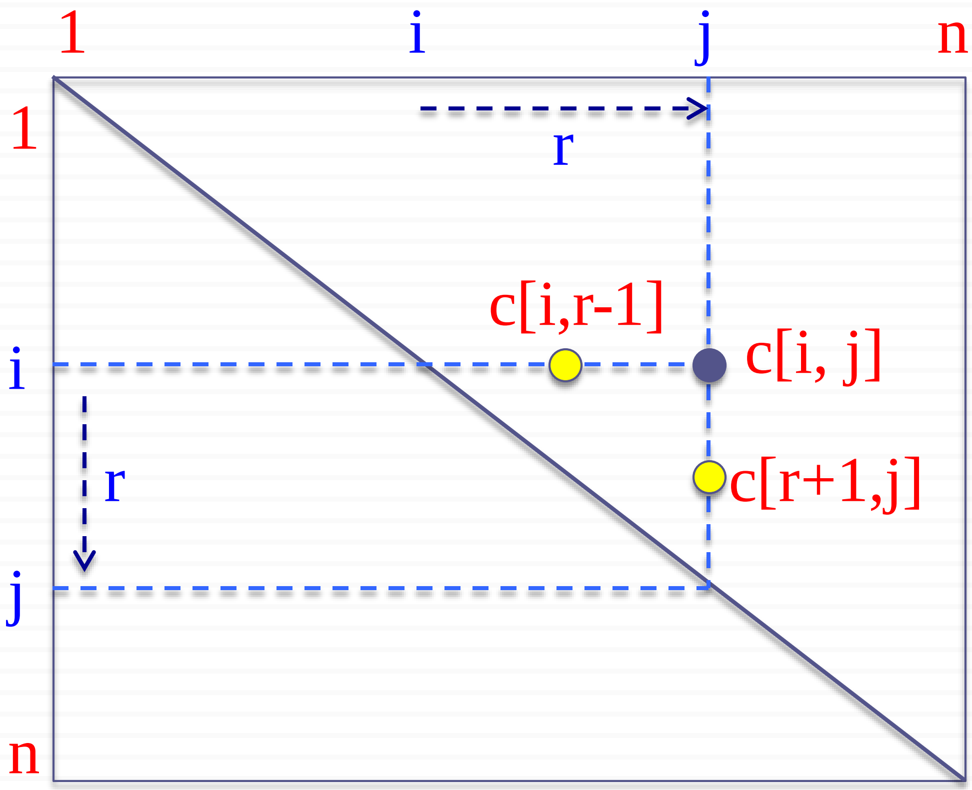
Bottom-up computation

$$c[i,j] = \begin{cases} 0 & \textit{if } \mathrm{i} > \mathrm{j} \\ \min\limits_{i \le r \le j} \{c[i,r-1] + c[r+1,j] + P_{ij}\} & \textit{otherwise} \end{cases}$$

How to choose the order in which we process c[i, j] values?

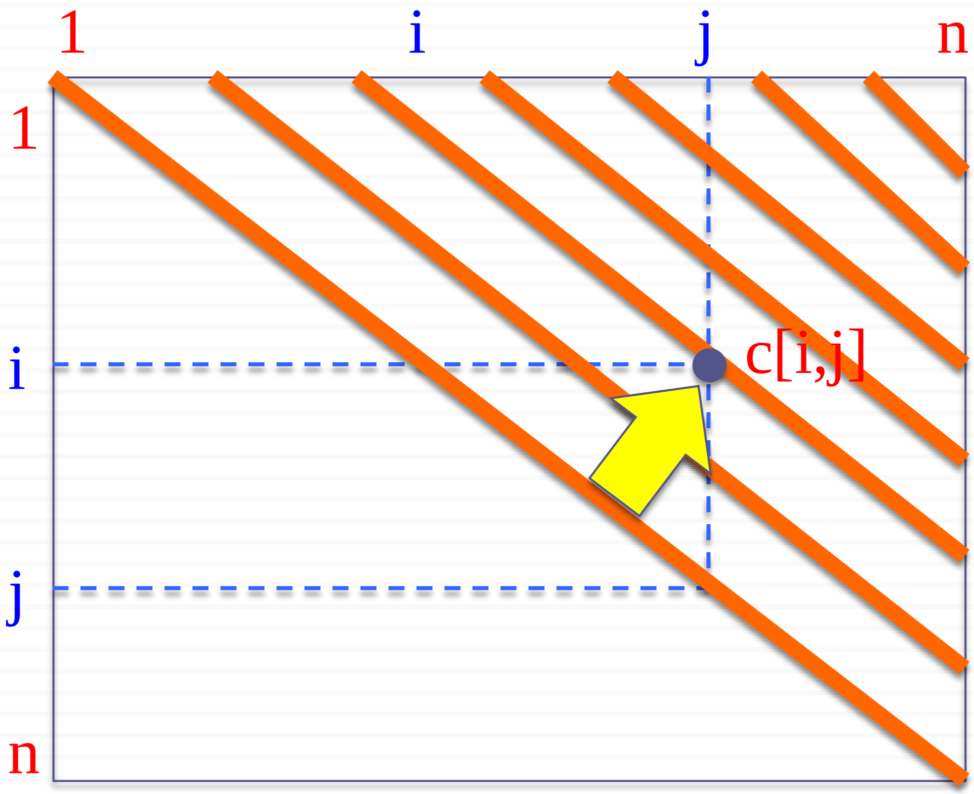
Before computing c[i, j], we have to make sure that the values for c[i, r-1] and c[r+1,j] have been computed for all r.

$$c[i,j] = \begin{cases} 0 & \textit{if } i > j \\ \min\limits_{i \le r \le j} \{c[i, r-1] + c[r+1, j] + P_{ij}\} & \textit{otherwise} \end{cases}$$



c[i,j] must be processed after c[i,r-1] and c[r+1,j]

$$c[i,j] = \begin{cases} 0 & \textit{if } i > j \\ \min\limits_{i \le r \le j} \{ c[i,r-1] + c[r+1,j] + P_{ij} \} & \textit{otherwise} \end{cases}$$



If the entries c[i,j] are computed in the shown order, then c[i,r-1] and c[r+1,j] values are guaranteed to be computed before c[i,j].

Computing the Optimal BST Cost

```
OPTIMAL-BST-COST (p, n)
   for i ← 1 to n do
      c[i, i-1] ← 0
      c[i, i]   ← p[i]
      R[i, j]   ← i
   PS[1] ← p[1]   // PS[i]: prefix_sum(i): Sum of all p[j] values for j ≤ i
   for i ← 2 to n do
      PS[i] ← p[i] + PS[i-1]   // compute the prefix sum
   for d ← 1 to n−1 do // BSTs with d+1 consecutive keys
      for i ← 1 to n – d do
         j ← i + d
         c[i, j] ← ∞
         for r ← i to j do
            q ← min{c[i,r-1] + c[r+1, j]} + PS[j] – PS[i-1]}
            if q < c[i, j] then
               c[i, j] ← q
               R[i, j] ← r
   return c[1, n], R
```

Note on Prefix Sum

- We need P_{ij} values for each i, j ($1 \leq i \leq n$ and $1 \leq j \leq n$),

  where: $$P_{ij} = \sum_{h=i}^{j} p_h$$

- If we compute the summation directly for every (i, j) pair, the runtime would be $\Theta(n^3)$.

- Instead, we spend $O(n)$ time in preprocessing to compute the prefix sum array PS. Then we can compute each P_{ij} in $O(1)$ time using PS.

Note on Prefix Sum

In preprocessing, compute for each i:

$PS[i]$: the sum of $p[j]$ values for $1 \leq j \leq i$

Then, we can compute P_{ij} in $O(1)$ time as follows:

$$P_{ij} = PS[i] - PS[j-1]$$

Example:

	1	2	3	4	5	6	7	8
p:	0.05	0.02	0.06	0.07	0.20	0.05	0.08	0.02
PS:	0.05	0.07	0.13	0.20	0.40	0.45	0.53	0.55

$P_{27} = PS[7] - PS[1] = 0.53 - 0.05 = 0.48$

$P_{36} = PS[6] - PS[2] = 0.45 - 0.07 = 0.38$