CS473 - Algorithms I

Lecture 11
Greedy Algorithms

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Activity Selection Problem

Activity Selection Problem

- We have:
  - A set of activities with fixed start and finish times
  - One shared resource (only one activity can use at any time)
- Objective: Choose the **max number** of compatible activities

Note: Objective is to maximize the number of activities, not the total time of activities.

- Example:

  Activities: Meetings with fixed start and finish times

  Shared resource: A meeting room

  Objective: Schedule the max number of meetings

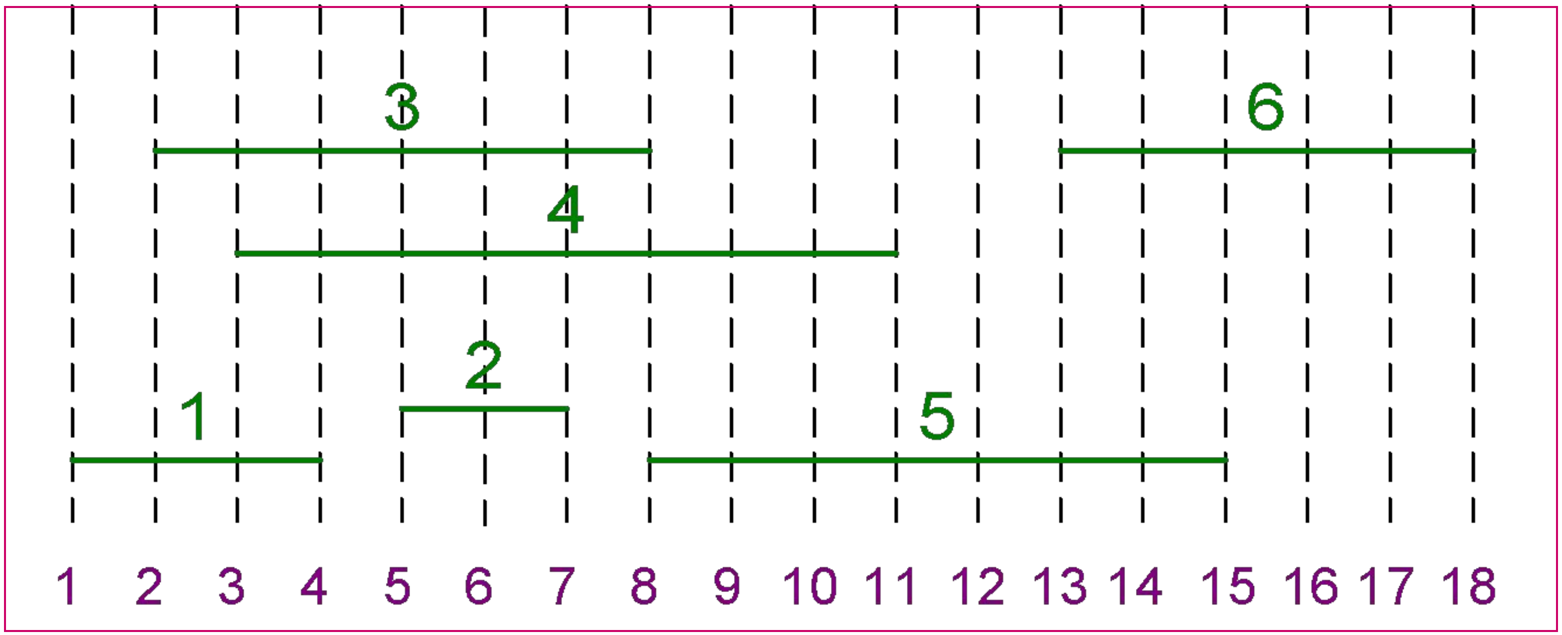
Activity Selection Problem

- Input: a set $S = \{a_1, a_2, \ldots, a_n\}$ of n activities
  - s_i : Start time of activity a_i,
  - f_i : Finish time of activity a_i

  Activity i takes place in $[s_i, f_i)$
- Aim: Find max-size subset A of mutually *compatible* activities
  - Max number of activities, not max time spent in activities
  - Activities i and j are compatible if intervals $[s_i, f_i)$ and $[s_j, f_j)$ do not overlap, i.e., either $s_i \geq f_j$ or $s_j \geq f_i$

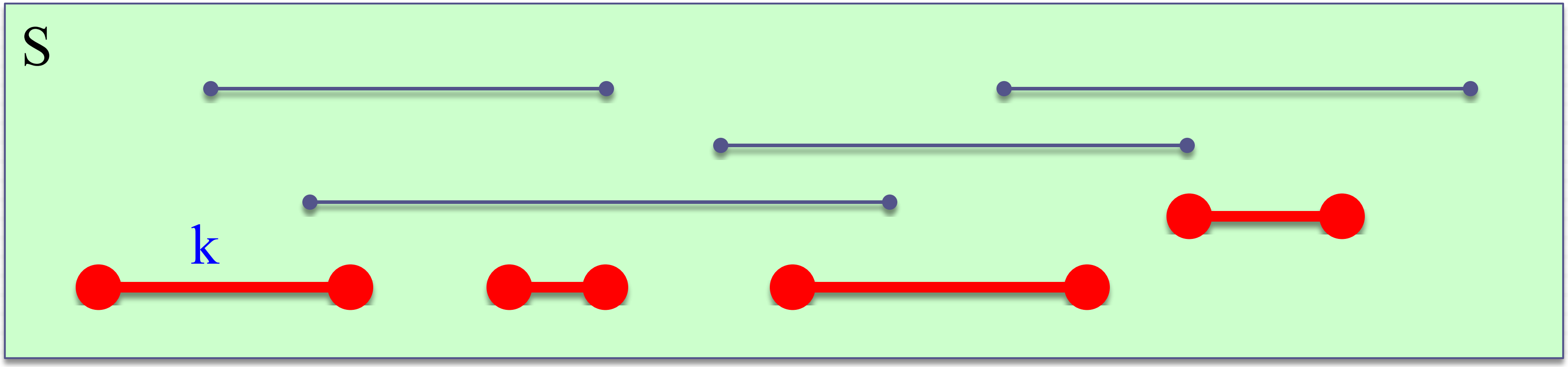
Activity Selection Problem: An Example

$S = \{[1, 4), [5, 7), [2, 8), [3, 11), [8, 15), [13, 18)\}$



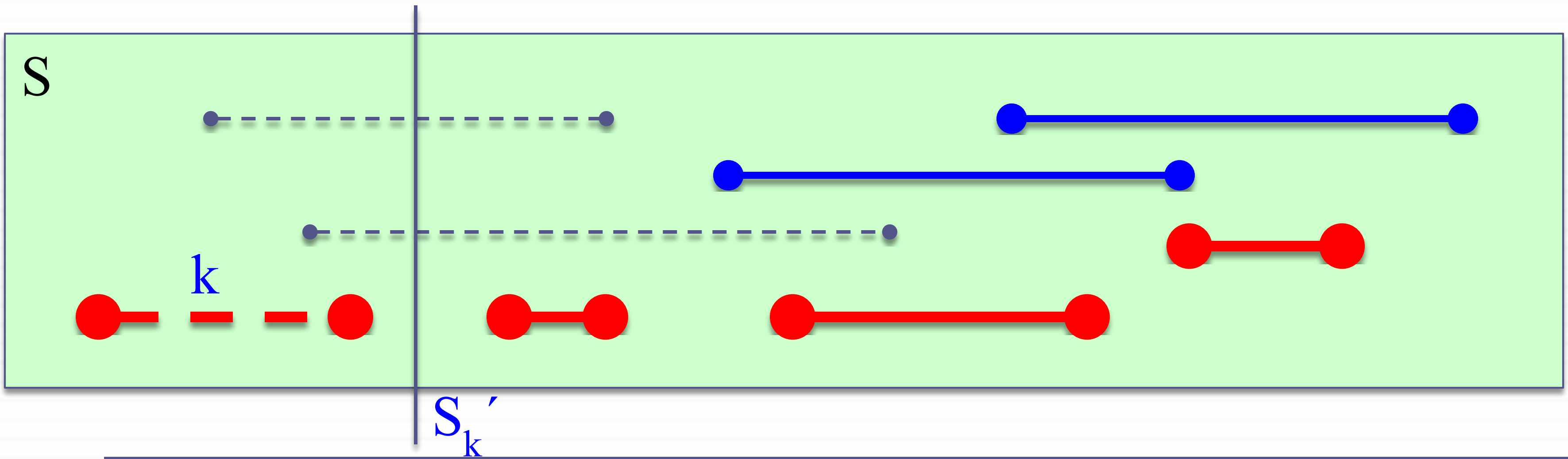
Optimal Substructure Property

- Consider an optimal solution A for activity set S.
- Let k be the activity in A with the earliest finish time



Optimal Substructure Property

- Consider an optimal solution A for activity set S.
- Let k be the activity in A with the earliest finish time
- Now, consider the subproblem S_k' that has the activities that start after k finishes, i.e. $S_k' = \{a_i \in S: s_i \geq f_k\}$
- What can we say about the optimal solution to S_k' ?



Optimal Substructure Property

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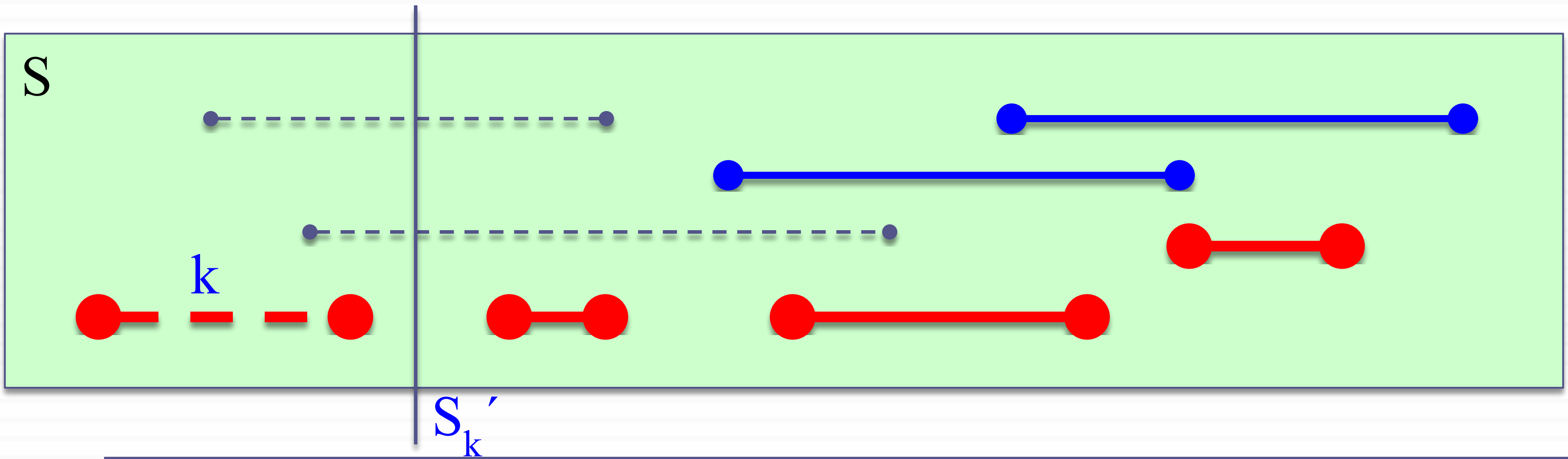
- Let k be the activity in A with the earliest finish time
- Now, consider the subproblem S_k' that has the activities that start after k finishes, i.e. $S_k' = \{a_i \in S: s_i \geq f_k\}$
- A-{k} is an optimal solution for S_k'. Why?



Optimal Substructure

Theorem: Let k be the activity with the earliest finish time in an optimal soln $A \subseteq S$ then

$A-\{k\}$ is an optimal solution to subproblem

$S_k' = \{a_i \in S: s_i \geq f_k\}$

Proof (by contradiction):

> Let B' be an optimal solution to S_k' and $|B'| > |A-\{k\}| = |A| - 1$

> Then, $B = B' \cup \{k\}$ is compatible and $|B| = |B'|+1 > |A|$

Contradiction to the optimality of A Q.E.D.

Optimal Substructure

- Recursive formulation: Choose the first activity k, and then solve the remaining subproblem S_k'

- How to choose the first activity k?

  DP, memoized recursion?

  i.e. choose the k value that will have the max size for S_k'

- DP would work,

  but is it necessary to try all possible values for k?

Greedy Choice Property

- Assume (without loss of generality) $f_1 \leq f_2 \leq \ldots \leq f_n$
  - If not, sort activities according to their finish times in non-decreasing order

- *Greedy choice property*: a sequence of locally optimal (greedy) choices $\Rightarrow$ an optimal solution

- How to choose the first activity greedily without losing optimality?

Greedy Choice Property - Theorem

Let activity set $S = \{a_1, a_2, \ldots a_n\}$, where $f_1 \leq f_2 \leq \ldots \leq f_n$

Theorem: There exists an optimal solution $A \subseteq S$ such that $a_1 \in A$

In other words, the activity with the earliest finish time is guaranteed to be in an optimal solution.

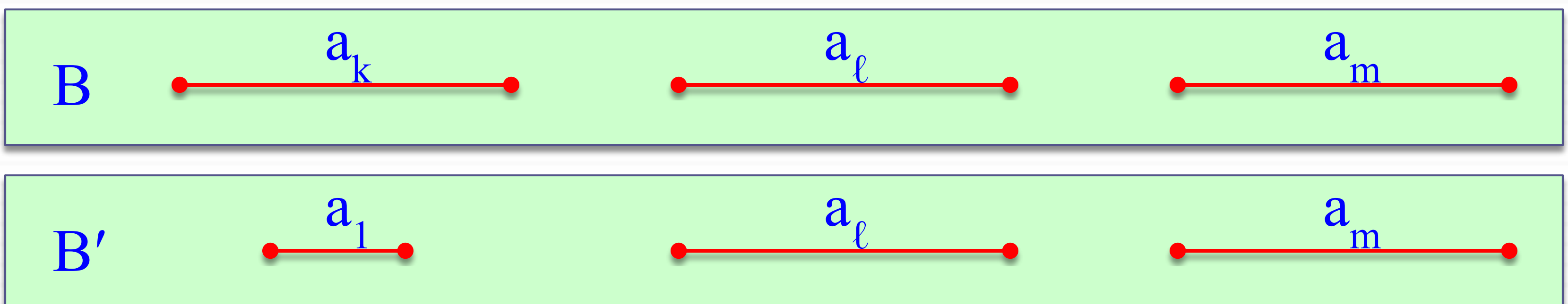
Greedy Choice Property - Proof

Theorem: There exists an optimal solution $A \subseteq S$ such that $a_1 \in A$

Proof: Consider an arbitrary optimal solution $B = \{a_k, a_\ell, a_m, \ldots\}$, where $f_k < f_\ell < f_m < \ldots$

If $k = 1$, then B starts with a_1, and the proof is complete

If $k > 1$, then create another solution B' by replacing a_k with a_1. Since $f_1 \leq f_k$, B' is guaranteed to be valid, and $|B'| = |B|$, hence also optimal



Greedy Algorithm

- So far, we have:
  - Optimal substructure property: If $A = \{a_k, \ldots\}$ is an optimal solution, then $A - \{a_k\}$ must be optimal for subproblem S_k', where $S_k' = \{a_i \in S: s_i \geq f_k\}$

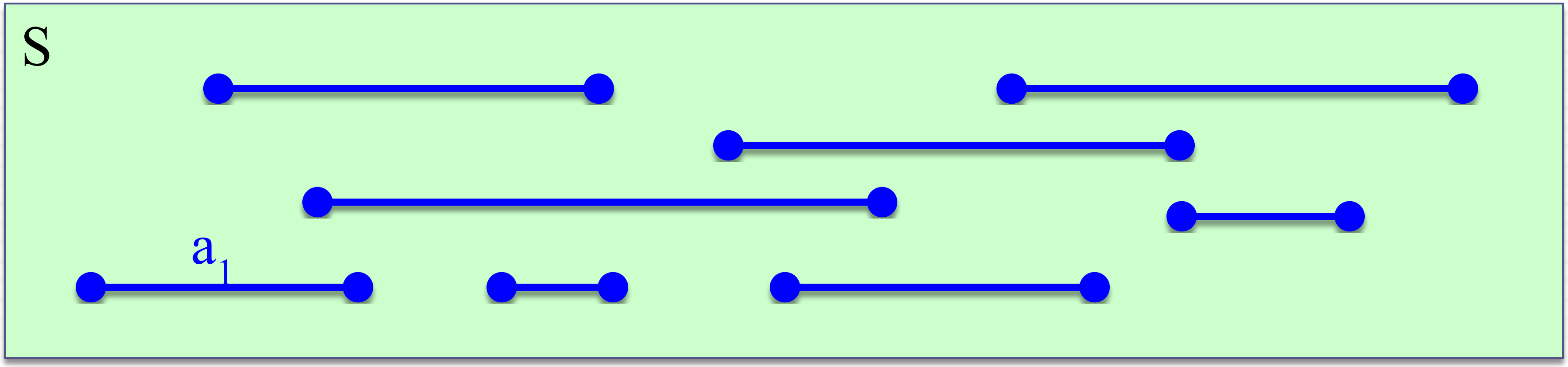
    Note: a_k is the activity with the earliest finish time in A

  - Greedy choice property: There is an optimal solution A that contains a_1

    Note: a_1 is the activity with the earliest finish time in S

Greedy Algorithm

- Basic idea of the greedy algorithm:
  1. Add a_1 to A
  2. Solve the remaining subproblem S_1', and then append the result to A



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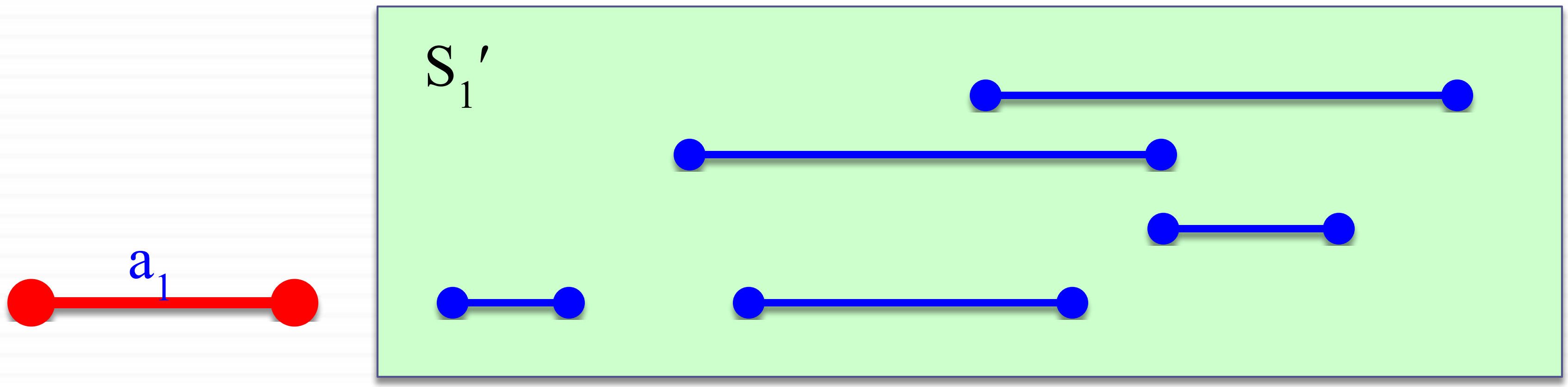


Greedy Algorithm for Activity Selection

j: specifies the index of most recent activity added to A

$f_j = \text{Max}\{f_k : k \in A\}$, max finish time of any activity in A; because activities are processed in non-decreasing order of finish times

Thus, "$s_i \geq f_j$" checks the compatibility of i to current A

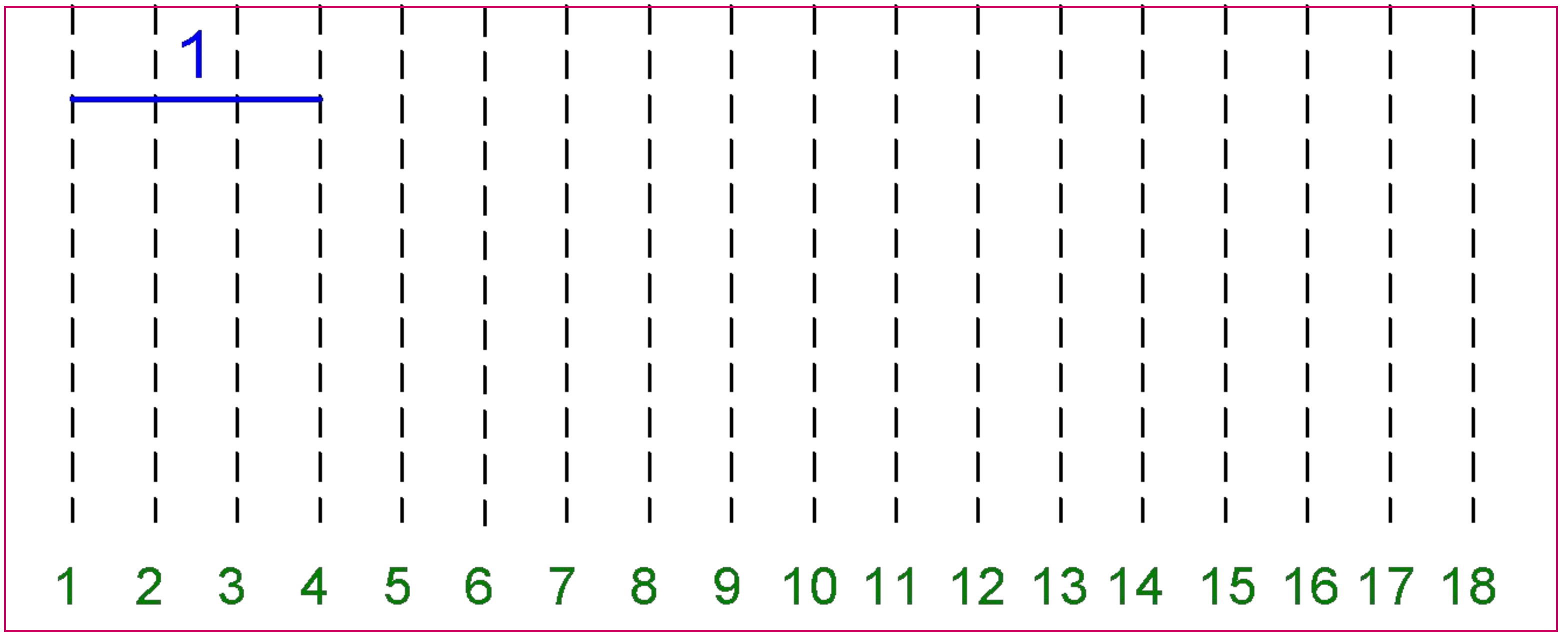
Running time: $\Theta(n)$ assuming that the activities were already sorted

GAS (s, f, n)
$A \leftarrow \{1\}$
$j \leftarrow 1$
for $i \leftarrow 2$ **to** n **do**
if $s_i \geq f_j$ **then**
$A \leftarrow A \cup \{i\}$
$j \leftarrow i$
return A

Activity Selection Problem: An Example

$S=\{[1, 4), [5, 7), [2, 8), [3, 11), [8, 15), [13, 18)\}$

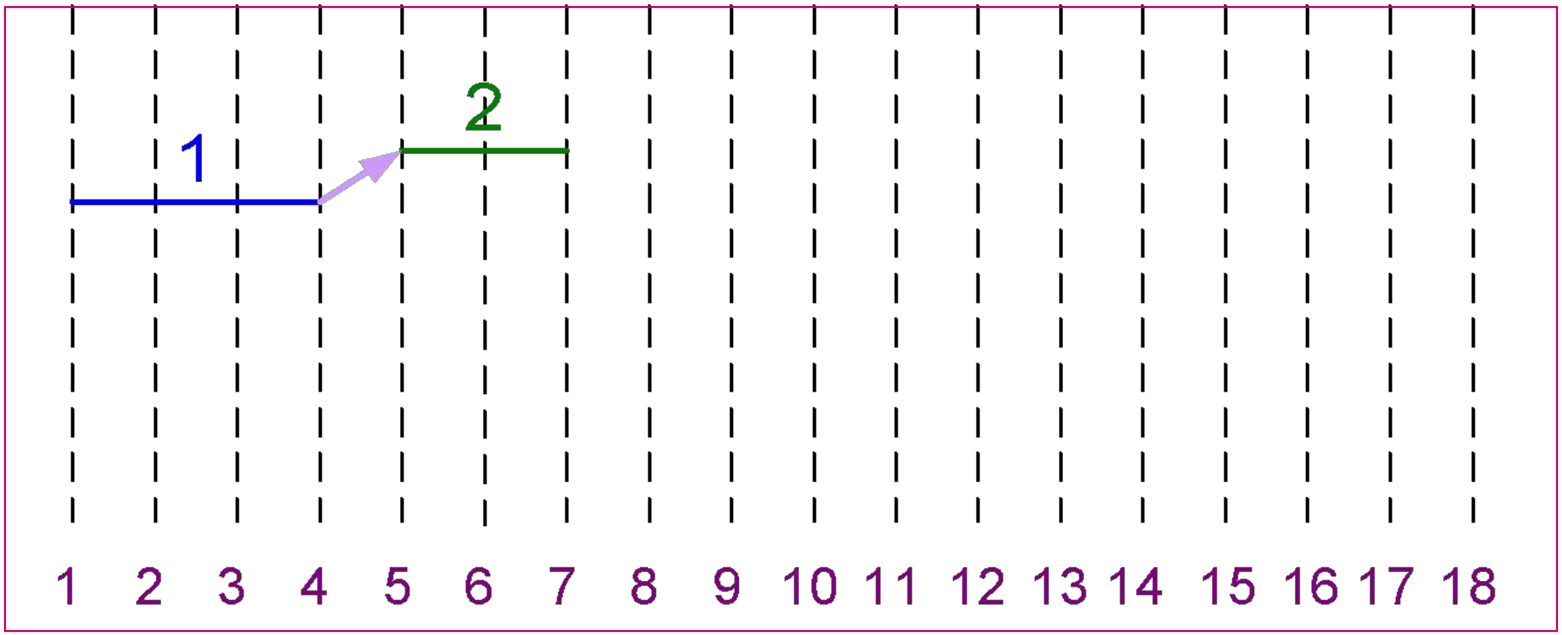
$f_j=0$



Activity Selection Problem: An Example

$S=\{[1, 4), [5, 7), [2, 8), [3, 11), [8, 15), [13, 18)\}$

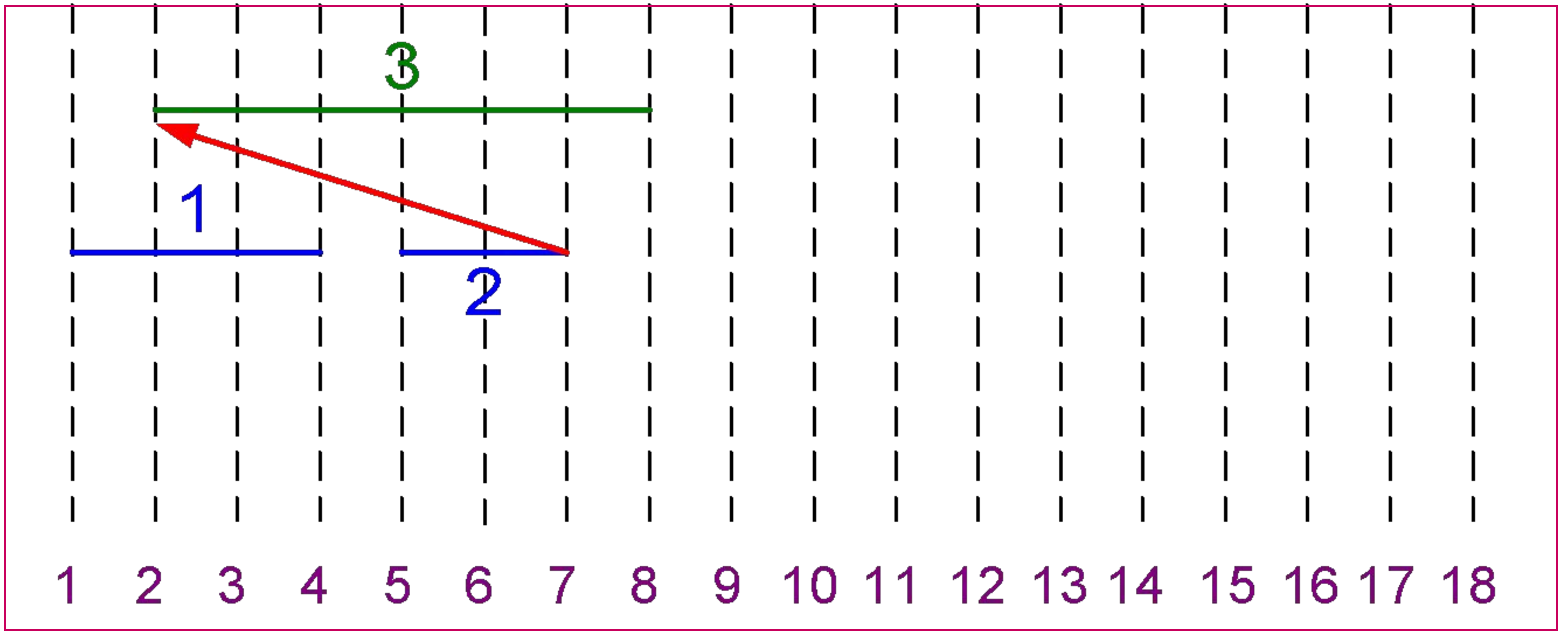
$f_j=4$



Activity Selection Problem: An Example

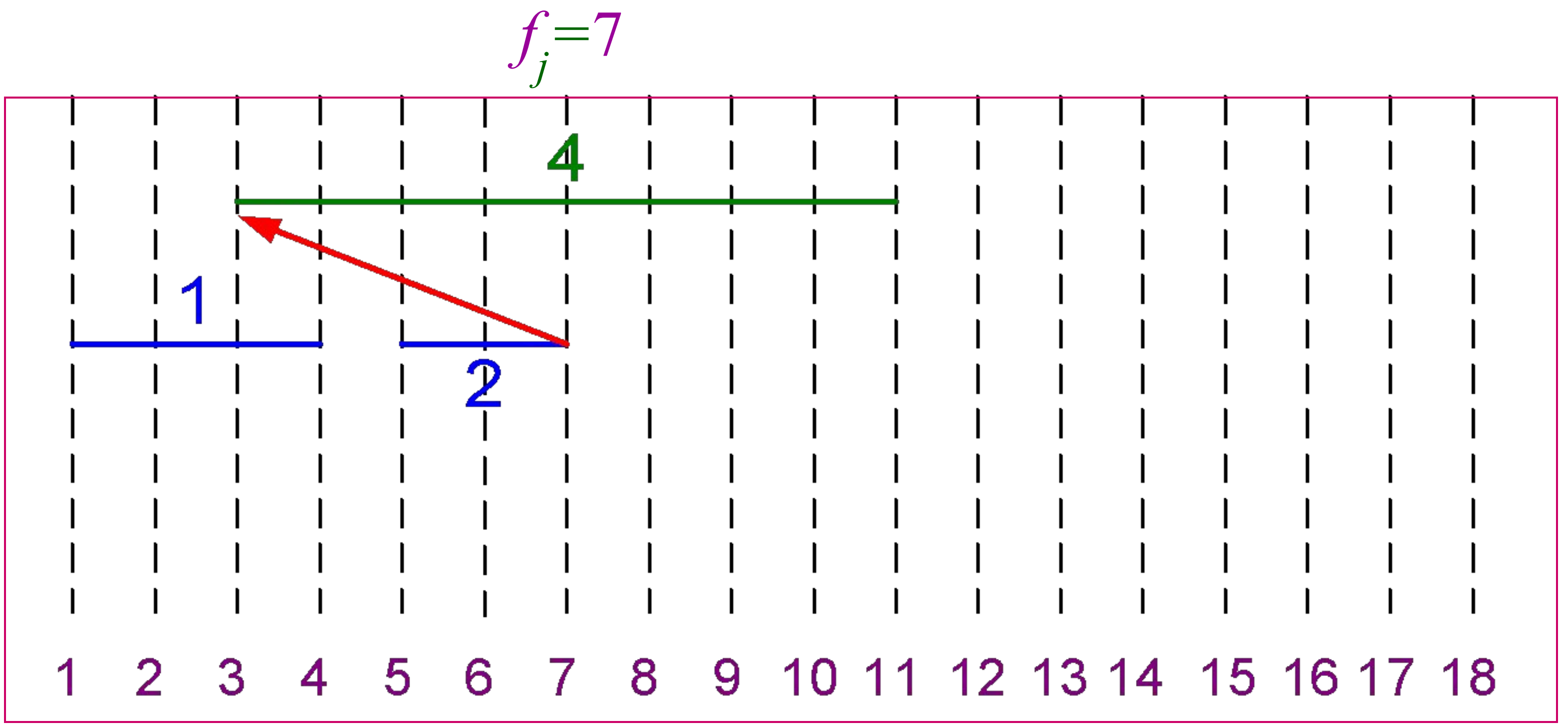
$S = \{[1, 4), [5, 7), [2, 8), [3, 11), [8, 15), [13, 18)\}$

$f_j = 7$



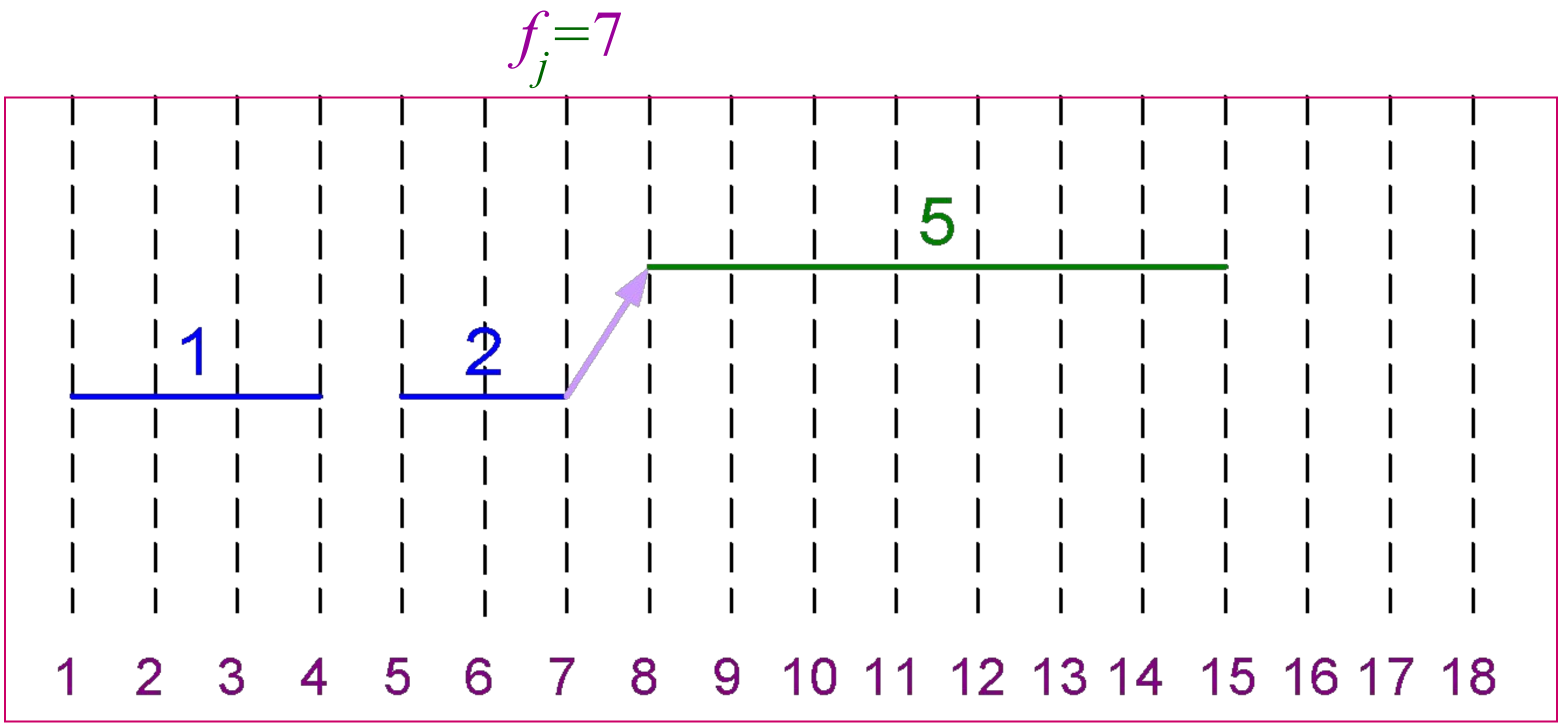
Activity Selection Problem: An Example

S={[1, 4), [5, 7), [2, 8), [3, 11), [8, 15), [13, 18)}



Activity Selection Problem: An Example

$S=\{[1, 4), [5, 7), [2, 8), [3, 11), [8, 15), [13, 18)\}$



Activity Selection Problem: An Example

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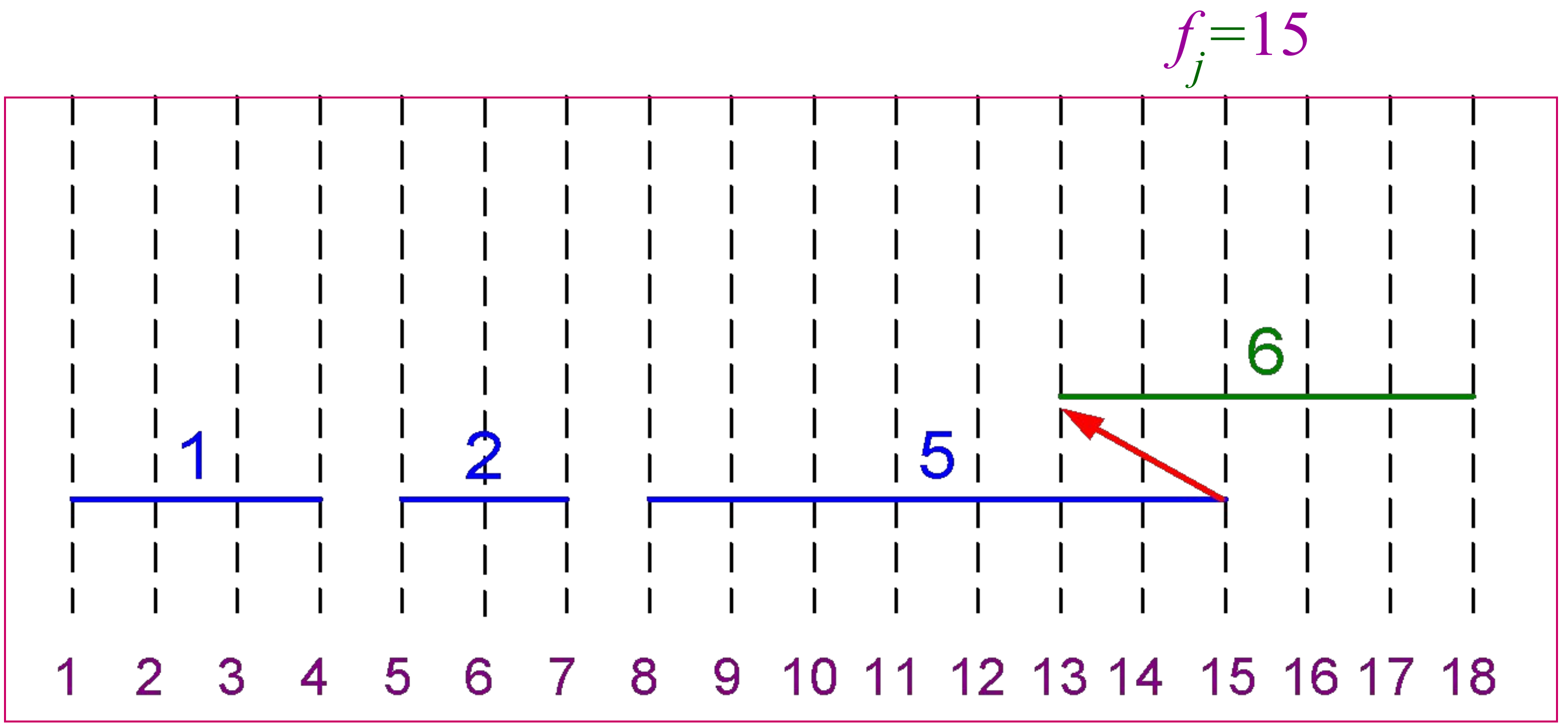
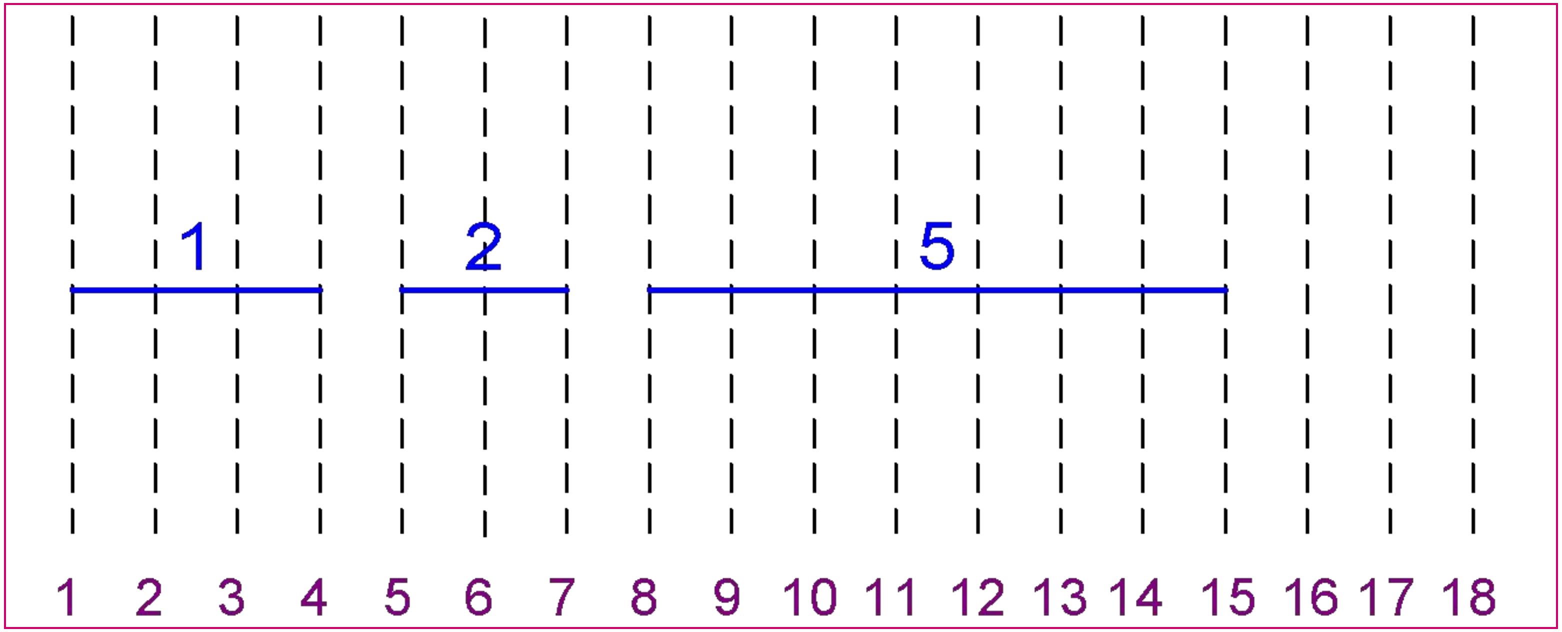


Activity Selection Problem: An Example

$S=\{[1, 4), [5, 7), [2, 8), [3, 11), [8, 15), [13, 18)\}$

$A=\{1, 2, 5\}$



Comparison of DP and Greedy Algorithms

Reminder: DP-Based Matrix Chain Order

$$m_{ij} = \min_{i \le k < j} \left\{ m_{ik} + m_{k+1,j} + p_{i-1} p_k p_j \right\}$$

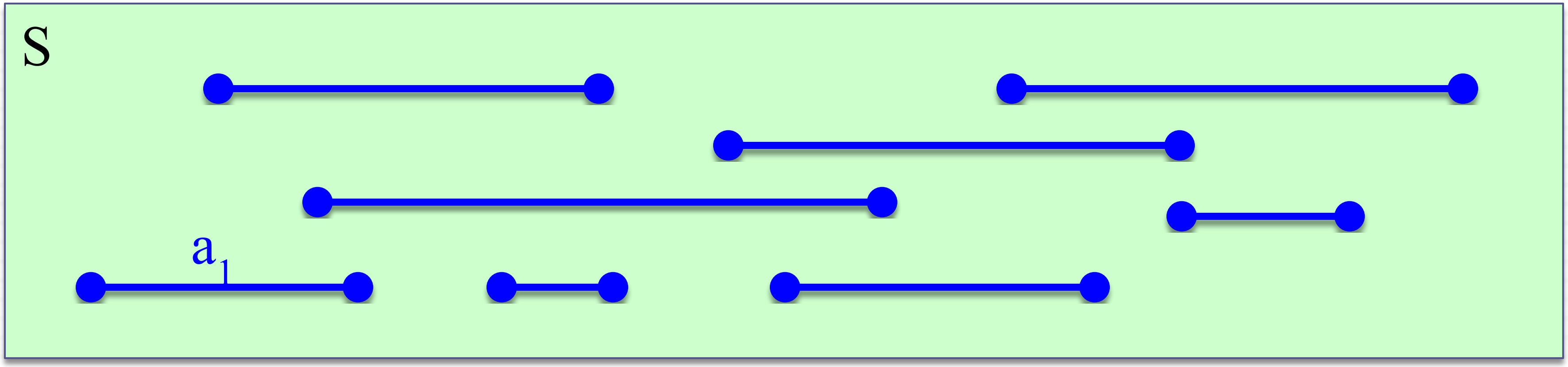
- We don't know ahead of time which k value to choose.
- We first need to compute the results of subproblems m_{ik} and $m_{k+1,j}$ before computing m_{ij}
- The selection of k is done based on the results of the subproblems.

Greedy Algorithm for Activity Selection

1. Make a greedy selection in the beginning:

   Choose a_1 (the activity with the earliest finish time)

2. Solve the remaining subproblem S_1' (all activities that start after a_1)



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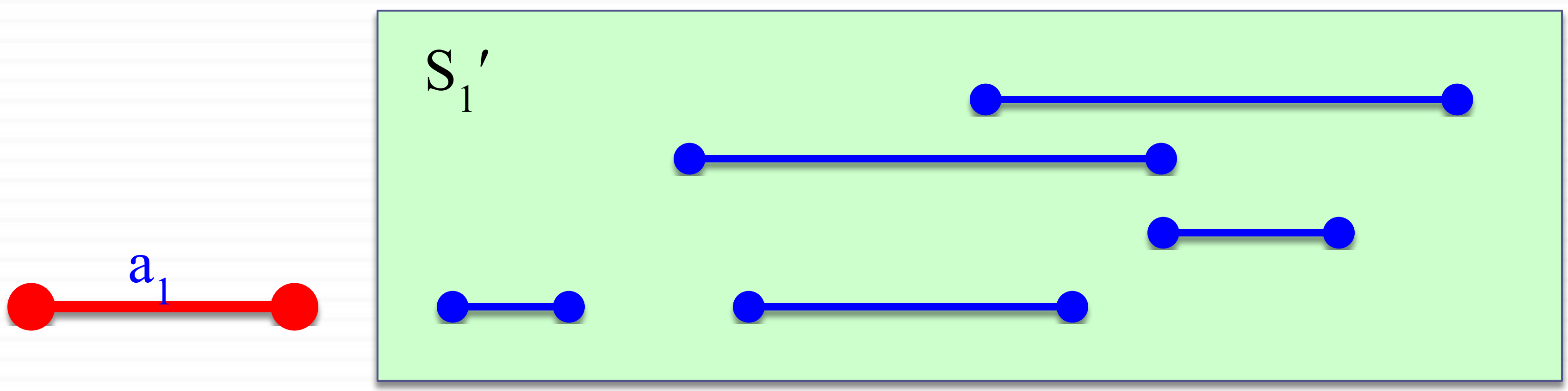


Greedy vs Dynamic Programming

- Optimal substructure property exploited by both Greedy and DP strategies
- Greedy Choice Property: A sequence of locally optimal choices $\Rightarrow$ an optimal solution
  - We make the choice that seems best at the moment
  - Then solve the subproblem arising after the choice is made
- DP: We also make a choice/decision at each step, but the choice may depend on the optimal solutions to subproblems
- Greedy: The choice may depend on the choices made so far, but it cannot depend on any future choices or on the solutions to subproblems

Greedy vs Dynamic Programming

- DP is a bottom-up strategy
- Greedy is a top-down strategy
  - each greedy choice in the sequence iteratively reduces each problem to a similar but smaller problem

Proof of Correctness of Greedy Algorithms

- Examine a globally optimal solution
- Show that this soln can be modified so that
  1) A greedy choice is made as the first step
  2) This choice reduces the problem to a similar but smaller problem
- Apply induction to show that a greedy choice can be used at every step
- Showing (2) reduces the proof of correctness to proving that the problem exhibits optimal substructure property

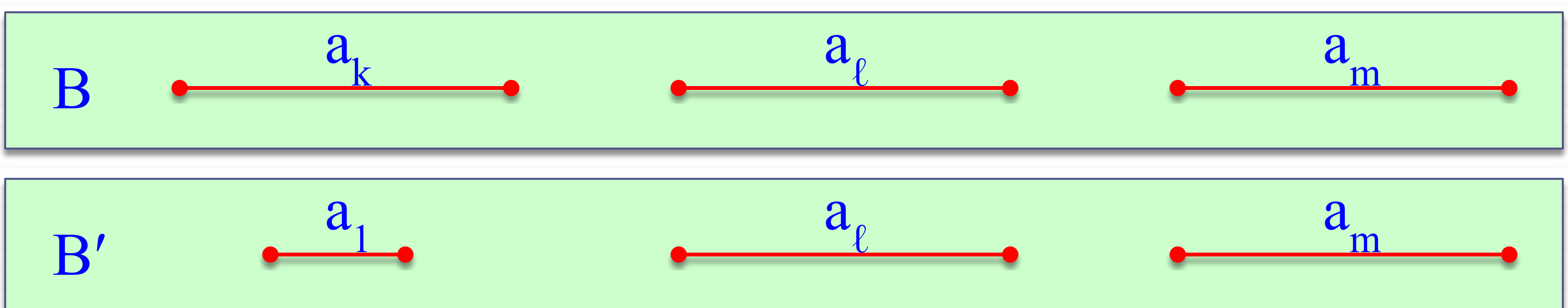
Greedy Choice Property - Proof

Theorem: There exists an optimal solution $A \subseteq S$ such that $a_1 \in A$

Proof: Consider an arbitrary optimal solution $B = \{a_k, a_\ell, a_m, \ldots\}$, where $f_k < f_\ell < f_m < \ldots$

If $k = 1$, then B starts with a_1, and the proof is complete

If $k > 1$, then create another solution B' by replacing a_k with a_1. Since $f_1 \le f_k$, B' is guaranteed to be valid, and $|B'| = |B|$, hence also optimal



Elements of Greedy Strategy

- How can you judge whether
- A greedy algorithm will solve a particular optimization problem?

Two key ingredients

- Greedy choice property
- Optimal substructure property

Key Ingredients of Greedy Strategy

- Greedy Choice Property: A globally optimal solution can be arrived at by making locally optimal (greedy) choices
- In DP,we make a choice at each step but the choice may depend on the solutions to subproblems
- In Greedy Algorithms, we make the choice that seems best at that moment then solve the subproblems arising after the choice is made
  - The choice may depend on choices so far, but it cannot depend on any future choice or on the solutions to subproblems
- DP solves the problem bottom-up
- Greedy usually progresses in a top-down fashion by making one greedy choice after another reducing each given problem instance to a smaller one

Key Ingredients: Greedy Choice Property

- We must prove that a greedy choice at each step yields a globally optimal solution
- The proof examines a globally optimal solution
- Shows that the soln can be modified so that a greedy choice made as the first step reduces the problem to a similar but smaller subproblem
- Then induction is applied to show that a greedy choice can be used at each step
- Hence, this induction proof reduces the proof of correctness to demonstrating that an optimal solution must exhibit optimal substructure property

Key Ingredients: Greedy Choice Property

- How to prove the greedy choice property?
  1. Consider the greedy choice c
  2. Assume that there is an optimal solution B that doesn't contain c.
  3. Show that it is possible to convert B to another optimal solution B′, where B′ contains c.
- *Example*: Activity selection algorithm

  Greedy choice: a_1 (the activity with the earliest finish time)

  Consider an optimal solution B without a_1

  Replace the first activity in B with a_1 to construct B′

  Prove that B′ must be an optimal solution

Key Ingredients: Optimal Substructure

- A problem exhibits optimal substructure if an optimal solution to the problem contains within it optimal solutions to subproblems

Example: Activity selection problem S

If an optimal solution A to S begins with activity a_1 then the set of activities

$$A' = A - \{a_1\}$$

is an optimal solution to the activity selection problem

$$S' = \{a_i \in S: s_i \geq f_1\}$$

Key Ingredients: Optimal Substructure

- Optimal substructure property is exploited by both Greedy and dynamic programming strategies
- Hence one may
  - Try to generate a dynamic programming soln to a problem when a greedy strategy suffices inefficient
  - Or, may mistakenly think that a greedy soln works when in fact a DP soln is required incorrect

Example: Knapsack Problems(S, w)

Knapsack Problems

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Knapsack Problem

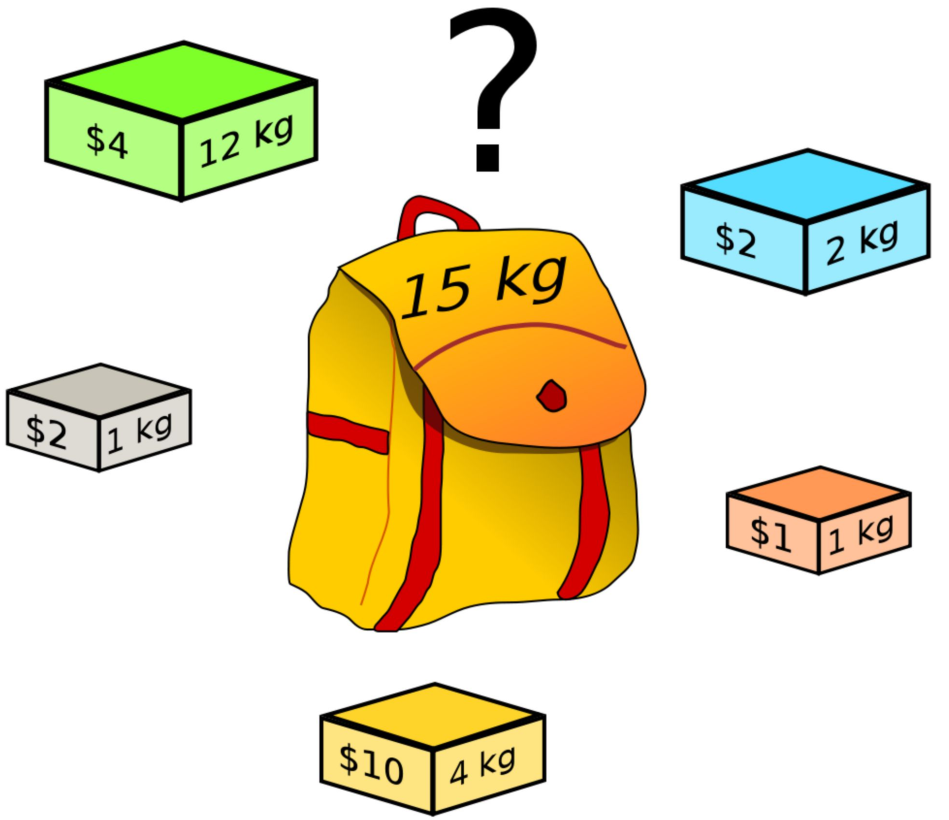


Image source: Wikimedia Commons

- Each item i has:
  weight w_i
  value v_i
- A thief has a knapsack of weight capacity w
- Which items to choose to maximize the value of the items in the knapsack?

Knapsack Problem: Two Versions

- The 0-1 knapsack problem:

  Each item is discrete.

  Each item either chosen as a whole or not chosen.

  Examples: TV, laptop, gold bricks, etc.

- The fractional knapsack problem:

  Can choose fractional part of each item.

  If item i has weight w_i, we can choose any amount $\leq w_i$

  Examples: Gold dust, silver dust, rice, etc.

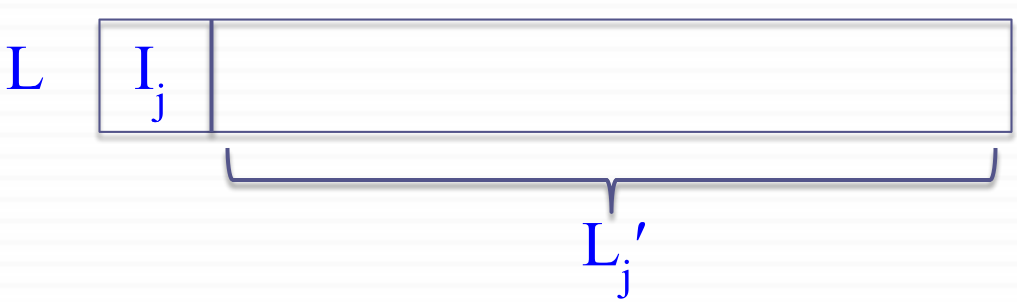
Knapsack Problems

- The 0-1Knapsack Problem(S, W)
  - A thief robbing a store finds n items $S = \{I_1, I_2, \ldots, I_n\}$, the ith item is worth v_i dollars and weighs w_i pounds, where v_i and w_i are integers
  - He wants to take as valuable a load as possible, but he can carry at most W pounds in his knapsack, where W is an integer
  - The thief cannot take a fractional amount of an item
- The Fractional Knapsack Problem (S, W)
  - The scenario is the same
  - But, the thief can take fractions of items rather than having to make binary (0-1) choice for each item

Optimal Substructure Property for the 0-1 Knapsack Problem (S, W)

- Consider an optimal load L for the problem (S, W).
- Let I_j be an item chosen in L with weight w_j
- Assume we remove I_j from L, and let:

$$L_j' = L - \{I_j\}$$
$$S_j' = S - \{I_j\}$$
$$W_j' = W - w_j$$



What can we say about the optimal substructure property?

Optimal Substructure Property for the 0-1 Knapsack Problem (S, W)

$L_j' = L - \{I_j\}$

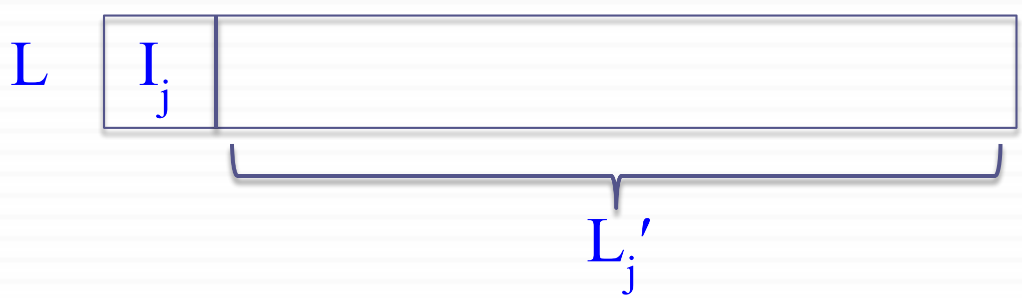
$S_j' = S - \{I_j\}$

$W_j' = W - w_j$

Optimal substructure property:

L_j' must be an optimal solution for (S_j', W_j')

Why?



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Optimal Substructure Property for the 0-1 Knapsack Problem (S, W)

$L_j' = L - \{I_j\}$ $S_j' = S - \{I_j\}$ $W_j' = W - w_j$

Optimal substructure: L_j' must be an optimal solution for (S_j', W_j')

Proof: By contradiction, assume there is a solution B_j' for (S_j', W_j'), which is better than L_j'.

We can construct a solution B for the original problem (S, W) as: $B = B_j' \cup \{I_j\}$.

The total value of B is now higher than L, which is a contradiction because L is optimal for (S, W).

Q.E.D.

Optimal Substructure Property for the Fractional Knapsack Problem (S, W)

- Consider an optimal solution L for (S, W)
- If we remove a weight $0 < w \le w_j$ of item j from optimal load L

  The remaining load

  $$L_j' = L - \{w \text{ pounds of } I_j\}$$

  must be a most valuable load weighing at most

  $$W_j' = W - w$$

  pounds that the thief can take from

  $$S_j' = S - \{I_j\} \cup \{w_j - w \text{ pounds of } I_j\}$$

- That is, L_j' should be an optimal soln to the

  Fractional Knapsack Problem(S_j', W_j')

Knapsack Problems

- Two different problems:
  - 0-1 knapsack problem
  - Fractional knapsack problem

- The problems are similar.
- Both problems have optimal substructure property.

- Which algorithm to solve each problem?

Fractional Knapsack Problem

- Can we use a greedy algorithm?
- Greedy choice: Take as much as possible from the item with the largest value per pound v_i/w_i

- Does the greedy choice property hold?

  Let j be the item with the largest value per pound v_j/w_j

  Need to prove that there is an optimal load that has as much j as possible.

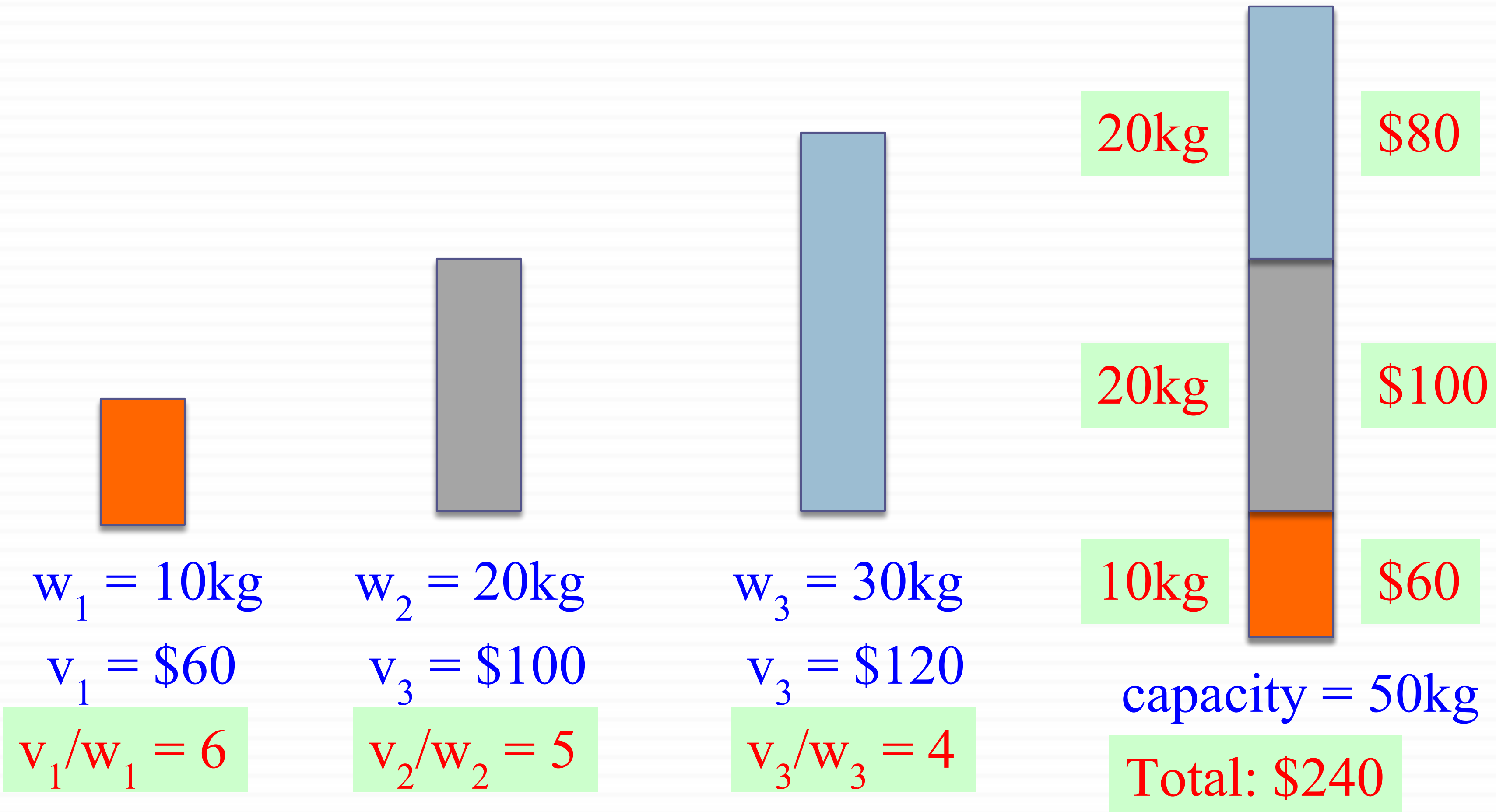
  Proof: Consider an optimal solution L that does not have the maximum amount of item j. We could replace the items in L with item j until L has maximum amount of j. L would still be optimal, because item j has the highest value per pound.

Greedy Solution to Fractional Knapsack

1) Compute the value per pound v_i / w_i for each item
2) The thief begins by taking, as much as possible, of the item with the greatest value per pound
3) If the supply of that item is exhausted before filling the knapsack, then he takes, as much as possible, of the item with the next greatest value per pound
4) Repeat (2-3) until his knapsack becomes full

- Thus, by sorting the items by value per pound the greedy algorithm runs in $O(n \lg n)$ time

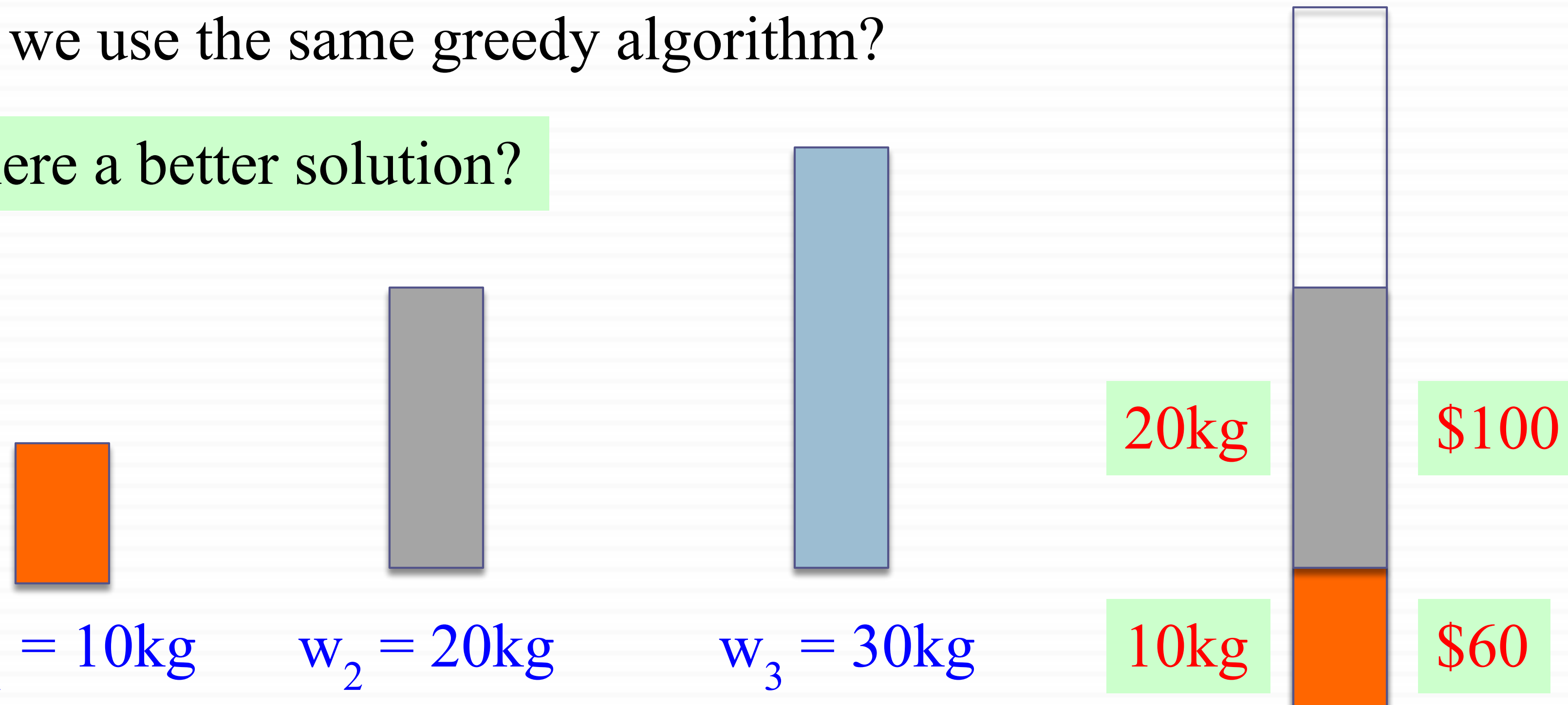
Fractional Knapsack Problem: Example



0-1 Knapsack Problem

Can we use the same greedy algorithm?

Is there a better solution?



$w_1 = 10\text{kg}$ $w_2 = 20\text{kg}$ $w_3 = 30\text{kg}$

$v_1 = \$60$ $v_3 = \$100$ $v_3 = \$120$

$v_1/w_1 = 6$ $v_2/w_2 = 5$ $v_3/w_3 = 4$

capacity = 50kg

Total: $160

0-1 Knapsack Problem

The optimal solution for this problem is:

This solution cannot be obtained using the greedy algorithm

30kg $120

20kg $100

$w_1 = 10\text{kg}$

$v_1 = \$60$

$v_1/w_1 = 6$

$w_2 = 20\text{kg}$

$v_3 = \$100$

$v_2/w_2 = 5$

$w_3 = 30\text{kg}$

$v_3 = \$120$

$v_3/w_3 = 4$

capacity = 50kg

Total: $220

0-1 Knapsack Problem

- When we consider an item I_j for inclusion we must compare the solutions to two subproblems
  - Subproblems in which I_j is included and excluded
- The problem formulated in this way gives rise to many

  overlapping subproblems (a key ingredient of DP)

  In fact, dynamic programming can be used to solve the 0-1 Knapsack problem

0-1 Knapsack Problem

- A thief robbing a store containing n articles $\{a_1, a_2, \ldots, a_n\}$
  - The value of ith article is v_i dollars (v_i is integer)
  - The weight of ith article is w_i kg (w_i is integer)
- Thief can carry at most W kg in his knapsack
- Which articles should he take to maximize the value of his load?
- Let $K_{n,W}=\{a_1, a_2, \ldots, a_n : W\}$ denote 0-1 knapsack problem
- Consider the solution as a sequence of n decisions
  - i.e., ith decision: whether thief should pick a_i for optimal load

Optimal Substructure Property

- Notation: $K_{n,W}$:

  The items to choose from: $\{a_1, \ldots, a_n\}$

  The knapsack capacity: W

- Consider an optimal load L for problem $K_{n,W}$

- Let's consider two cases:

  1) a_n is in L

  2) a_n is not in L

Optimal Substructure Property

- Case 1: If $a_n \in L$
  What can we say about the optimal substructure?
  $L - \{a_n\}$ must be optimal for $K_{n-1,W-wn}$
  $K_{n-1,W-wn}$:
  The items to choose from $\{a_1, \ldots a_{n-1}\}$
  The knapsack capacity: $W - w_n$
- Case 2: If $a_n \notin L$
  What can we say about the optimal substructure?
  L must be optimal for $K_{n-1,W}$
  $K_{n-1,W}$:
  The items to choose from $\{a_1, \ldots a_{n-1}\}$
  The knapsack capacity: W

Optimal Substructure Property

- In other words, optimal solution to $K_{n,W}$ contains an optimal solution to:

either: $K_{n-1, W-wn}$ (if a_n is selected)

or: $K_{n-1, W}$ (if a_n is not selected)

Recursive Formulation

c[i, w]: The value of an optimal solution to $K_{i,w}$

where $K_{i,w}$: $\{a_1, \ldots a_i: w\}$

$$c[i,w] = \begin{cases} 0, & \text{if } i = 0 \text{ or } w = 0 \\ c[i-1,w], & \text{if } w_i > w \\ max\{v_i + c[i-1,w-w_i]\,,\, c[i-1,w]\} & \text{o/w} \end{cases}$$

0-1 Knapsack Problem

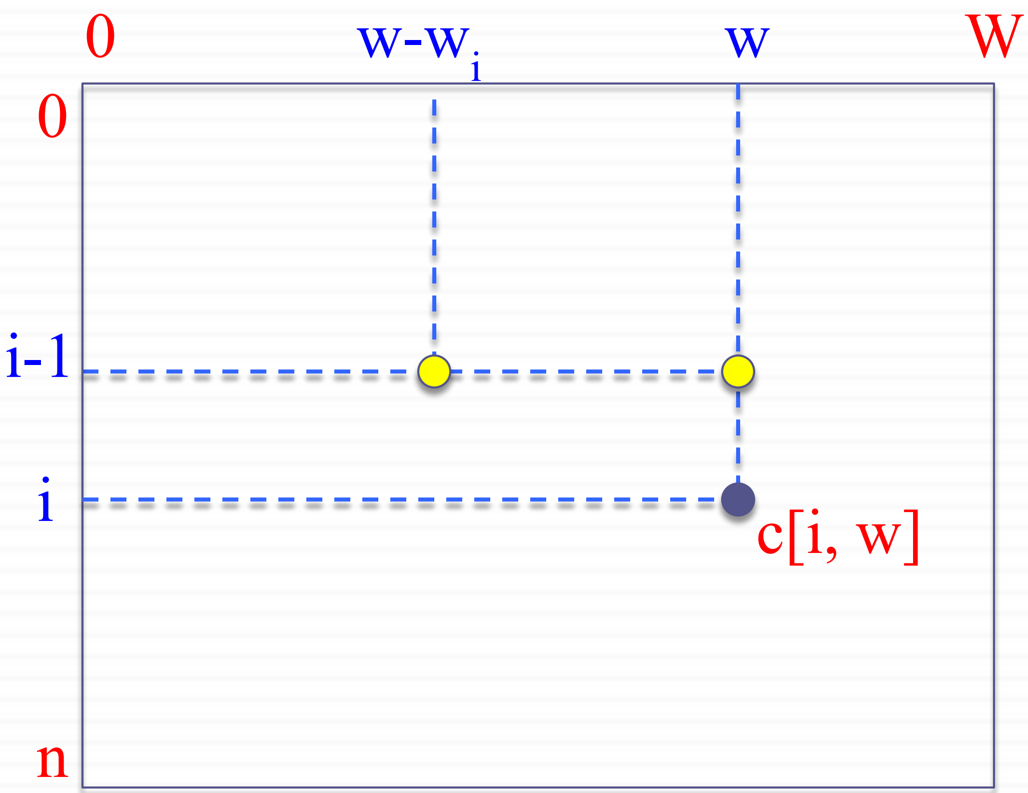
Recursive definition for value of optimal soln:

This recurrence says that an optimal solution $S_{i,w}$ for $K_{i,w}$

- either contains $a_i \Rightarrow c(S_{i,w}) = v_i + c(S_{i-1,w-wi})$
- or does not contain $a_i \Rightarrow c(S_{i,w}) = c(S_{i-1,w})$

- If thief decides to pick a_i
  - He takes v_i value and he can choose from $\{a_1, a_2, \ldots, a_{i-1}\}$ up to the weight limit $w - w_i$ to get $c[i-1, w - w_i]$
- If he decides not to pick a_i
  - He can choose from $\{a_1, a_2, \ldots, a_{i-1}\}$ up to the weight limit w to get $c[i-1, w]$
- The better of these two choices should be made

Bottom-up Computation



Need to process:

c[i, w]

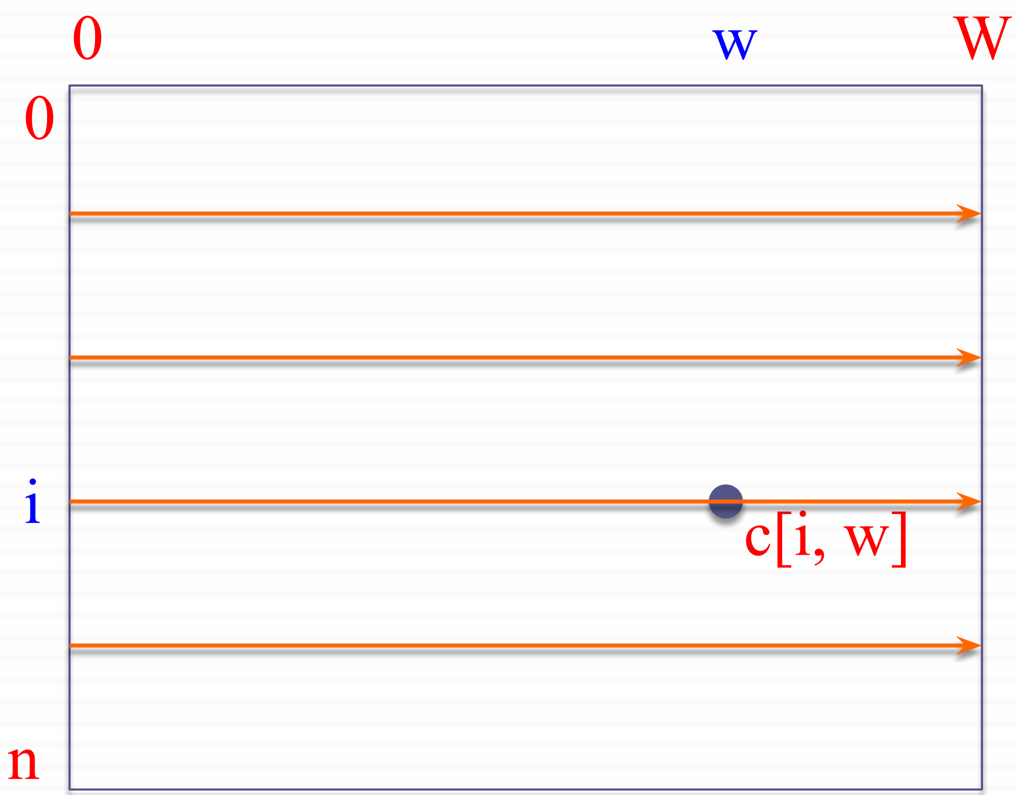
after computing:

c[i-1, w],

c[i-1, w-w$_i$]

for all w$_i$ < w

Bottom-up Computation



for i ← 1 **to** n
    for w ← 1 **to** W

....

....

c[i, w] =

DP Solution to 0-1 Knapsack

KNAP0-1(v, w, n, W)

for $\omega \leftarrow 0$ **to** W **do**
 $c[0, \omega] \leftarrow 0$

for $i \leftarrow 1$ **to** n **do**
 $c[i, 0] \leftarrow 0$

for $i \leftarrow 1$ **to** n **do**
 for $\omega \leftarrow 1$ **to** W **do**
  if $w_i \leq \omega$ **then**
   $c[i, \omega] \leftarrow max\{v_i + c[i-1, \omega - w_i], c[i-1, \omega]\}$
  else
   $c[i, \omega] \leftarrow c[i-1, \omega]$

return $c[n, W]$

c is an $(n+1)\times(W+1)$ array; $c[0..n : 0..W]$

Note: table is computed in row-major order

Run time: $T(n) = \Theta(nW)$

Constructing an Optimal Solution

- No extra data structure is maintained to keep track of the choices made to compute c[i, w]

  i.e. The choice of whether choosing item i or not

- Possible to understand the choice done by comparing c[i, w] with c[i-1, w]

  If c[i,w] = c[i-1, w] then it means item i was assumed to be not chosen for the best c[i, w]

Finding the Set S of Articles in an Optimal Load

SOLKNAP0-1(a, v, w, n, W, c)

$i \leftarrow n$; $\omega \leftarrow W$
$S \leftarrow \varnothing$

while $i > 0$ **do**
  if $c[i, \omega] = c[i-1, \omega]$ **then**
    $i \leftarrow i - 1$
  else
    $S \leftarrow S \cup \{a_i\}$
    $\omega \leftarrow \omega - w_i$
    $i \leftarrow i - 1$

return S

0-1 Knapsack Example

Item i	1	2	3	4
Value ($)	100	20	60	40
Weight (kg)	3	2	4	1

c(i,w)	0	1	2	3	4	W=5
0	0	0	0	0	0	0
1	0					
2	0					
3	0					
n=4	0					

0-1 Knapsack Example

Item i	**1**	**2**	**3**	**4**
Value ($)	100	20	60	40
Weight (kg)	3	2	4	1

c(i,w)	**0**	**1**	**2**	**3**	**4**	**W=5**
0	0	**0**	0	0	0	0
1	0	0	0	100	**100**	100
2	0	0	20	100	**100**	120
3	0	0	20	100	**100**	120
n=4	0	40	40	100	140	**140**

Huffman Codes

Huffman Codes for Compression

- Widely used and very effective for data compression
- Savings of 20% - 90% typical

  (depending on the characteristics of the data)

- In summary: Huffman's greedy algorithm uses a table of frequencies of character occurrences to build up an optimal way of representing each character as a binary string.

Binary String Representation - Example

- Consider a data file with:
  - 100K characters
  - Each character is one of {a, b, c, d, e, f}
- Frequency of each character in the file:

	a	b	c	d	e	f
frequency	45K	13K	12K	16K	9K	5K

- *Binary character code*: Each character is represented by a unique binary string.
- *Intuition*: Frequent characters ⇔ shorter codewords
  Infrequent characters ⇔ longer codewords

Binary String Representation - Example

	a	*b*	*c*	*d*	*e*	*f*
frequency	45K	13K	12K	16K	9K	5K
fixed-length	000	001	010	011	100	101
variable-length(1)	0	101	100	111	1101	1100
variable-length(2)	0	10	110	1110	11110	11111

How many total bits needed for fixed-length codewords?

100K * 3 = 300K bits

How many total bits needed for variable-length(1) codewords?

45K*1 + 13K*3 + 12K*3 + 16K*3 + 9K*4 + 5K*4 = 224K

How many total bits needed for variable-length(2) codewords?

45K*1 + 13K*2 + 12K*3 + 16K*4 + 9K*5 + 5K*5 = 241K

Prefix Codes

- *Prefix codes*: No codeword is also a prefix of some other codeword
- Example:

	a	*b*	*c*	*d*	*e*	*f*
codeword	0	101	100	111	1101	1100

- It can be shown that:

Optimal data compression is achievable with a prefix code

- In other words, optimality is not lost due to prefix-code restriction.

Prefix Codes: Encoding

	a	*b*	*c*	*d*	*e*	*f*
codeword	0	101	100	111	1101	1100

- ***Encoding***: Concatenate the codewords representing each character of the file

- ***Example***: Encode file "abc" using the codewords above

  abc $\Rightarrow$ 0.101.100 $\Rightarrow$ 0101100

Note: "." denotes the concatenation operation. It is just for illustration purposes, and does not exist in the encoded string

Prefix Codes: Decoding

- Decoding is quite simple with a prefix code
- The first codeword in an encoded file is unambiguous

  because no codeword is a prefix of any other

- ***Decoding algorithm***:
  1. Identify the initial codeword
  2. Translate it back to the original character
  3. Remove it from the encoded file
  4. Repeat the decoding process on the remainder of the encoded file.

Prefix Codes: Decoding - Example

	a	*b*	*c*	*d*	*e*	*f*
codeword	0	101	100	111	1101	1100

Example: Decode encoded file 001011101

$001011101 \Rightarrow 0.01011101 \Rightarrow 0.0.1011101$
$0.0.101.1101 \Rightarrow 0.0.101.1101 \Rightarrow$ aabe

Prefix Codes

Convenient representation for the prefix code:

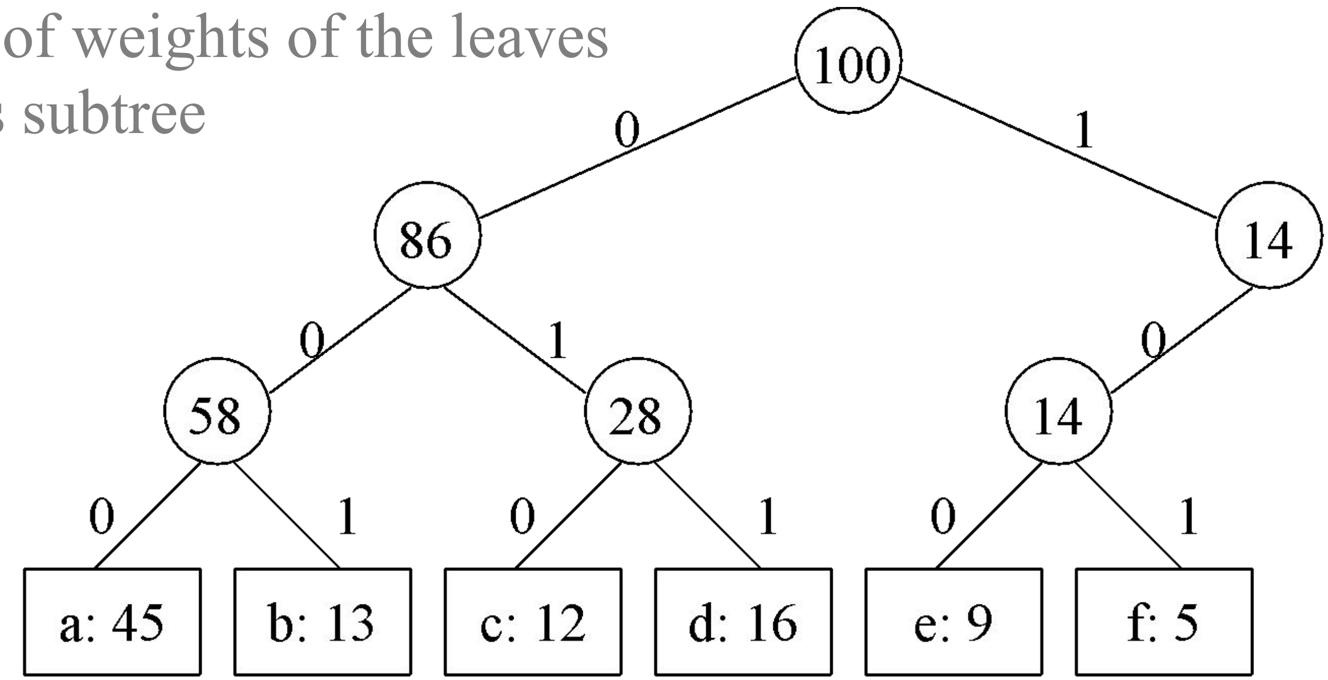
a binary tree whose leaves are the given characters

Binary codeword for a character is the path from the root to that character in the binary tree

“0” means “go to the left child”
“1” means “go to the right child”

Binary Tree Representation of Prefix Codes

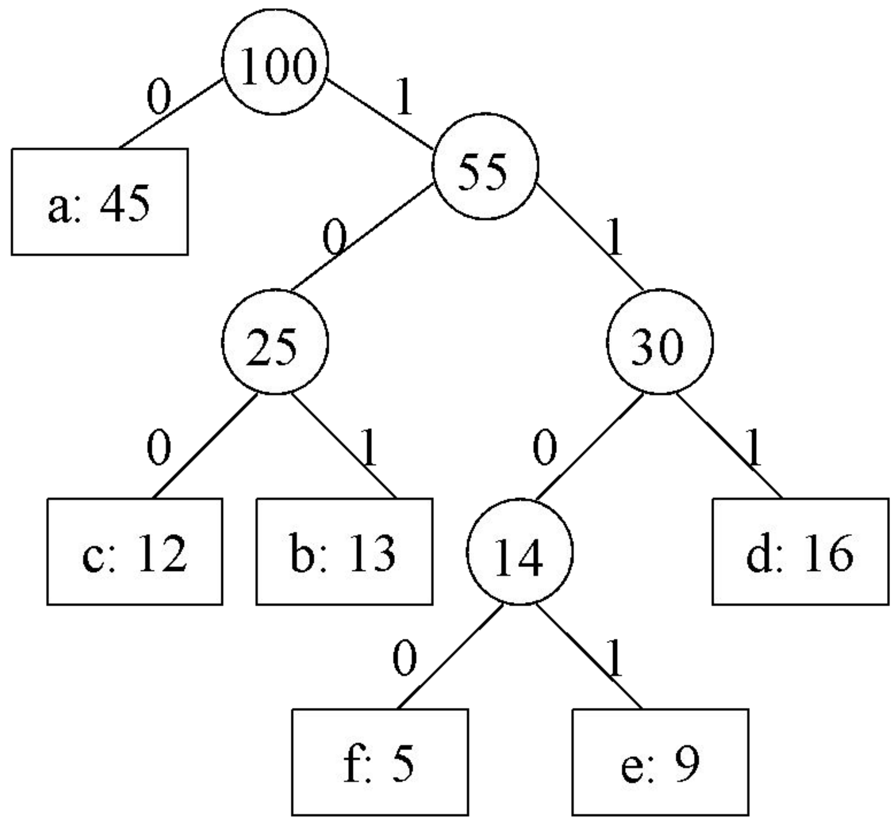
Weight of an internal node:
sum of weights of the leaves
in its subtree



The binary tree corresponding to the fixed-length code

Binary Tree Representation of Prefix Codes

Weight of an internal node: sum of weights of the leaves in its subtree



The binary tree corresponding to the optimal variable-length code

An optimal code for a file is always represented by a full binary tree

Full Binary Tree Representation of Prefix Codes

Consider an FBT corresponding to an optimal prefix code

It has $|C|$ leaves (external nodes)

One for each letter of the alphabet where C is the alphabet from which the characters are drawn

Lemma: An FBT with $|C|$ external nodes has exactly $|C|-1$ internal nodes

Full Binary Tree Representation of Prefix Codes

- Consider an FBT T, corresponding to a prefix code.
- Notation:

  f(c): frequency of character c in the file

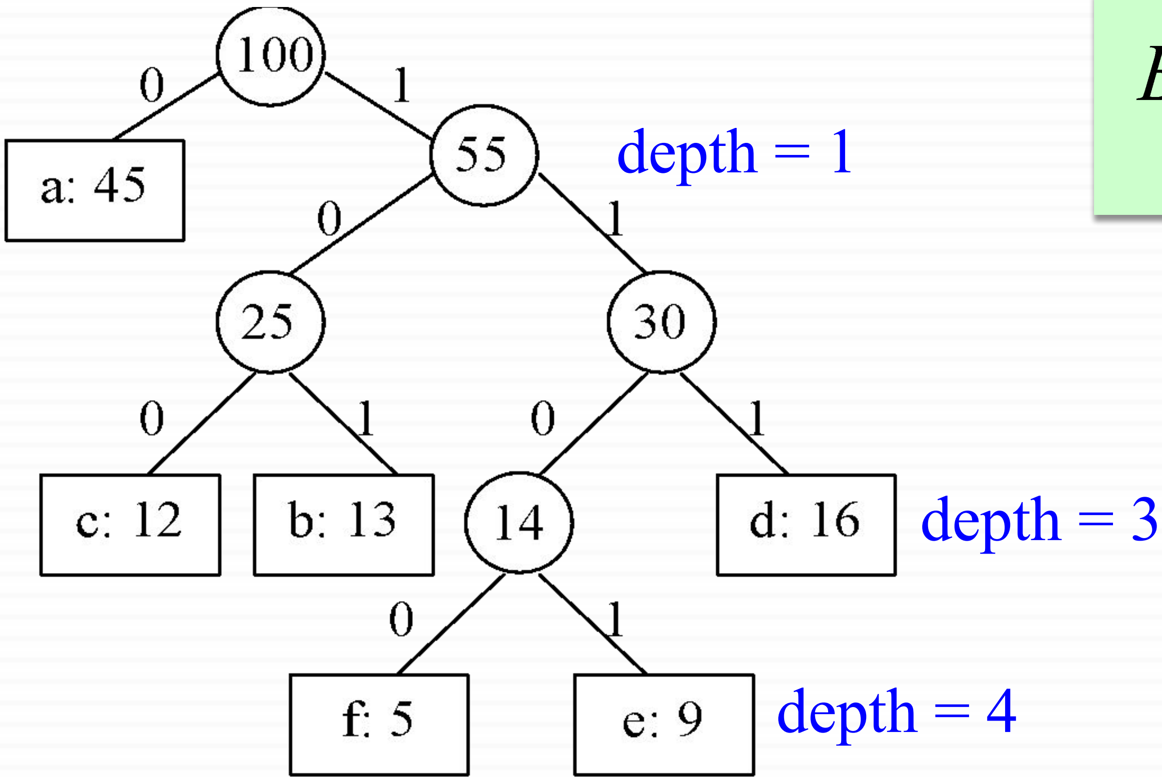
  $d_T(c)$: depth of c's leaf in the FBT T

  B(T): the number of bits required to encode the file
- What is the length of the codeword for c?

  $d_T(c)$, same as the depth of c in T
- How to compute B(T), cost of tree T?

$$B(T) = \sum_{c \in C} f(c)\, d_T(c)$$

Cost Computation - Example



$$B(T) = \sum_{c \in C} f(c)\, d_T(c)$$

B(T) = 45*1 + 12*3 + 13*3 + 16*3 + 5*4 + 9*4 = 224

Prefix Codes

Lemma: Let each internal node i is labeled with the sum of the weight $w(i)$ of the leaves in its subtree

Then $B(T) = \sum_{c \in C} f(c)\, d_T(c) = \sum_{i \in I_T} w(i)$ where

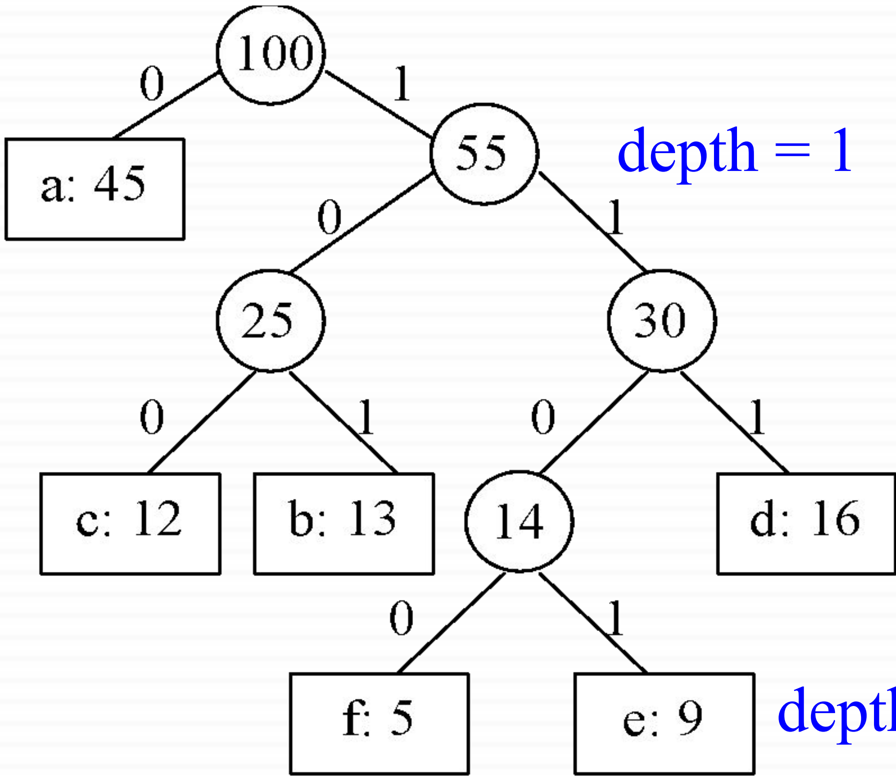
I_T denotes the set of internal nodes in T

Proof: Consider a leaf node c with $f(c)$ & $d_T(c)$

Then, $f(c)$ appears in the weights of $d_T(c)$ internal node along the path from c to the root

Hence, $f(c)$ appears $d_T(c)$ times in the above summation

Cost Computation - Example



depth = 1

depth = 3

depth = 4

$$B(T) = \sum_{i \in I_T} w(i)$$

B(T) = 100 + 55 + 25 + 30 + 14 = 224

Constructing a Huffman Code

Problem Formulation: For a given character set C, construct an optimal prefix code with the minimum total cost

Huffman invented a greedy algorithm that constructs an optimal prefix code called a Huffman code

The greedy algorithm

- builds the FBT corresponding to the optimal code in a bottom-up manner
- begins with a set of $|C|$ leaves
- performs a sequence of $|C|-1$ "merges" to create the final tree

Constructing a Huffman Code

A priority queue Q, keyed on f, is used to identify the two least-frequent objects to merge

The result of the merger of two objects is a new object

- inserted into the priority queue according to its frequency
- which is the sum of the frequencies of the two objects merged

Constructing a Huffman Code

HUFFMAN(C)
  $n \leftarrow |C|$
  $Q \leftarrow$ BUILD-HEAP(C)
  for $i \leftarrow 1$ to $n - 1$ do
    $z \leftarrow$ ALLOCATE-NODE()
    $x \leftarrow$ left[z] $\leftarrow$ EXTRACT-MIN(Q)
    $y \leftarrow$ right[z] $\leftarrow$ EXTRACT-MIN(Q)
    $f[z] \leftarrow f[x] + f[y]$
    INSERT(Q, z)
  return EXTRACT-MIN(Q) Δ only one object left in Q

Priority queue is implemented as a binary heap

Initiation of Q (BUILD-HEAP): O(n) time

EXTRACT-MIN & INSERT take O(lg$n$) time on Q with n objects

$$T(n) = \sum_{i=1}^{n} \lg i = O(\lg(n!)) = O(n \lg n)$$

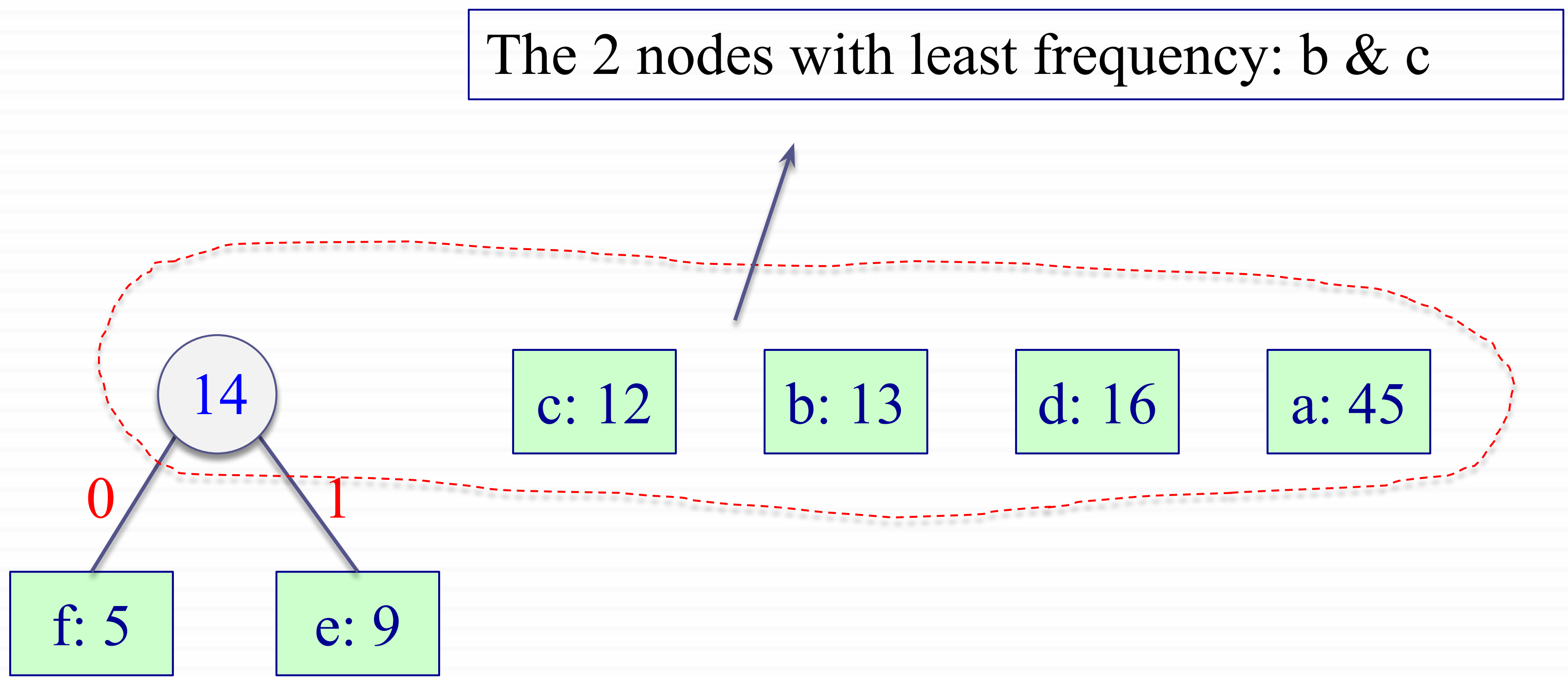
Constructing a Huffman Code - Example

Start with one leaf node for each character

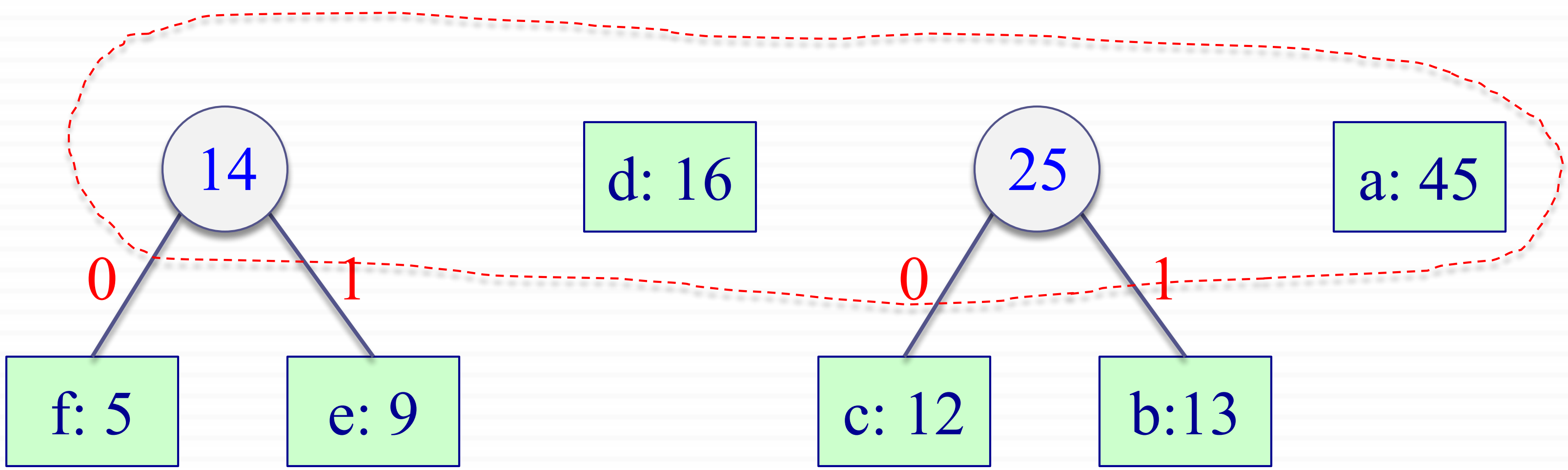
The 2 nodes with the least frequencies: f & e
Merge f & e and create an internal node
Set the internal node frequency to 5 + 9 = 14

f: 5 | e: 9 | c: 12 | b: 13 | d: 16 | a: 45

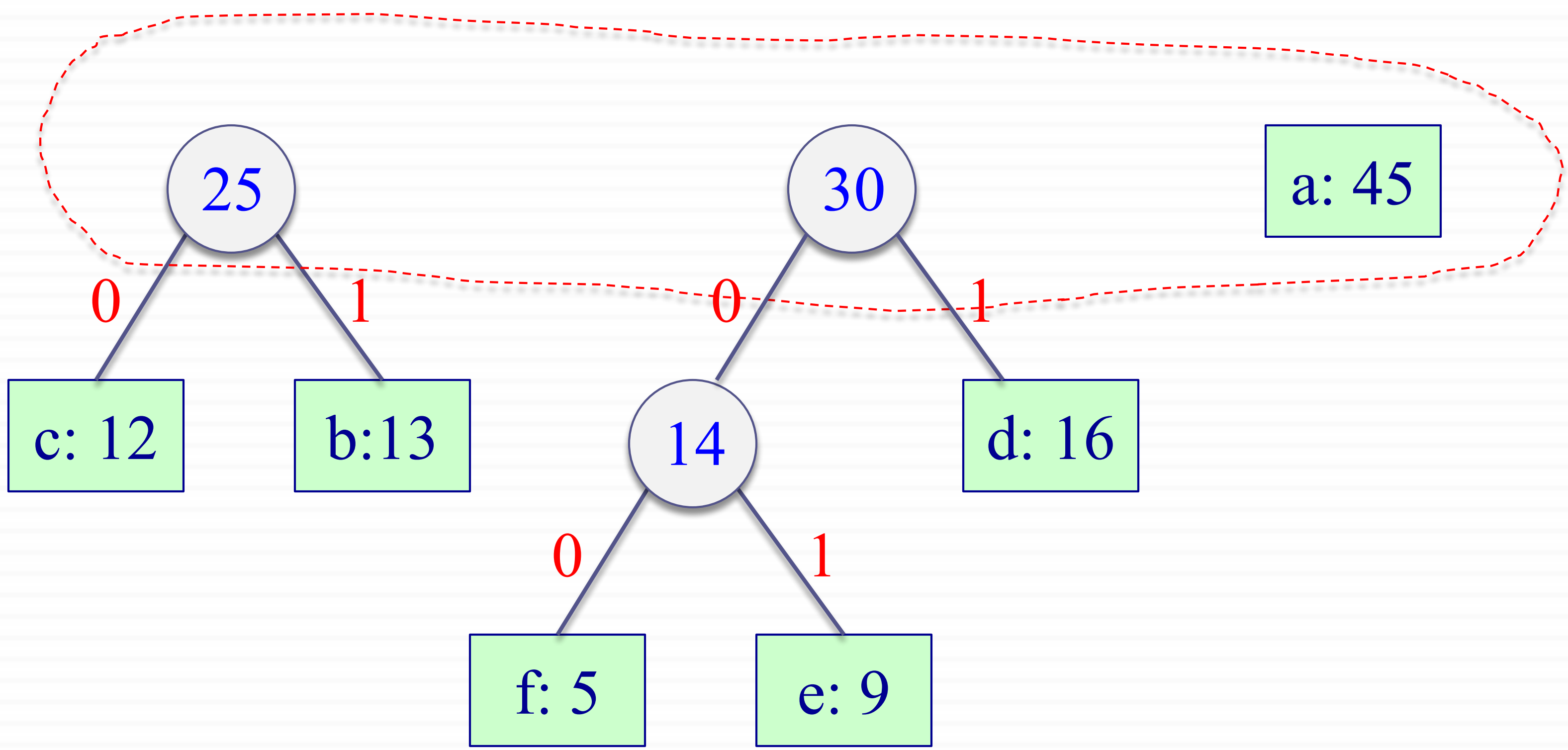
Constructing a Huffman Code - Example



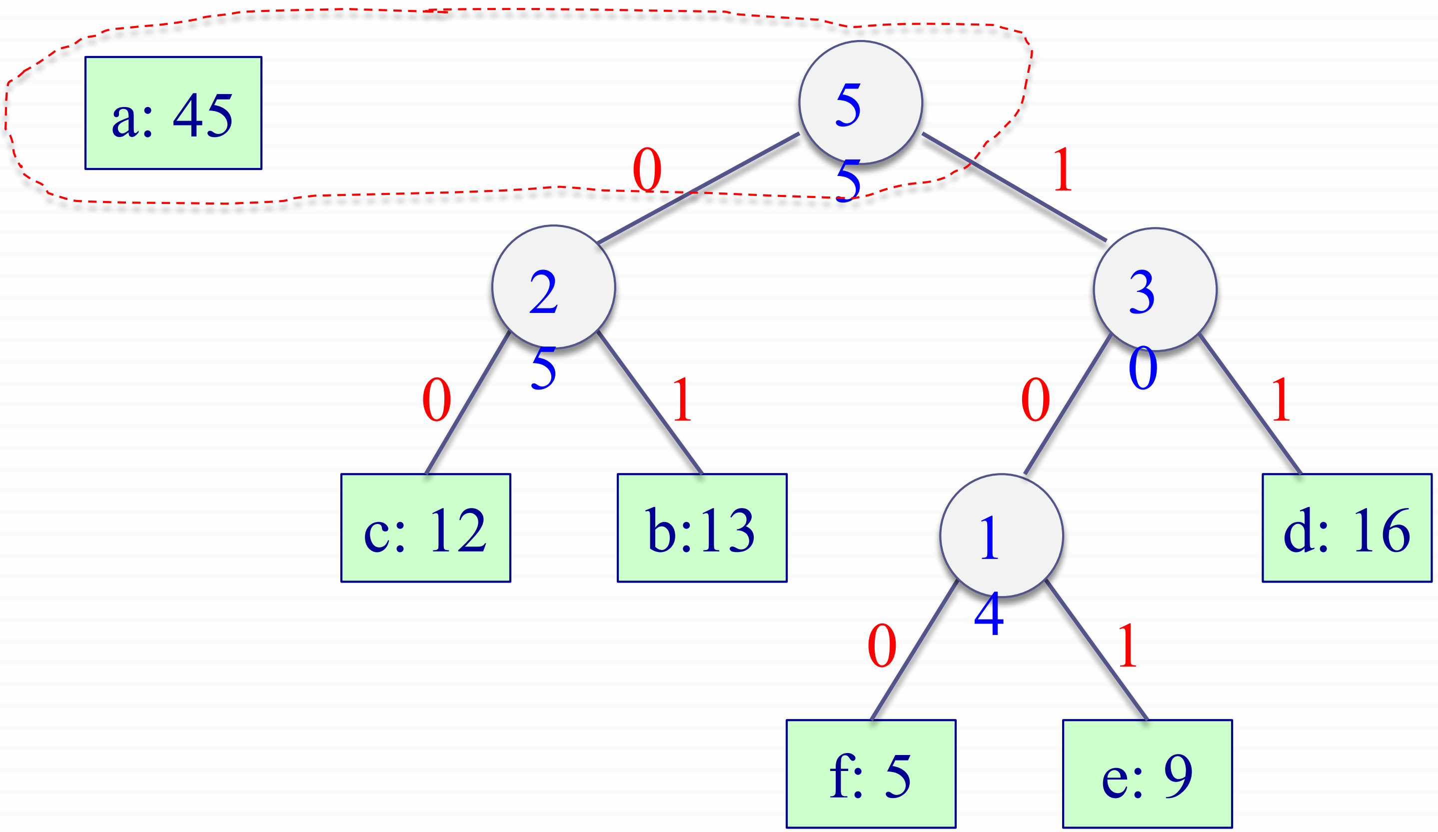
Constructing a Huffman Code - Example



Constructing a Huffman Code - Example



Constructing a Huffman Code - Example



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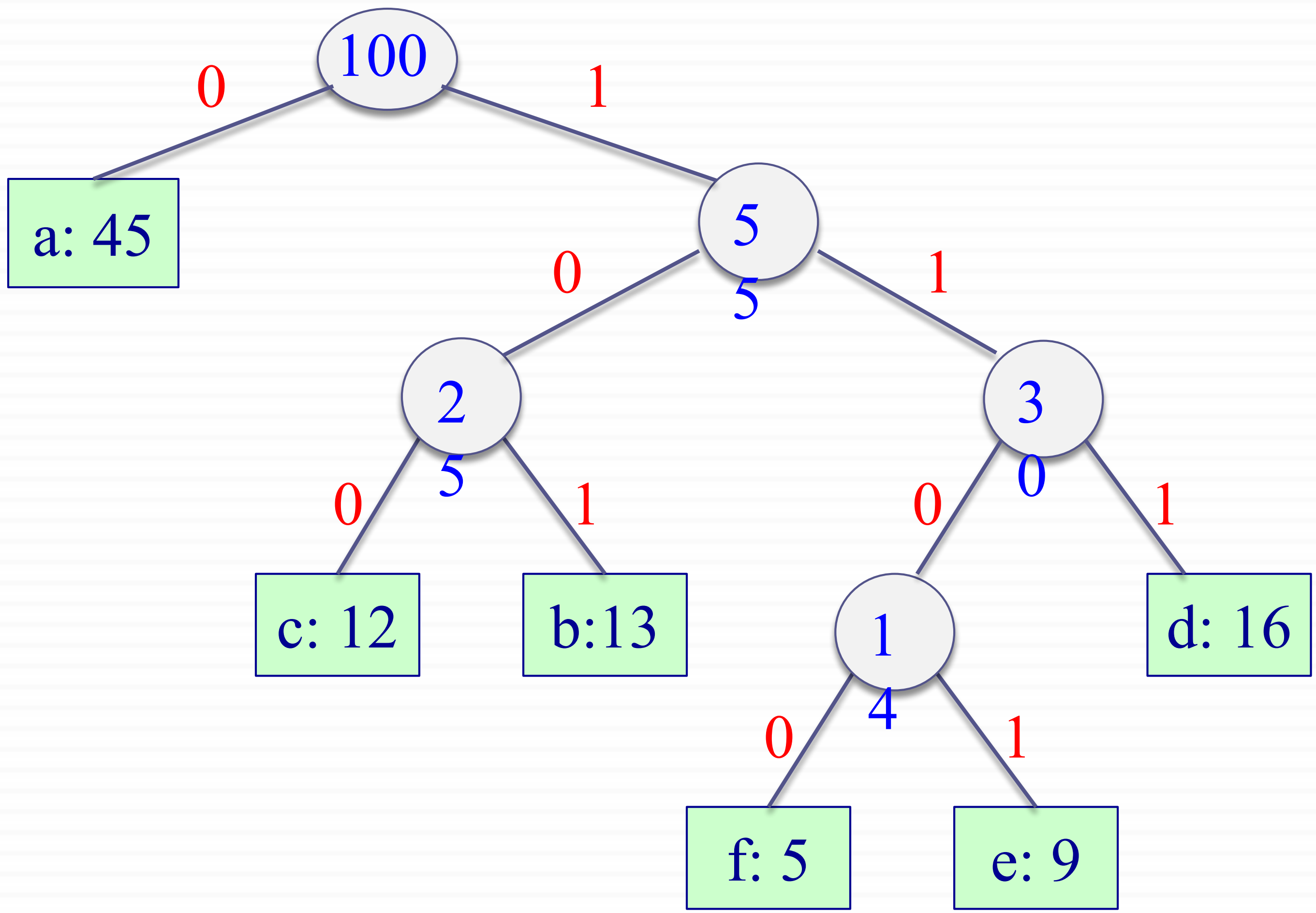
Constructing a Huffman Code - Example



Correctness Proof of Huffman's Algorithm

- We need to prove:
  - The greedy choice property
  - The optimal substructure property

- What is the greedy step in Huffman's algorithm?

  Merging the two characters with the lowest frequencies

- We will first prove the greedy choice property

Greedy Choice Property

Lemma 1: Let x & y be two characters in C having the lowest frequencies.

Then, $\exists$ an optimal prefix code for C in which the codewords for x & y have the same length and differ only in the last bit

Note: If x & y are merged in Huffman's algorithm, their codewords are guaranteed to have the same length and they will differ only in the last bit. Lemma 1 states that there exists an optimal solution where this is the case.

Greedy Choice Property - Proof

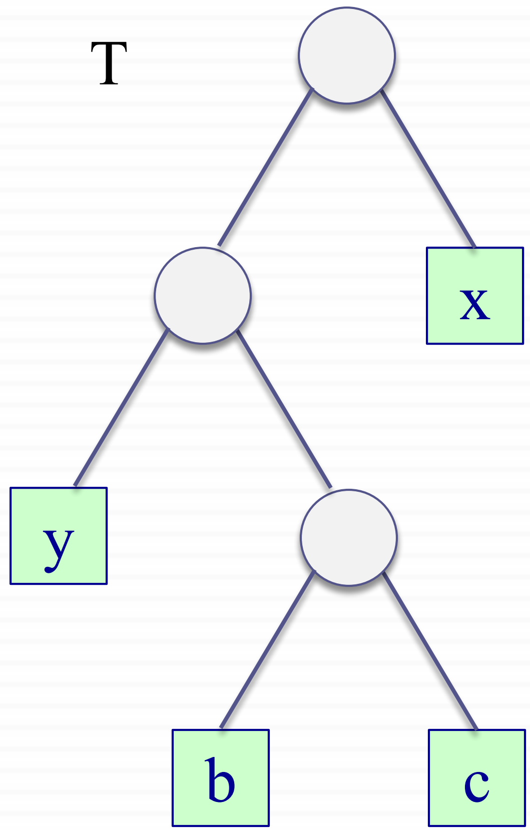
- Outline of the proof:
  - Start with an arbitrary optimal solution
  - Convert it to an optimal solution that satisfies the greedy choice property.

- *Proof*: Let T be an arbitrary optimal solution where:

  b & c are the sibling leaves with the max depth

  x & y are the characters with the lowest frequencies

Greedy Choice Property - Proof



Reminder:

b & c are the nodes with max depth

x & y are the nodes with min freq.

Without loss of generality, assume:

$f(x) \leq f(y)$

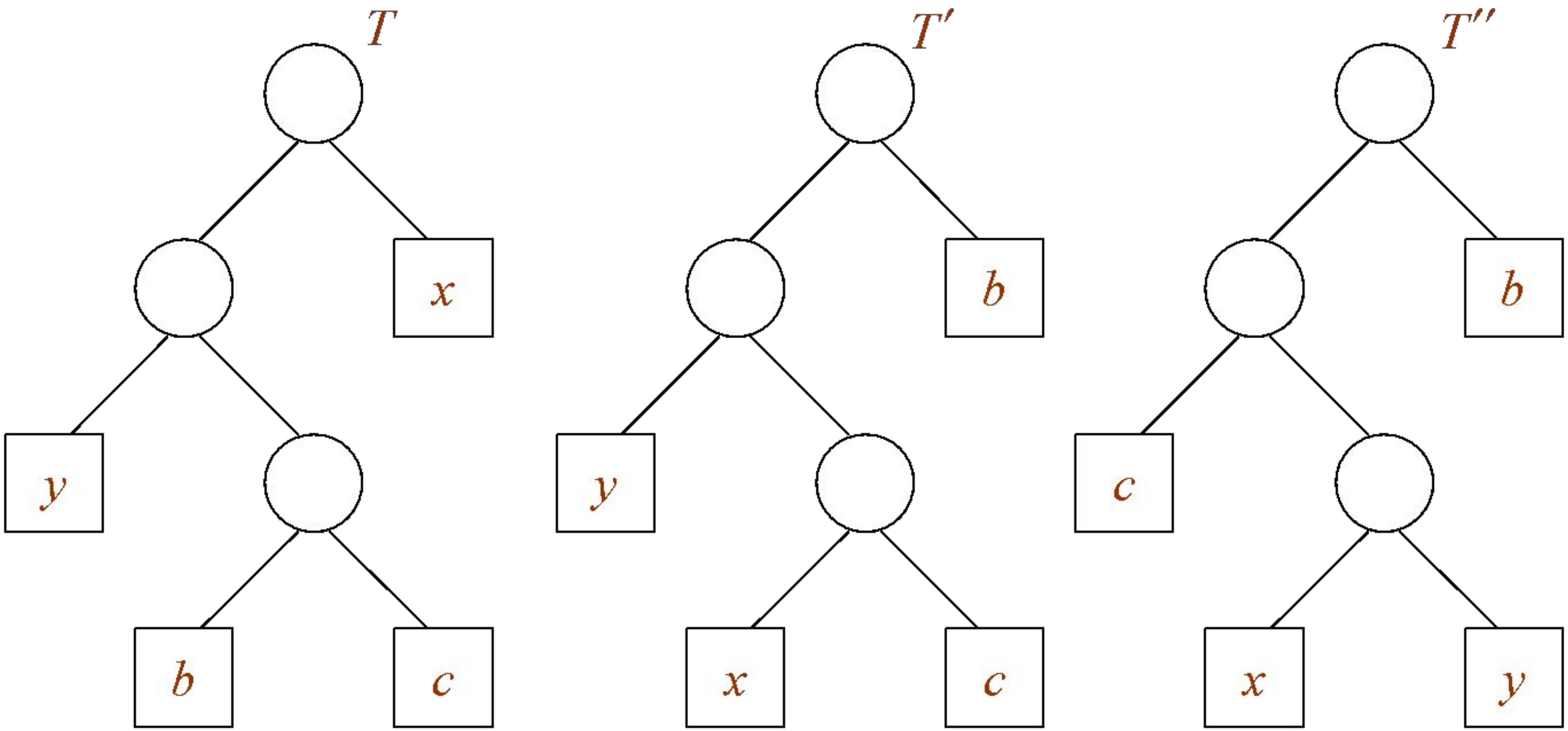
$f(b) \leq f(c)$

Then, it must be the case that:

$f(x) \leq f(b)$

$f(y) \leq f(c)$

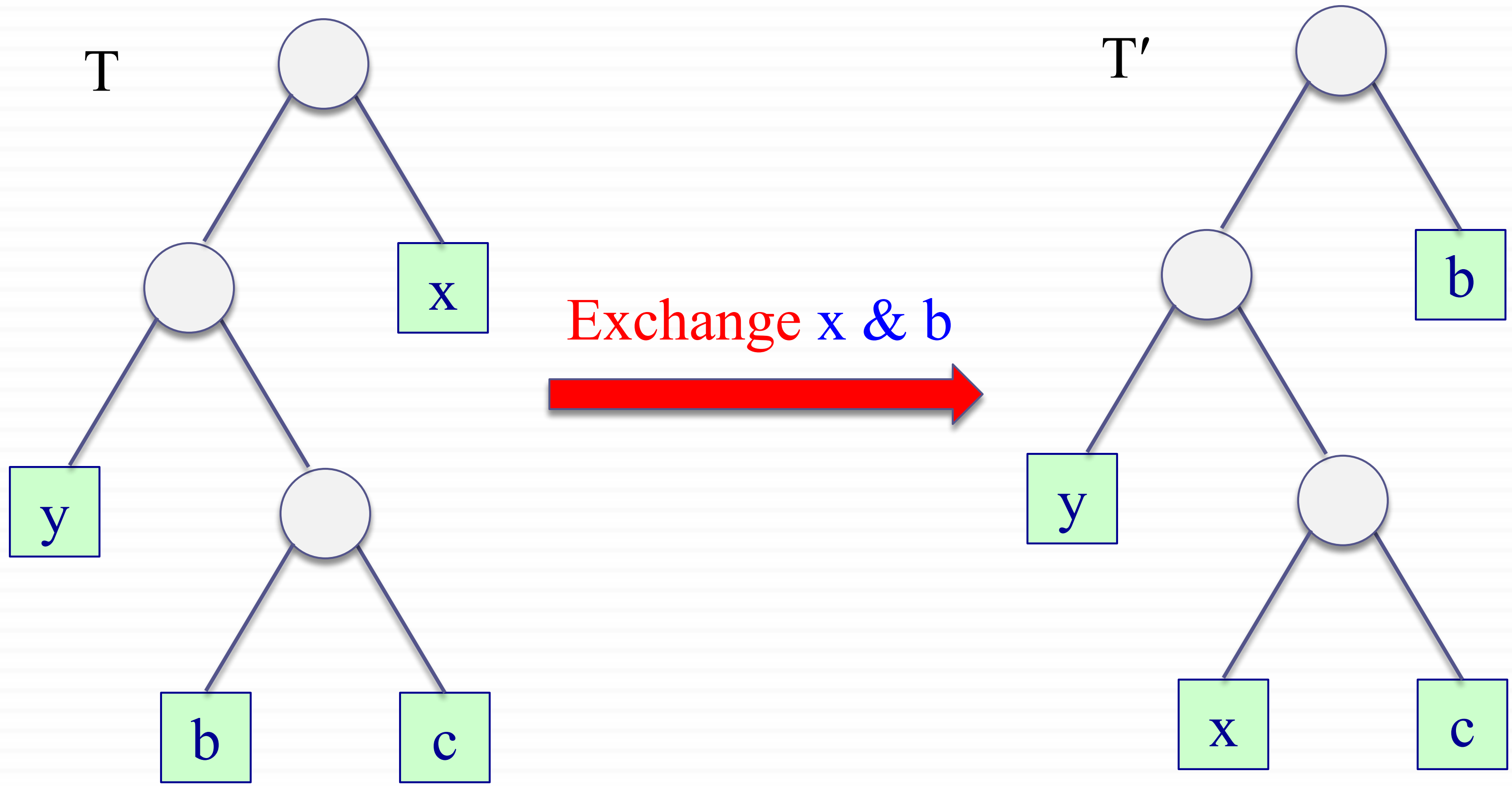
Greedy Choice Property - Proof



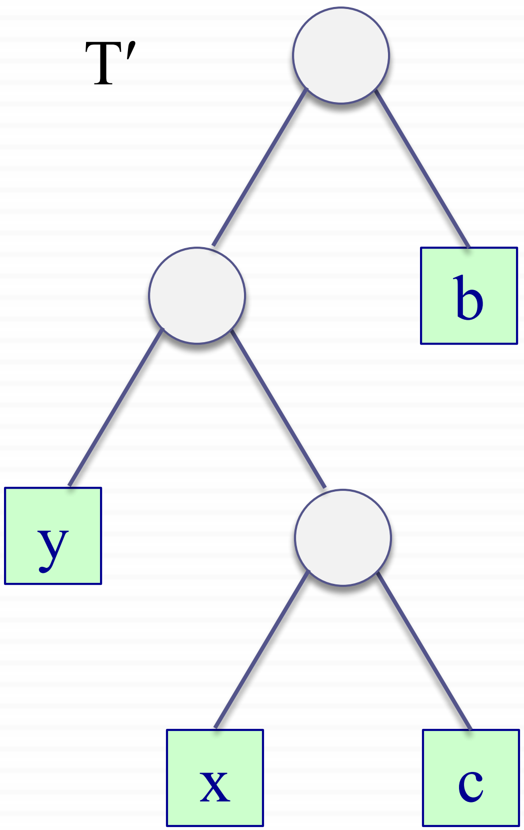
$T \Rightarrow T'$: exchange the positions of the leaves b & x

$T' \Rightarrow T''$: exchange the positions of the leaves c & y

Greedy Choice Property - Proof



Greedy Choice Property - Proof



Reminder: Cost of tree T':

$$B(T') = \sum_{c \in C} f(c)\, d_{T'}(c)$$

How does B(T′) compare to B(T)?

Reminder: $f(x) \leq f(b)$

$d_{T'}(x) = d_T(b)$ and $d_{T'}(b) = d_T(x)$

Greedy Choice Property - Proof

Reminder: $f(x) \leq f(b)$

$d_{T'}(x) = d_T(b)$ and $d_{T'}(b) = d_T(x)$

The difference in cost between T and T′:

$$B(T) - B(T') = \sum_{c \in C} f(c) d_T(c) - \sum_{c \in C} f(c) d_{T'}(c)$$

$$= f[x] d_T(x) + f[b] d_T(b) - f[x] d_{T'}(x) - f[b] d_{T'}(b)$$

$$= f[x] d_T(x) + f[b] d_T(b) - f[x] d_T(b) - f[b] d_T(x)$$

$$= f[b](d_T(b) - d_T(x)) - f[x](d_T(b) - d_T(x))$$

$$= (f[b] - f[x])(d_T(b) - d_T(x))$$

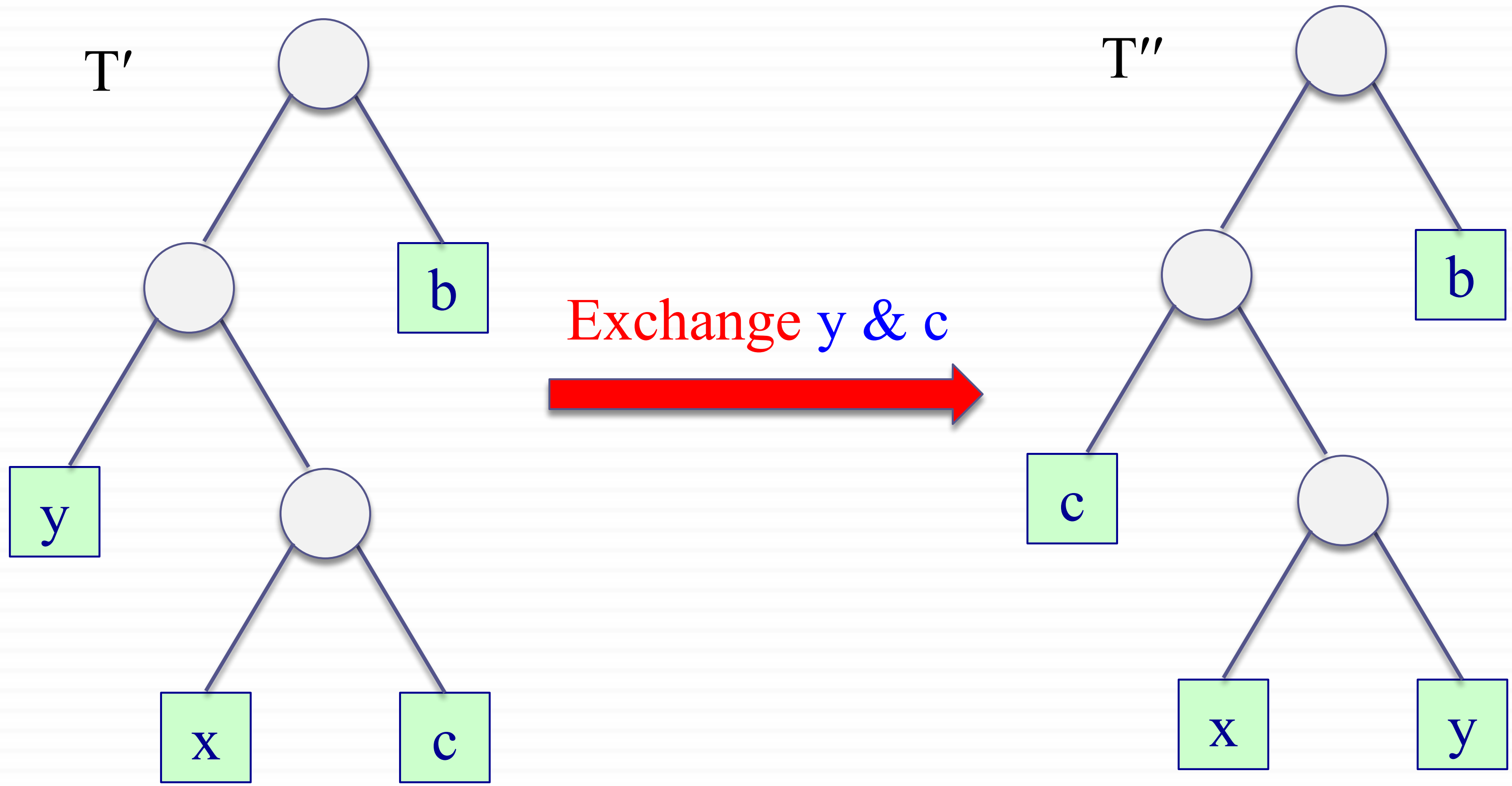
Greedy Choice Property - Proof

$$B(T) - B(T') = (f[b] - f[x])(d_T(b) - d_T(x))$$

Since $f[b] - f[x] \geq 0$ and $d_T(b) \geq d_T(x)$
therefore $B(T') \leq B(T)$

In other words, T′ is also optimal

Greedy Choice Property - Proof



Greedy Choice Property - Proof

- We can similarly show that

  $B(T') - B(T'') \geq 0 \Rightarrow B(T'') \leq B(T')$

  which implies $B(T'') \leq B(T)$
- Since T is optimal $\Rightarrow B(T'') = B(T) \Rightarrow T''$ is also optimal

- *Note*: T″ contains our greedy choice:

  Characters x & y appear as sibling leaves of max-depth in T″

- Hence, the proof for the greedy choice property is complete

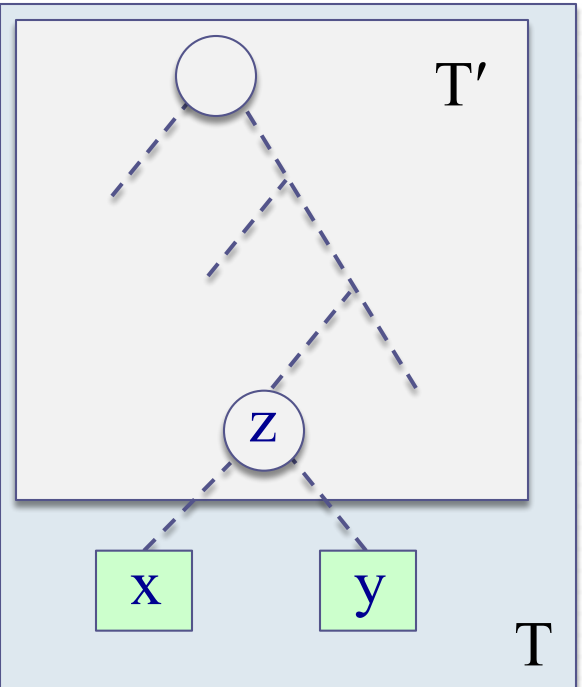
Greedy-Choice Property of Determining an Optimal Code

Lemma 1 implies that
process of building an optimal tree by mergers can begin with the greedy choice of merging those two characters with the lowest frequency

We have already proved that $B(T) = \sum_{i \in I_T} w(i)$, that is,
the total cost of the tree constructed
is the sum of the costs of its mergers (internal nodes) of all possible mergers

At each step Huffman chooses the merger that incurs the least cost

Optimal Substructure Property



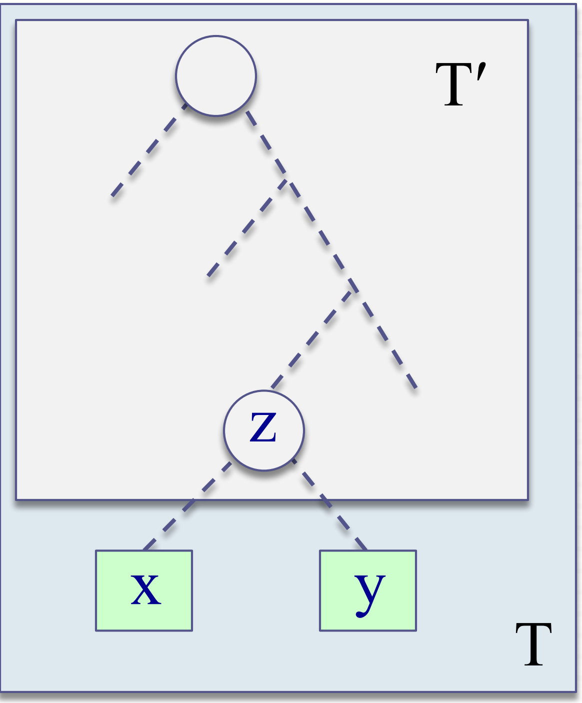
Consider an optimal solution T for alphabet C.
Let x and y be any two sibling leaf nodes in T.
Let z be the parent node of x and y in T.

Consider the subtree T′ where $T' = T - \{x, y\}$.
Here, consider z as a new character, where

$$f[z] = f[x] + f[y]$$

Optimal substructure property: T′ must be optimal for the alphabet C′, where $C' = C - \{x, y\} \cup \{z\}$

Optimal Substructure Property - Proof



Reminder:

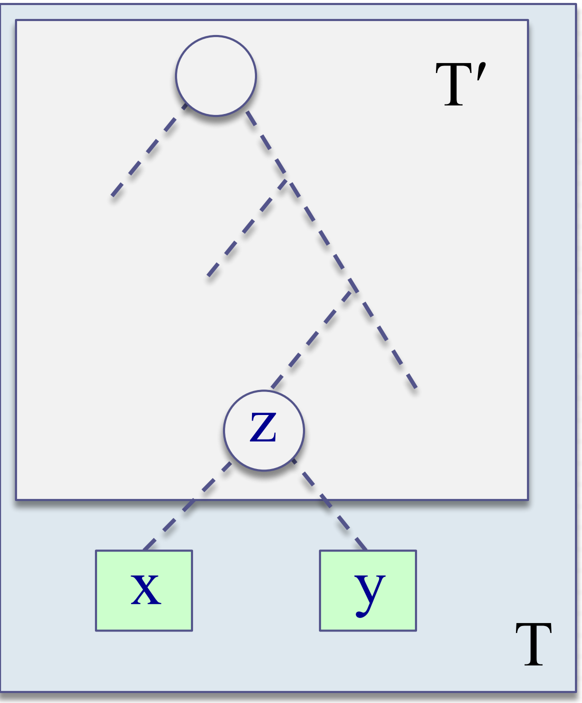
$$B(T) = \sum_{c \in C} f[c]\, d_T(c)$$

Try to express B(T) in terms of B(T′).

Note: All characters in C′ have the same depth in T and T′.

B(T) = B(T′) – cost(z) + cost(x) + cost(y)

Optimal Substructure Property - Proof



Reminder:

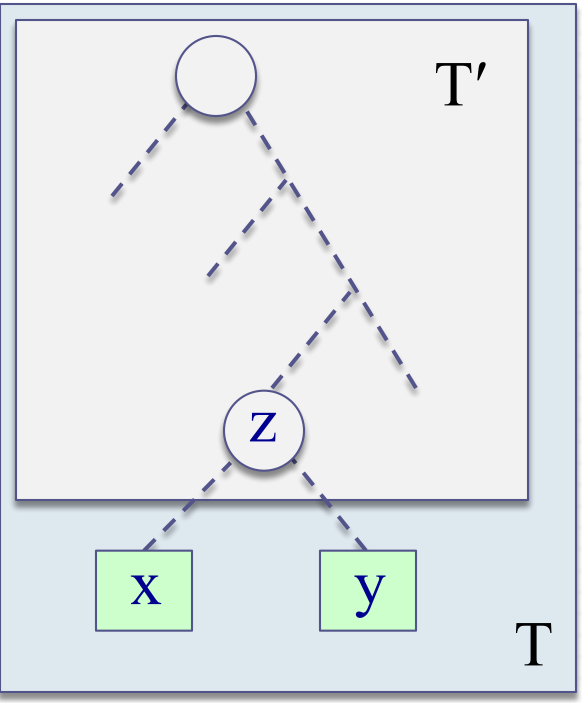
$$B(T) = \sum_{c \in C} f[c]\, d_T(c)$$

$$\begin{aligned}
B(T) &= B(T') - \text{cost}(z) + \text{cost}(x) + \text{cost}(y) \\
&= B(T') - f[z].d_T(z) + f[x].d_T(x) + f[y].d_T(y) \\
&= B(T') - f[z].d_T(z) + (f[x] + f[y])\,(d_T[z]+1) \\
&= B(T') - f[z].d_T(z) + f[z]\,(d_T[z]+1) \\
&= B(T') + f[z]
\end{aligned}$$

$d_T(x) = d_T(z) + 1$
$d_T(y) = d_T(z) + 1$

$$\boxed{B(T) = B(T') + f[x] + f[y]}$$

Optimal Substructure Property - Proof



$B(T) = B(T') + f[x] + f[y]$

We want to prove that T' is optimal for

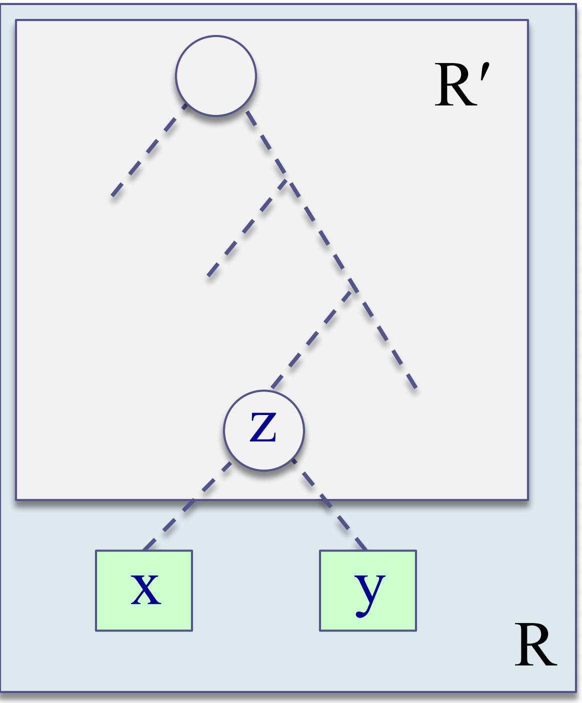
$$C' = C - \{x, y\} \cup \{z\}$$

Assume by contradiction that that there exists another solution for C' with smaller cost than T'. Call this solution R':

$$B(R') < B(T')$$

Let us construct another prefix tree R by adding x & y as children of z in R'

Optimal Substructure Property - Proof



Let us construct another prefix tree R by adding x & y as children of z in R′.

We have:

$$B(R) = B(R') + f[x] + f[y]$$

In the beginning, we assumed that:

$$B(R') < B(T')$$

So, we have:

$$B(R) < B(T') + f[x] + f[y] = B(T)$$

Contradiction! $\Rightarrow$ Proof complete

Greedy Algorithm for Huffman Coding - Summary

- For the greedy algorithm, we have proven that:
  - The greedy choice property holds.
  - The optimal substructure property holds.

- So, the greedy algorithm is optimal.